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ASYMPTOTICS OF THE POISSON PROBLEM IN DOMAINS WITH CURVED ROUGH BOUNDARIES*

ALEXANDRE L. MADUREIRA† AND FRÉDÉRIC VALENTIN†

Abstract. Effective boundary conditions (wall laws) are commonly employed to approximate PDEs in domains with rough boundaries, but it is neither easy to design such laws nor to estimate the related approximation error. A two-scale asymptotic expansion based on a domain decomposition result is used here to mitigate such difficulties, and as an application we consider the Poisson equation. The proposed scheme considers rough curved boundaries and allows a complete asymptotic expansion for the solution, highlighting the influence of the boundary curvature. The derivation and estimation of high order effective conditions is a corollary of such development. Sharp estimates for first and second order wall law approximations are considered for different Sobolev norms and show superior convergence rates in the interior of the domain. A numerical test illustrates several of the results obtained here.

Key words. Poisson equation, rough boundary, effective boundary conditions, asymptotic expansion, wall laws, curved domain

AMS subject classifications. 58J37, 35J05, 35J25

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1. Introduction.

In several applications, it is necessary to solve PDEs in domains with boundaries that are rough. Analytic solutions are rarely available, and direct numerical computations are usually out of reach since the rapidly varying wrinkles and the domain have different length scales. The traditional remedy is to pose special boundary conditions on a mollified domain to capture the geometrical influence of the wrinkles. The development of such conditions is cumbersome in general, and modeling error estimates can be out of reach. The aim of this paper is to investigate and explicate such issues.

Problems posed on domains with rough boundary pervade many fields of research. In aerodynamics, aircrafts and space shuttles are often covered with tiles, hence their walls have an array of periodic gaps [22].

Similarly, small air injecting nozzles are periodically introduced over wings of aircrafts to decrease the drag [6]. Another interesting example in the fluid mechanics is the flow field around golf balls, in which the wrinkles associated to the curvature decrease the gap between the air-pressure behind and in front of the ball. Finally, in hemodynamics, the cell surfaces of the endothelium modifies the wall shear stress produced by the blood flow, and realistic computer simulations must take this effect into account [32].

To avoid discretizing such intricate boundaries, practitioners start resorting to *wall laws*, which are effective boundary conditions that try to emulate the effect of the wrinkles without actually resolving them. Ingenious methods were developed, some in ad hoc fashion, but many of them based on firm mathematical ground.

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†Departamento de Matemática Aplicada, Laboratório Nacional de Computação Científica, Av. Getúlio Vargas 333, CEP 25651-070 Petrópolis - RJ, Brazil (alm@lncc.br, valentin@lncc.br). The first author was partially supported by the CNPq/Brazil Project 306104/2004-0, and the second author by the CNPq/Brazil Project 300348/2003-7.

Nonetheless, even when mathematics played a significant role and error estimates were found, some qualitative aspects of the resulting models that were observed numerically were missed by the theory. For instance, close to the wrinkles, the exact solution wiggles, where the model solution does not. Hence the approximation there is precarious in the H^1 norm, but fine in the L^2 norm. However, far from the boundary, where the solution is "smooth," a better approximation occurs, even when derivatives are considered. Previously, some authors considered some of these effects, but it seems hard to generalize their results to other operators, or to second order, curvature dependent approximations.

There are several papers devoted to finding good wall laws as well as the corresponding modeling errors estimates. Most of the articles fit in the framework of two-scale asymptotic expansions [14, 23, 29, 30]. For fluids, [9, 11] deal with the Stokes equations, and [5, 7, 13, 24] focus on the steady and unsteady incompressible Navier–Stokes equations. An interesting alternative way to derive effective boundary conditions is based on domain decomposition strategies as introduced in [3] for the Laplace operator, and extended to other operators [4, 7]; see also [31] for a survey and [10, 27, 28] for related techniques and problems.

The previous references considered the wrinkles to be laid upon a flat line or surface, or considered first order approximations only. In [1] first and second order models for diffraction of an electromagnetic wave by a cylindrical curved grating were considered, and some H^1 norm estimates were obtained. The references [2, 20, 21] also developed wall laws for wave scattering problems.

We extend here the results of [25], where we considered first and second order wall laws for general curved boundaries. We estimate the modeling errors in the L^2 and H^1 norms, both on the whole domain and in its interior, confirming several numerical predictions. Our mathematical framework mixes two-scale asymptotic expansions and domain decomposition ideas. Using such procedure, wall laws of arbitrary order come by naturally, and we derive first and second order effective boundary conditions. Local boundary fitted coordinates expose how the exact solution depends on the curvature of the boundary (in a sense that we make clear in what follows). This is crucial to develop high order models, which depend on the curvature. We believe that our approach is quite general and can handle more sophisticated operators.

We now outline the contents of this paper. In the next section, we introduce basic definitions and highlight the main ingredients of the approach. Section 3 presents wall laws of different orders, along with error estimates and a summary of effective conditions. Section 4 contains the development of the asymptotic expansion, and the details necessary to define the boundary layer terms are in section 5. The errors associated with the asymptotic expansion are considered in section 6. Finally, in section 7 we validate the models numerically.

We now briefly introduce and explain some basic notation that we use throughout this paper. As usual, if D is an open set, then ∂D denotes its boundary, $\overline{D}$ its closure, $L^2(D)$ is the set of square integrable functions in D, and $H^s(D)$ is the corresponding Sobolev space of order s, for a real number s. We denote the norms of those spaces by $\|\cdot\|_{L^2(D)}$ and $\|\cdot\|_{H^s(D)}$. Also, the symbol $\cdot|_D$ denotes the restriction of a function to the domain D. Without loss of generality, we have chosen to work in two dimensions. Nonetheless, all that follows can be generalized to the three-dimensional case. Bold fonts indicate two-dimensional vectors, and the symbol $\partial_{\boldsymbol{n}}$ indicates the (outward) normal derivative with respect to the domain Ω_s. Similarly, ∂_x denotes the derivative with respect to the variable x, etc. We denote by c a generic constant (not necessarily the same in all occurrences) which is independent of ε, but may depend on Ω_s and Sobolev norms of f.

2. Definitions and main results. We denote the domain of interest by $\Omega^\varepsilon \subset \mathbb{R}^2$, which is open, bounded, and ε-dependent. Here, ε indicates the length scale of the roughness element. It is convenient to consider $\Omega^\varepsilon = \Omega_s \cup \Omega_r^\varepsilon \cup \Gamma$, where the limit domain Ω_s is open and ε-independent, the open set Ω_r^ε depends on ε with $\Omega_s \cap \Omega_r^\varepsilon = \emptyset$, and the interface $\Gamma = \partial\Omega_s \cap \partial\Omega_r^\varepsilon$. The precise definition of these subdomains follow.

We assume that Ω_s has its boundary $\partial\Omega_s$ constituted of two disjoint parts, a smooth inner boundary Γ, and a Lipschitz-continuous outer boundary. We arc length parameterize the smooth curve Γ by an ε-independent function $\boldsymbol{\psi} : \mathbb{R} \to \mathbb{R}^2$, which is periodic with period L, and injective in $(0, L)$. In other words, Γ is a simple closed curve with length L, and it bounds a region of the plane, which we call its interior. We orient Γ in such a way that it is positively oriented, i.e., going along the direction of increasing parameter, the interior of the curve stays on the left. We assume that $\varepsilon = L/N$, for some positive integer N.

The domain Ω_r^ε has Γ as its outer boundary and Γ_r^ε as its inner boundary. The curve Γ_r^ε, which is also closed, is defined as a perturbation of Γ, and is parameterized by

$$\boldsymbol{\psi}^\varepsilon(\theta) = \boldsymbol{\psi}(\theta) + \varepsilon\big[d_0 - \psi_r(\varepsilon^{-1}\theta)\big]\boldsymbol{n}(\theta),$$

where $\boldsymbol{n}$ is the normal vector pointing towards the interior of Γ, and $d_0 > 1$ is such that $d_0\varepsilon$ is smaller than the minimum radius of curvature of Γ. The function $\psi_r : \mathbb{R} \to \mathbb{R}$ is independent of ε, Lipschitz-continuous with $\psi_r(0) = 0$, and periodic with period 1. Without loss of generality, we assume that $\|\psi_r\|_{L^\infty(\mathbb{R})} = 1$. Formally,

$$\Omega_r^\varepsilon \;=\; \{\, \mathbf{x} = \boldsymbol{\psi}(\theta) + (\varepsilon\, d_0 - \rho)\, \boldsymbol{n}(\theta) \;:\; \theta \in [0, L),\; \rho \in (\varepsilon\, \psi_r(\varepsilon^{-1}\theta), \varepsilon\, d_0)\,\}.$$

Hence, Ω^ε has its boundary constituted of two parts, a rough inner boundary Γ_r^ε, and a Lipschitz-continuous outer boundary that is independent of ε and does not intersect Γ. Note that Γ splits the original domain Ω^ε into two regions Ω_s and Ω_r^ε, one ε-independent, and the other containing the wrinkles. The subdomain Ω_r^ε is the set of points between Γ and Γ_r^ε, and Ω_s is the ε-independent domain comprehended between Γ and the outer boundary of Ω^ε. This is the set of all points at least "slightly away" from the wrinkles. See Figure 1. Finally, we denote a typical point in it by $\mathbf{x} = (x_1, x_2)$.

We consider the problem

$$\begin{aligned} -\,\Delta\, u^\varepsilon &= f \quad \text{in } \Omega^\varepsilon, \\ u^\varepsilon &= 0 \quad \text{on } \partial\Omega^\varepsilon, \end{aligned} \tag{1}$$

where f has support in Ω_s.

It is clear that the solution u^ε depends in a nontrivial way on the small parameter ε. It is our goal to unfold this dependence and show how to develop models for (1). It is possible to expand u^ε in a formal power series with respect to ε. This expansion is far from trivial since it has to take into account effects from the wrinkles as well as from the curvature. To focus on the main steps of our approach, we start by presenting the first few terms of this expansion, and only in Ω_s. The details of the asymptotics are considered in section 4.

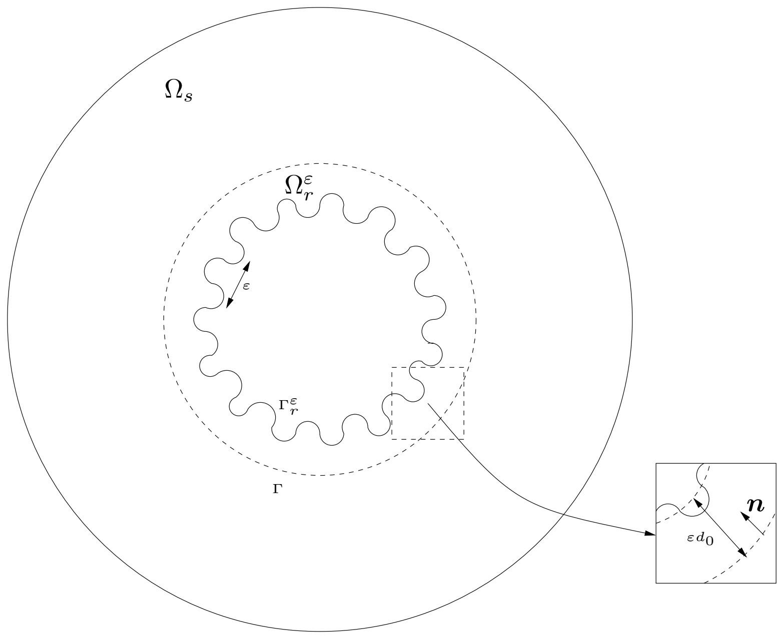

FIG. 1. *The domain* Ω^ε.

The asymptotic expansion of u^ε in Ω_s is a formal combination of an ε-independent part and a highly oscillatory part which decays exponentially to zero away from Γ,

$$u^\varepsilon \sim u^0 + \varepsilon\, u^1 + \varepsilon\, W^{1,0} + \dots \quad \text{in } \Omega_s. \tag{2}$$

While u^0, u^1 are ε-independent, the oscillatory function $W^{1,0}$ depends on ε, but only in a trivial manner.

It is natural to define the first term of the asymptotic such that

$$\begin{aligned} -\Delta\, u^0 &= f \quad \text{in } \Omega_s, \\ u^0 &= 0 \quad \text{on } \partial\Omega_s. \end{aligned} \tag{3}$$

To continue the description of the expansion, it is necessary to introduce a *cell problem*. This is no different from other singularly perturbed problems, perhaps elliptic PDEs with highly oscillatory coefficients being the most notorious. Such cell problems are a essential part in up-scaling procedures and brings information related to the small scale geometry into the large scale behavior of the solution.

In the present case, the cell problem is defined in the semi-infinite strip Ω_r, which "contains" the geometry of the wrinkles,

$$\Omega_r = \{\, (\hat\theta, \hat\rho) \in \mathbb{R}^2 \,:\, \hat\theta \in (0,1),\ \hat\rho \in (\psi_r(\hat\theta), +\infty) \,\},$$

i.e., Ω_r occupies the region delimited by straight lateral boundaries at $\hat\theta = 0$ and $\hat\theta = 1$, and by the lower boundary $\Gamma_r = \{\, (\hat\theta, \psi_r(\hat\theta)) \,:\, \hat\theta \in (0,1) \,\}$; see Figure 2.

We define $C^\infty_{\mathrm{per}}(\Omega_r)$ by restricting to Ω_r the functions in $C^\infty(\mathbb{R}^2)$ which are one-periodic with respect to $\hat\theta$. Let $H^1_{\mathrm{per}}(\Omega_r)$ be the closure of $C^\infty_{\mathrm{per}}(\Omega_r)$ with respect to the $H^1(\Omega_r)$ norm. We also introduce the space of exponentially decaying functions

$$S(\Omega_r) = \{ w \in H^1_{\mathrm{per}}(\Omega_r) \,:\, w\, e^{\alpha\hat\rho} \in H^1(\Omega_r) \text{ for some } \alpha > 0 \}\,.$$

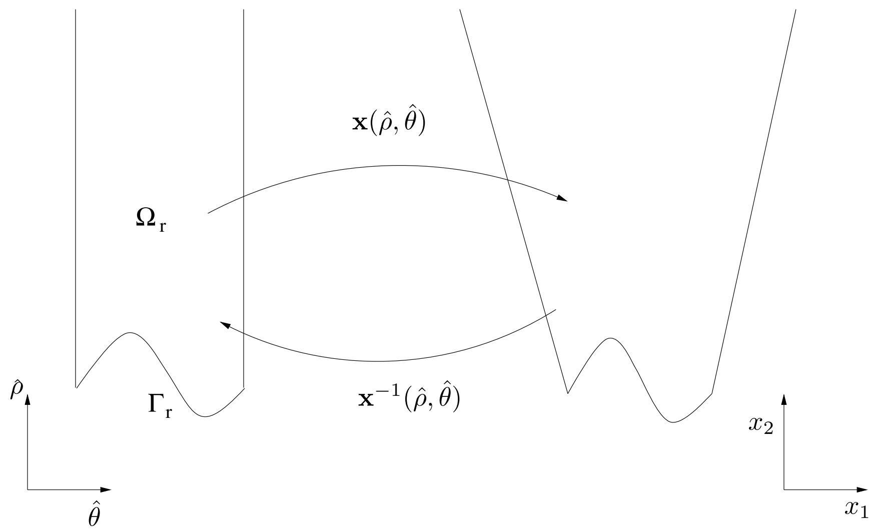

Fig. 2. *The cell domain.*

The following result guarantees that certain Poisson problems posed in Ω_r are well posed, and the solutions have nice properties. The reference [5] deals with related questions for the Stokes operator.

Lemma 1. *Let $F \in S^*(\Omega_r)$, the dual space of $S(\Omega_r)$. Then there is a unique solution $w \in H^1_{loc}(\Omega_r)$ that is one-periodic with respect to $\hat\theta$, and such that $\nabla w \in L^2(\Omega_r)$, and*

$$(\partial_{\hat\theta\hat\theta} + \partial_{\hat\rho\hat\rho})w = F \quad \text{in } \Omega_r, \qquad w = \hat\rho \quad \text{on } \Gamma_r. \tag{4}$$

Moreover, there exists a unique constant z such that $w - z \in S(\Omega_r)$, and, if $F \equiv 0$,

$$z \le \|\psi_r\|_{L^\infty(\mathbb{R})}.$$

Proof. A simple modification of the beautiful arguments of [8, Lemma 4.4] guarantees well posedness and yields a proof of the decaying behavior of the solution towards a constant. Assume now that w is harmonic, and for $t \ge \|\psi_r\|_{L^\infty(\mathbb{R})}$, let $\gamma_t = (0,1) \times \{t\}$. Then Green's identity yields that $\int_{\gamma_t} \partial_{\boldsymbol{n}}\, w$ is constant with respect to t. Letting $t \to \infty$, we have that actually $\int_{\gamma_t} \partial_{\boldsymbol{n}}\, w = 0$ for all $t \ge \|\psi_r\|_{L^\infty(\mathbb{R})}$. Using again Green's identity in $S_{t,\tilde t} = (0,1) \times (t, \tilde t)$, for $\tilde t > t \ge \|\psi_r\|_{L^\infty(\mathbb{R})}$, we gather that

$$\int_{\partial S_{t,\tilde t}} w\, \partial_{\boldsymbol{n}}\, \hat\rho = \int_{\partial S_{t,\tilde t}} \hat\rho\, \partial_{\boldsymbol{n}}\, w = 0.$$

Thus $\int_{\gamma_t} w = \int_{\gamma_{\tilde t}} w$, and letting $\tilde t \to \infty$, we see that $z = \int_{\gamma_t} w$. Then $z \le \|w\|_{L^\infty(\gamma_t)}$, and we conclude from the maximum principle [17] that $z \le \|\psi_r\|_{L^\infty(\mathbb{R})}$. ∎

Remark 1. Note that $S^*(\Omega_r)$ contains, for instance, functions that grow at most algebraically with respect to $\hat\rho$.

We define $w^{0,0} \in S(\Omega_r)$, and the constant $z^{0,0}$ as the solution of

$$(\partial_{\hat\theta\hat\theta} + \partial_{\hat\rho\hat\rho})\, w^{0,0} = 0 \quad \text{in } \Omega_r, \qquad w^{0,0} = \hat\rho - z^{0,0} \quad \text{on } \Gamma_r. \tag{5}$$

It follows immediately from Lemma 1 that (5) is well defined. Both $z^{0,0}$ and $w^{0,0}$ are related to the boundary layers that naturally appear in the original problem.

To incorporate the influence of the cell problem into the asymptotic expansion (2), we introduce boundary fitted coordinates (θ, ρ) for points "close enough" to Γ; see [12]. Let ρ_0 be a positive number smaller than the minimum radius of curvature of Γ. For a given $\theta \in [0, L)$ and $\rho \in (\varepsilon\,\psi_r(\varepsilon^{-1}\theta), \varepsilon\, d_0 + \rho_0)$, we have

$$\mathbf{x}(\theta, \rho) = \boldsymbol{\psi}(\theta) + (\varepsilon\, d_0 - \rho)\, \boldsymbol{n}(\theta) \in \Omega^\varepsilon.$$

Note that $|\rho - \varepsilon\, d_0| = \operatorname{dist}(\mathbf{x}, \Gamma)$ is the distance between $\mathbf{x}$ and Γ and that the above map defines a local diffeomorphism. The change of coordinates $\mathbf{x} \to (\rho, \theta)$ is not well defined globally in Ω^ε, but only for points with distance from Γ smaller than the minimum radius of curvature of Γ.

In such a new system of coordinates, we simply write the normal derivative of u^0 at a point $\mathbf{x} \in \Gamma$ as $\partial_{\boldsymbol{n}}\, u^0(\theta, \varepsilon\, d_0)$, where θ is such that $\mathbf{x} = \boldsymbol{\psi}(\theta)$. We set

$$W^{1,0}(\theta, \rho) = \Upsilon(\varepsilon d_0 + \rho)\, w^{0,0}(\varepsilon^{-1}\theta, \varepsilon^{-1}\rho)\, \partial_{\boldsymbol{n}}\, u^0(\theta, \varepsilon\, d_0) \tag{6}$$

in the formal expansion (2), where $\Upsilon(\cdot)$ is a smooth ε-independent cutoff function, such that $\Upsilon(\rho)$ equals one if ρ is smaller than a fixed number smaller than ρ_0, and vanishes for $\rho \geq \rho_0$. For instance, we may set Υ identically equal to one in $(-\infty, \rho_0/3]$ and vanishing in $[\rho_0, +\infty)$. The following estimates follow from standard regularity results and scaling arguments (see also Lemma 3):

$$\|W^{1,0}\|_{L^2(\Omega^\varepsilon)} \leq c\, \varepsilon^{1/2}, \qquad \|W^{1,0}\|_{H^1(\Omega^\varepsilon)} \leq c\, \varepsilon^{-1/2}. \tag{7}$$

Finally, let

$$\begin{gathered} -\Delta\, u^1 = 0 \quad \text{in } \Omega_s, \\ u^1 = \left(-d_0 + z^{0,0}\right)\, \partial_{\boldsymbol{n}}\, u^0 \quad \text{on } \Gamma, \qquad u^1 = 0 \quad \text{on } \partial\Omega_s \backslash \Gamma. \end{gathered} \tag{8}$$

Albeit (2) is formal, we show below (Theorem 4) that if

$$e = u^\varepsilon - u^0 - \varepsilon\, u^1 - \varepsilon\, W^{1,0},$$

then there exists an ε-independent constant c such that

$$\|e\|_{H^1(\Omega_s)} \leq c\, \varepsilon^{3/2}. \tag{9}$$

Several other estimates follow from a combination of (9), the triangle inequality, and (7). For instance, we easily find that

$$\|u^\varepsilon - u^0 - \varepsilon\, u^1\|_{H^1(\Omega_s)} \leq \|e\|_{H^1(\Omega_s)} + \|\varepsilon\, W^{1,0}\|_{H^1(\Omega_s)} \leq c\, \varepsilon^{1/2}, \tag{10}$$

$$\|u^\varepsilon - u^0 - \varepsilon\, u^1\|_{L^2(\Omega_s)} \leq \|e\|_{H^1(\Omega_s)} + \|\varepsilon\, W^{1,0}\|_{L^2(\Omega_s)} \leq c\, \varepsilon^{3/2}. \tag{11}$$

The culprit for the low convergence rates in some of the estimates above are the boundary layers. Hence, *interior estimates*, i.e., estimates that bound the errors in domains that are away from the boundary ought to show better rates. It is possible to obtain such estimates by adding a higher order boundary layer term similar to (6) to the expansion. The new term, which we denote by $\check{\mathrm{W}}$, behaves like $W^{1,0}$, i.e., decays exponentially fast to zero with ρ/ε, and

$$\|\check{W}\|_{L^2(\Omega^\varepsilon)} \leq c\, \varepsilon^{1/2}, \qquad \|\check{W}\|_{H^1(\Omega^\varepsilon)} \leq c\, \varepsilon^{-1/2}.$$

TABLE 1
Relative error convergence rates for a zeroth order model.

quantity	$L^2(\Omega_s)$ error	$L^2(\Omega_s^{int})$ error
u	$O(\varepsilon)$	$O(\varepsilon)$
∇u	$O(\varepsilon^{1/2})$	$O(\varepsilon)$

For the sake of simplicity, we do not describe $\check{W}$ now. Such a function is defined after two different cell problems are solved; see (19) and (20). It is now possible to derive a better estimate in the H^1 norm:

$$\|u^\varepsilon - u^0 - \varepsilon\, u^1 - \varepsilon\, W^{1,0} - \varepsilon^2\, \check{W}\|_{H^1(\Omega_s)} \le c\,\varepsilon^2. \tag{12}$$

Finally, let $\Omega_s^{int} \subset \Omega_s$ be such that $\overline{\Omega_s^{int}} \cap \overline{\Gamma} = \emptyset$. Then

$$\begin{aligned} &\|u^\varepsilon - u^0 - \varepsilon\, u^1\|_{H^1(\Omega_s^{int})} \\ &\le \|u^\varepsilon - u^0 - \varepsilon\, u^1 - \varepsilon\, W^{1,0} - \varepsilon^2\, \check{W}\|_{H^1(\Omega_s)} + \|\varepsilon\, W^{1,0}\|_{H^1(\Omega_s^{int})} + \|\varepsilon^2\, \check{W}\|_{H^1(\Omega_s^{int})} \le c\,\varepsilon^2. \end{aligned} \tag{13}$$

Note in (13) that the exponential decay of both $W^{1,0}$ and $\check{W}$ guarantees that their $H^1(\Omega_s^{int})$ norms are also exponentially small and hence bounded by $c\,\varepsilon^2$.

3. Derivation of wall laws.

3.1. Zeroth order wall law. A first attempt to approximate u^ε would use u^0. It immediately follows from (10), (11) and regularity estimates for u^1 that

$$\begin{gathered} \|u^\varepsilon - u^0\|_{H^1(\Omega_s)} \le \|u^\varepsilon - u^0 - \varepsilon\, u^1\|_{H^1(\Omega_s)} + \|\varepsilon\, u^1\|_{H^1(\Omega_s)} \le c\,\varepsilon^{1/2}, \\ \|u^\varepsilon - u^0\|_{L^2(\Omega_s)} \le \|u^\varepsilon - u^0 - \varepsilon\, u^1\|_{L^2(\Omega_s)} + \|\varepsilon\, u^1\|_{L^2(\Omega_s)} \le c\,\varepsilon. \\ \|u^\varepsilon - u^0\|_{L^2(\Omega_s^{int})} \le \|u^\varepsilon - u^0 - \varepsilon\, u^1\|_{H^1(\Omega_s^{int})} + \|\varepsilon\, u^1\|_{H^1(\Omega_s)} \le c\,\varepsilon. \end{gathered}$$

The $O(\varepsilon^{1/2})$ error in the $H^1(\Omega_s)$ norm is due to the inability of this approximation to capture the oscillatory behavior of the solution close to the wrinkles. This explains the better performance in the $L^2(\Omega_s)$ and interior norms. Table 1 presents various relative error estimates with respect to ε, including interior estimates.

3.2. First order wall law. Inspired by (2), (10)–(13), we would like to approximate u^ε by the first terms of its asymptotic expansion, but without solving the PDEs that define these terms. A first step in this direction is to consider only the functions that actually have influence in the interior of the domain, i.e., we assume

$$u^\varepsilon \approx u^0 + \varepsilon\, u^1. \tag{14}$$

Thus, over Γ, from (3), (8), and (14),

$$u^\varepsilon \approx \varepsilon\,(-d_0 + z^{0,0})\,\partial_{\boldsymbol{n}}\, u^0, \qquad \partial_{\boldsymbol{n}}\, u^\varepsilon \approx \partial_{\boldsymbol{n}}\, u^0 + \varepsilon\ \partial_{\boldsymbol{n}}\, u^1. \tag{15}$$

So

$$u^\varepsilon + \varepsilon\,(d_0 - z^{0,0})\,\partial_{\boldsymbol{n}}\, u^\varepsilon \approx \varepsilon^2\,(d_0 - z^{0,0})\,\partial_{\boldsymbol{n}}\, u^1,$$

TABLE 2
Relative error convergence rates for a first order model.

quantity	$L^2(\Omega_s)$ error	$L^2(\Omega_s^{int})$ norm error
u	$O(\varepsilon^{3/2})$	$O(\varepsilon^2)$
∇u	$O(\varepsilon^{1/2})$	$O(\varepsilon^2)$

on Γ, and this amount can be small enough for certain applications. We define then $\bar{u} \in H^1(\Omega_s)$ approximating u^ε in Ω_s by

$$
\begin{gathered}
-\Delta \bar{u} = f \quad \text{in } \Omega_s, \\
\bar{u} + \varepsilon\,(d_0 - z^{0,0})\,\partial_{\boldsymbol{n}}\,\bar{u} = 0 \quad \text{on } \Gamma, \qquad \bar{u} = 0 \quad \text{on } \partial\Omega_s \backslash \Gamma.
\end{gathered} \tag{16}
$$

It follows from Lemma 1 that $z^{0,0} \le \|\psi_r\|_{L^\infty(\mathbb{R})}$, and since $\|\psi_r\|_{L^\infty(\mathbb{R})} = 1 < d_0$, the difference $d_0 - z^{0,0}$ is positive. Thus (16) is well posed for all positive ε.

To estimate the modeling error, we first note that if $\bar{e} = \bar{u} - u^0 - \varepsilon\, u^1$, then

$$
\begin{gathered}
-\Delta \bar{e} = 0 \quad \text{in } \Omega_s, \\
\bar{e} + \varepsilon\,(d_0 - z^{0,0})\,\partial_{\boldsymbol{n}}\,\bar{e} = -\varepsilon^2\,(d_0 - z^{0,0})\,\partial_{\boldsymbol{n}}\,u^1 \quad \text{on } \Gamma, \qquad \bar{e} = 0 \quad \text{on } \partial\Omega_s \backslash \Gamma.
\end{gathered} \tag{17}
$$

It follows from regularity estimates [16, Theorem 4.24] that there exists an ε-independent constant c such that

$$
\|\bar{e}\|_{H^1(\Omega_s)} \le c\,\varepsilon^2.
$$

The modeling error estimates are then as follows:

$$
\|u^\varepsilon - \bar{u}\|_{H^1(\Omega_s)} \le \|u^\varepsilon - u^0 - \varepsilon\, u^1\|_{H^1(\Omega_s)} + \|\bar{u} - u^0 - \varepsilon\, u^1\|_{H^1(\Omega_s)} \le c\,\varepsilon^{1/2},
$$

where we used the triangle inequality and (10).

Analogously, using (11), (13), we obtain L^2 and interior estimates

$$
\|u^\varepsilon - \bar{u}\|_{L^2(\Omega_s)} \le c\,\varepsilon^{3/2} \qquad \|u^\varepsilon - \bar{u}\|_{H^1(\Omega_s^{int})} \le c\,\varepsilon^2.
$$

We summarize the convergence results in Table 2.

3.3. Second order wall law. The derivation of higher order approximations to u^ε follows the same *modus operandi* as in the previous subsection. We first consider only the terms that have influence away from Γ_r^ε and assume that

$$
u^\varepsilon \approx u^0 + \varepsilon\, u^1 + \varepsilon^2\, u^2. \tag{18}
$$

To define the term u^2 above we introduce two new cell problems, seeking $w^{1,0}$ and $w^{1,1}$ in $S(\Omega_r)$, and the constants $z^{1,0}$ and $z^{1,1}$ satisfying

$$
-(\partial_{\hat{\theta}\hat{\theta}} + \partial_{\hat{\rho}\hat{\rho}})\, w^{1,0} = \chi - \partial_{\hat{\rho}}\, w^{0,0} + 2\,\hat{\rho}\,\partial_{\hat{\theta}\hat{\theta}}\, w^{0,0} \quad \text{in } \Omega_r, \qquad w^{1,0} = -z^{1,0} \quad \text{on } \Gamma_r, \tag{19}
$$

$$
-(\partial_{\hat{\theta}\hat{\theta}} + \partial_{\hat{\rho}\hat{\rho}})\, w^{1,1} = 2\,\partial_{\hat{\theta}}\, w^{0,0} \quad \text{in } \Omega_r, \qquad w^{1,1} = z^{1,1} \quad \text{on } \Gamma_r, \tag{20}
$$

where $\chi(\hat\rho) = 1$ if $\hat\rho < d_0$, and $\chi(\hat\rho) = 0$ if $\hat\rho \geq d_0$. The previous cell problems are well posed, as Lemma 1 guarantees. The expression of $\check{W}$ mentioned on page 1455 is as follows:

$$\begin{aligned}\check{W}(\theta,\rho) = \Upsilon(\varepsilon d_0+\rho)\big[&w^{1,0}(\varepsilon^{-1}\theta,\varepsilon^{-1}\rho)\kappa(\theta)\,\partial_{\boldsymbol{n}}\,u^0(\theta,\varepsilon\, d_0)\\ &+ w^{1,1}(\varepsilon^{-1}\theta,\varepsilon^{-1}\rho)\partial_\theta\,\partial_{\boldsymbol{n}}\,u^0(\theta,\varepsilon\, d_0) + w^{0,0}(\varepsilon^{-1}\theta,\varepsilon^{-1}\rho)\,\partial_{\boldsymbol{n}}\,u^1(\theta,\varepsilon\, d_0)\big],\end{aligned}$$

where $\kappa(\theta)$ is the curvature of Γ at the point $\boldsymbol{\psi}(\theta)$.

Next, we define u^2 by

$$\begin{gathered}-\,\Delta\, u^2 = 0 \quad \text{in } \Omega_s,\\ u^2 = (-d_0 + z^{0,0})\,\partial_{\boldsymbol{n}}\,u^1 + z^{1,0}\kappa\,\partial_{\boldsymbol{n}}\,u^0 + z^{1,1}\partial_\theta\,\partial_{\boldsymbol{n}}\,u^0 \quad \text{on } \Gamma, \qquad u^2 = 0 \quad \text{on } \partial\Omega_s\backslash\Gamma,\end{gathered}$$

and the estimates below follow:

$$\|u^\varepsilon - u^0 - \varepsilon\, u^1 - \varepsilon^2\, u^2\|_{H^1(\Omega_s)} \leq c\,\varepsilon^{1/2}, \tag{21}$$

$$\|u^\varepsilon - u^0 - \varepsilon\, u^1 - \varepsilon^2\, u^2\|_{L^2(\Omega_s)} \leq c\,\varepsilon^{3/2}, \tag{22}$$

$$\|u^\varepsilon - u^0 - \varepsilon\, u^1 - \varepsilon^2\, u^2\|_{H^1(\Omega_s^{int})} \leq c\,\varepsilon^{3}. \tag{23}$$

If (18) holds, then

$$\begin{aligned}u^\varepsilon &\approx \varepsilon\,(-d_0 + z^{0,0} + \varepsilon\, z^{1,0}\kappa)\,\partial_{\boldsymbol{n}}\,u^0 + \varepsilon^2\,(-d_0 + z^{0,0})\,\partial_{\boldsymbol{n}}\,u^1 + \varepsilon^2\, z^{1,1}\partial_\theta\,\partial_{\boldsymbol{n}}\,u^0\\ &= \varepsilon\,(-d_0 + z^{0,0} + \varepsilon\, z^{1,0}\kappa)\,\partial_{\boldsymbol{n}}\,u^0 + \varepsilon^2\,(-d_0 + z^{0,0})\,\partial_{\boldsymbol{n}}\,u^1 + \varepsilon^2\,\frac{z^{1,1}}{-d_0 + z^{0,0}}\partial_\theta u^1\end{aligned}$$

over Γ, where we used from (8) that $\partial_\theta u^1 = (-d_0 + z^{0,0})\partial_\theta\,\partial_{\boldsymbol{n}}\,u^0$ to obtain the last equality. We also have

$$\partial_{\boldsymbol{n}}\,u^\varepsilon \approx \partial_{\boldsymbol{n}}\,u^0 + \varepsilon\;\partial_{\boldsymbol{n}}\,u^1 + \varepsilon^2\;\partial_{\boldsymbol{n}}\,u^2, \qquad \partial_\theta u^\varepsilon \approx \partial_\theta u^0 + \varepsilon\,\partial_\theta u^1 + \varepsilon^2\,\partial_\theta u^2,$$

over Γ. Hence,

$$\begin{aligned}u^\varepsilon + (\varepsilon\, d_0 - \varepsilon\, z^{0,0} - \varepsilon^2\, z^{1,0}\,\kappa)\,\partial_{\boldsymbol{n}}\,u^\varepsilon - \varepsilon^2\,\frac{z^{1,1}}{-d_0 + z^{0,0}}\,\partial_\theta u^\varepsilon \approx -\varepsilon^3\, z^{1,0}\,\kappa\;\partial_{\boldsymbol{n}}\,u^1\\ -\,\varepsilon^3(-d_0 + z^{0,0} + \varepsilon\, z^{1,0}\,\kappa)\;\partial_{\boldsymbol{n}}\,u^2 - \varepsilon^3\,\frac{z^{1,1}}{-d_0 + z^{0,0}}\,\partial_\theta u^1 - \varepsilon^4\,\frac{z^{1,1}}{-d_0 + z^{0,0}}\,\partial_\theta u^2.\end{aligned}$$

So we define $\bar{\bar{u}} \in H^1(\Omega_s)$ approximating u^ε in Ω_s by

$$\begin{gathered}-\,\Delta\,\bar{\bar{u}} = f \quad \text{in } \Omega_s,\\ \bar{\bar{u}} + (\varepsilon\, d_0 - \varepsilon\, z^{0,0} - \varepsilon^2\, z^{1,0}\,\kappa)\;\partial_{\boldsymbol{n}}\,\bar{\bar{u}} - \varepsilon^2\,\frac{z^{1,1}}{-d_0 + z^{0,0}}\,\partial_\theta\bar{\bar{u}} = 0 \quad \text{on } \Gamma,\\ \bar{\bar{u}} = 0 \quad \text{on } \partial\Omega_s\backslash\Gamma.\end{gathered} \tag{24}$$

Since Γ is a closed curve,

$$\int_\Gamma \bar{\bar{u}}\partial_\theta\bar{\bar{u}}\,d\theta = 0,$$

TABLE 3
Relative error convergence rates for a second order model.

quantity	$L^2(\Omega_s)$ error	$L^2(\Omega_s^{int})$ norm error
u	$O(\varepsilon^{3/2})$	$O(\varepsilon^3)$
∇u	$O(\varepsilon^{1/2})$	$O(\varepsilon^3)$

and for ε small enough the well posedness of (24) follows from Lemma 1 and the Lax–Milgram's lemma. To estimate the modeling error we first define $\bar{\bar{e}} = \bar{\bar{u}} - u^0 - \varepsilon\, u^1 - \varepsilon^2\, u^2$. Thus

$$-\Delta\, \bar{\bar{e}} = 0 \quad \text{in } \Omega_s,$$
$$\bar{\bar{e}} + (\varepsilon\, d_0 - \varepsilon\, z^{0,0} - \varepsilon^2\, z^{1,0}\kappa)\, \partial_{\boldsymbol{n}}\, \bar{\bar{e}} - \varepsilon^2\, \frac{z^{1,1}}{-d_0 + z^{0,0}}\, \partial_\theta \bar{\bar{e}} = -\varepsilon^3 z^{1,0}\, \kappa\, \partial_{\boldsymbol{n}}\, u^1$$
$$-\, \varepsilon^3\, (-d_0 + z^{0,0} + \varepsilon\, z^{1,0}\, \kappa)\, \partial_{\boldsymbol{n}}\, u^2 - \varepsilon^3\, \frac{z^{1,1}}{-d_0 + z^{0,0}}\, \partial_\theta u^1 - \varepsilon^4\, \frac{z^{1,1}}{-d_0 + z^{0,0}}\, \partial_\theta u^2 \quad \text{on } \Gamma,$$
$$\bar{\bar{e}} = 0 \quad \text{on } \partial\Omega_s \backslash \Gamma.$$

Regularity estimates [16, Theorem 4.24] guarantee the existence of an ε-independent constant c such that

$$\|\bar{\bar{e}}\|_{H^1(\Omega_s)} \leq c\, \varepsilon^3.$$

Using the triangle inequality and (21), it is possible to estimate the $H^1(\Omega_s)$ norm modeling error,

$$\|u^\varepsilon - \bar{\bar{u}}\|_{H^1(\Omega_s)} \leq \|u^\varepsilon - u^0 - \varepsilon\, u^1 - \varepsilon^2\, u^2\|_{H^1(\Omega_s)} + \|\bar{\bar{u}} - u^0 - \varepsilon\, u^1 - \varepsilon^2\, u^2\|_{H^1(\Omega_s)} \leq c\, \varepsilon^{1/2}.$$

Analogously, using (22), (23), we obtain L^2 and interior estimates,

$$\|u^\varepsilon - \bar{\bar{u}}\|_{L^2(\Omega_s)} \leq c\, \varepsilon^{3/2} \qquad \|u^\varepsilon - \bar{\bar{u}}\|_{H^1(\Omega_s^{int})} \leq c\, \varepsilon^3.$$

These results are displayed in Table 3.

3.4. Summary: The proposed effective problems. The first order boundary value problem in Ω_s is the following: find $\bar{u} \in H^1(\Omega_s)$ such that

$$-\Delta\, \bar{u} = f \quad \text{in } \Omega_s,$$
$$\partial_{\boldsymbol{n}}\, \bar{u} = -\frac{1}{\varepsilon\, (d_0 - z^{0,0})}\, \bar{u} \quad \text{on } \Gamma, \qquad \bar{u} = 0 \quad \text{on } \partial\Omega_s \backslash \Gamma, \tag{25}$$

where $z^{0,0}$ is obtained from (5). For error estimates, see Table 2.

The second order boundary value problem in Ω_s is: find $\bar{\bar{u}} \in H^1(\Omega_s)$ such that

$$-\Delta\, \bar{\bar{u}} = f \quad \text{in } \Omega_s,$$
$$\partial_{\boldsymbol{n}}\, \bar{\bar{u}} = -C_1^\varepsilon\, \bar{\bar{u}} + C_2^\varepsilon\, \partial_\theta \bar{\bar{u}} \quad \text{on } \Gamma, \qquad \bar{\bar{u}} = 0 \quad \text{on } \partial\Omega_s \backslash \Gamma, \tag{26}$$

with

$$C_1^\varepsilon = \frac{1}{\varepsilon\, (d_0 - z^{0,0} - \varepsilon\, z^{1,0}\, \kappa)}, \quad C_2^\varepsilon = \frac{\varepsilon\, z^{1,1}}{(d_0 - z^{0,0} - \varepsilon\, z^{1,0}\, \kappa)(z^{0,0} - d_0)},$$

and where $z^{0,0}$ is computed from (5), and $z^{1,0}$, $z^{1,1}$ from (19), (20). For error estimates, see Table 3.

4. Asymptotic expansion definition. We now find and justify the terms presented previously. Consider a formal asymptotic expansion in the general form

$$u^\varepsilon \sim u^0 + \varepsilon\, u^1 + \varepsilon^2\, u^2 + \cdots + W^{BL}(\varepsilon) \quad \text{in } \Omega_s. \tag{27}$$

Here, $W^{BL}(\varepsilon)$ corresponds to the oscillatory part of the solution, which dies away exponentially fast with the distance to the boundary. Our procedure to find out the terms in the expansion uses a domain decomposition result that we state below.

It is convenient to introduce the jump function $[\![\cdot]\!]$ that assigns the *absolute value* of the jump over the interface Γ.

LEMMA 2. *Let Ω^ε, Ω_s, Ω_r^ε, and Γ be as above. Then there exists an ε-independent constant c such that*

$$\begin{aligned}\|e\|_{H^1(\Omega_r^\varepsilon)} + \|e\|_{H^1(\Omega_s)} \\ \le c\left(\|\,\Delta\, e\|_{L^2(\Omega_r^\varepsilon)} + \|\,\Delta\, e\|_{L^2(\Omega_s)} + \|[\![e]\!]\|_{H^{1/2}(\Gamma)} + \|[\![\partial_{\boldsymbol{n}}\, e]\!]\|_{H^{-1/2}(\Gamma)}\right)\end{aligned} \tag{28}$$

whenever $e|_{\Omega_r^\varepsilon} \in H^1(\Omega_r^\varepsilon)$, $\Delta\, e|_{\Omega_r^\varepsilon} \in L^2(\Omega_r^\varepsilon)$, and $e|_{\Omega_s} \in H^1(\Omega_s)$, $\Delta\, e|_{\Omega_s} \in L^2(\Omega_s)$, with $e = 0$ on $\partial\Omega_r^\varepsilon \backslash \Gamma \cup \partial\Omega_s \backslash \Gamma$.

Proof. We first define

$$e^- = e|_{\Omega_r^\varepsilon}, \qquad e^+ = e|_{\Omega_s}.$$

It follows from Green's identity that

$$\begin{aligned}\int_{\Omega_r^\varepsilon} |\,\nabla\, e^-|^2\, d\mathbf{x} &= -\int_{\Omega_r^\varepsilon} e^-\,\Delta\, e^-\, d\mathbf{x} - < e^-, \partial_{\boldsymbol{n}}\, e^- >_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)},\\ \int_{\Omega_s} |\,\nabla\, e^+|^2\, d\mathbf{x} &= -\int_{\Omega_s} e^+\,\Delta\, e^+\, d\mathbf{x} + < e^+, \partial_{\boldsymbol{n}}\, e^+ >_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)},\end{aligned}$$

where $< \cdot,\cdot >_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}$ indicates the duality pairing between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$. Combining both identities and then adding and subtracting the quantity $< e^-, \partial_{\boldsymbol{n}}\, e^+ >_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}$, we gather that

$$\begin{aligned}|e^-|^2_{H^1(\Omega_r^\varepsilon)} + |e^+|^2_{H^1(\Omega_s)} = -\int_{\Omega_r^\varepsilon} e^-\,\Delta\, e^-\, d\mathbf{x} - \int_{\Omega_s} e^+\,\Delta\, e^+\, d\mathbf{x}\\ + < e^-, \partial_{\boldsymbol{n}}\, e^+ - \partial_{\boldsymbol{n}}\, e^- >_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} + < e^+ - e^-, \partial_{\boldsymbol{n}}\, e^+ >_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}.\end{aligned}$$

When estimating the above quantities, a delicate question is how the constants depend on the domains. For $u \in H^{1/2}(\Gamma)$ and $v \in H^{-1/2}(\Gamma)$, the inequality

$$< u, v >_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} \le \|u\|_{H^{1/2}(\Gamma)} \|v\|_{H^{-1/2}(\Gamma)}$$

comes by naturally by inducing the operator norm in $H^{-1/2}(\Gamma)$. Thus, with the aid of the Cauchy–Schwarz inequality it follows that

$$\begin{aligned}|e^-|^2_{H^1(\Omega_r^\varepsilon)} + |e^+|^2_{H^1(\Omega_s)} \le \|e^-\|_{L^2(\Omega_r^\varepsilon)}\|\,\Delta\, e^-\|_{L^2(\Omega_r^\varepsilon)} + \|e^+\|_{L^2(\Omega_s)}\|\,\Delta\, e^+\|_{L^2(\Omega_s)}\\ + \|e^-\|_{H^{1/2}(\Gamma)}\|[\![\partial_{\boldsymbol{n}}\, e]\!]\|_{H^{-1/2}(\Gamma)} + \|[\![e]\!]\|_{H^{1/2}(\Gamma)}\|\,\partial_{\boldsymbol{n}}\, e^+\|_{H^{-1/2}(\Gamma)}.\end{aligned}$$

Next, let Ω' be the interior of Γ, i.e., the open domain circumvented by Γ. Hence $\Omega_r^\varepsilon \subset \Omega'$, and since the trace of e vanishes on Γ_r^ε, the extension (by zero) operator

$$P : \{v \in H^1(\Omega_r^\varepsilon) :\, v = 0 \text{ on } \Gamma_r^\varepsilon\} \to H^1(\Omega')$$

given by $Pv = v$ in Ω_r^ε and $Pv = 0$, otherwise, is an isometry [26, 19]. By construction, P preserves L^2 norms as well and we gather the trace and Poincaré inequalities,

$$\|v\|_{H^{1/2}(\Gamma)} \le c\|Pv\|_{H^1(\Omega')} = c\|v\|_{H^1(\Omega_r^\varepsilon)} \tag{29}$$

$$\|v\|_{L^2(\Omega_r^\varepsilon)} = \|Pv\|_{L^2(\Omega')} \le c|Pv|_{H^1(\Omega')} = c|v|_{H^1(\Omega_r^\varepsilon)} \tag{30}$$

for all $v \in H^1(\Omega_r^\varepsilon)$, where the constant c is independent of ε.

Using now the trace inequality $\|v\|_{H^{1/2}(\Gamma)} \le c\|v\|_{H^1(\Omega_s)}$ for all $v \in H^1(\Omega_s)$ and (29), it follows that

$$\begin{aligned} |e^-|^2_{H^1(\Omega_r^\varepsilon)} + |e^+|^2_{H^1(\Omega_s)} \le (\|\,\Delta\, e^-\|_{L^2(\Omega_r^\varepsilon)} + \|\,\Delta\, e^+\|_{L^2(\Omega_s)})\|e\|_{L^2(\Omega^\varepsilon)} \\ + \|e^-\|_{H^1(\Omega_r^\varepsilon)}\|[\![\partial_{\boldsymbol{n}}\, e]\!]\|_{H^{-1/2}(\Gamma)} + c\|[\![e]\!]\|_{H^{1/2}(\Gamma)}\|e^+\|_{H^1(\Omega_s)}. \end{aligned} \tag{31}$$

To conclude the proof, it is enough to use in (31) the Poincaré inequality in Ω_r^ε given by (30) and also in Ω_s. $\square$

We shall apply Lemma 2 repeatedly with e being the difference between u^ε and a truncated asymptotic expansion. Hence, to make such a difference as small as possible, we ought to minimize the L^2 norm of $\Delta\, e$ in Ω_r^ε and Ω_s and control the jumps of both e and $\partial_{\boldsymbol{n}}\, e$ over Γ.

A natural choice for the first term of the asymptotic of u^ε is u^0 given by (3), plus the condition $u^0 = 0$ in $\overline{\Omega_r^\varepsilon}$. Applying Lemma 2 with $e = u^\varepsilon - u^0$, we see that the source of error is the normal derivative jump $[\![\partial_{\boldsymbol{n}}\, u^0]\!]$. We remedy this by adding $\varepsilon\,\zeta^1$ to the asymptotic, where

$$\zeta^1(\mathbf{x}) = \begin{cases} -\varepsilon^{-1}\rho\ \partial_{\boldsymbol{n}}\, u^0(\theta, d_0\,\varepsilon) & \text{in } \Omega_r^\varepsilon, \\ 0 & \text{in } \Omega_s. \end{cases}$$

The function ζ^1 defined as above satisfies the following properties:

1. $\zeta^1 \equiv 0$ outside Ω_r^ε,
2. $[\![\varepsilon\,\partial_{\boldsymbol{n}}\,\zeta^1]\!] \equiv |\,\partial_{\boldsymbol{n}}\, u^0|$ on Γ,
3. $\zeta^1 = \psi_r(\varepsilon^{-1}\theta)\ \partial_{\boldsymbol{n}}\, u^0(\theta, d_0\,\varepsilon)$ on Γ_r^ε.

So, in general, the correction of the jump of the normal derivative on Γ violates the zero Dirichlet condition at Γ_r^ε.

Proceeding with the computations, we have to add a boundary corrector to compensate for the value of ζ^1 on Γ_r^ε. This is nontrivial since a "typical" boundary corrector does not decay to zero; see Lemma 1 and (5). A similar, but actually simpler situation occurs for the asymptotics of plates [18]. Thus, we add a boundary corrector that is the sum of two functions and is given by $\varepsilon\left[W^1(\varepsilon) + \chi_r^\varepsilon Z^1(\varepsilon)\right]$. One part, corresponding to $W^1(\varepsilon)$, decays exponentially fast to zero with $\varepsilon^{-1}\rho$ and undulates with $\varepsilon^{-1}\theta$. The other part, corresponding to $Z^1(\varepsilon)$, depends only on θ and is nonzero only in Ω_r^ε. Hence,

$$-\,\Delta\, W^1(\varepsilon) = \chi_r^\varepsilon\,\Delta\left[\zeta^1 + Z^1(\varepsilon)\right] \quad \text{in } \Omega^\varepsilon, \tag{32}$$

$$W^1(\varepsilon) = -\zeta^1 - Z^1(\varepsilon) \quad \text{on } \Gamma_r^\varepsilon, \tag{33}$$

and the characteristic function of Ω_r^ε is given by χ_r^ε, where

$$\chi_r^\varepsilon(\rho) = \begin{cases} 1 & \text{if } \rho < \varepsilon d_0, \\ 0 & \text{otherwise.} \end{cases}$$

At first sight, finding $W^1(\varepsilon)$ and $Z^1(\varepsilon)$ satisfying (32), (33) seems (at least!) as hard as solving the original problem (1). Nevertheless, it is possible to make use of the periodicity of the wrinkles, and formally recast (32), (33) as a sequence of ε-independent problems which are easier to solve. We write

$$W^1(\varepsilon) \sim W^{1,0} + \varepsilon\, W^{1,1} + \varepsilon^2\, W^{1,2} + \cdots, \tag{34}$$

$$Z^1(\varepsilon) \sim Z^{1,0} + \varepsilon\, Z^{1,1} + \varepsilon^2\, Z^{1,2} + \cdots. \tag{35}$$

We shall impose on Γ_r^ε that $W^{1,0} = -\zeta^1 - Z^{1,0}$ and that $W^{1,j} = -Z^{1,j}$ for $j \neq 0$. We postpone the precise definition of these terms for now, but add, formally, the term $\varepsilon\, \chi_r^\varepsilon(\mathbf{x}) Z^{1,0}(\theta) + \varepsilon\, W^{1,0}(\theta, \varepsilon^{-1}\theta, \varepsilon^{-1}\rho)$ to the asymptotic. The remaining terms of the expansion for $W^1(\varepsilon)$, $Z^1(\varepsilon)$ shall be added as we continue to develop the expansion.

So far the asymptotic reads as

$$\begin{cases} \varepsilon\, \zeta^1 + \varepsilon\, W^{1,0} + \varepsilon\, Z^{1,0} & \text{in } \Omega_r^\varepsilon, \\ u^0 + \varepsilon\, W^{1,0} & \text{in } \Omega_s. \end{cases} \tag{36}$$

Note that now the normal derivative of the difference between u^ε and the expression in (36) has zero jump on Γ, but the difference itself has nontrivial jump equal to $-d_0\, \varepsilon\, \partial_{\boldsymbol{n}}\, u^0 + \varepsilon\, Z^{1,0}$ on Γ. Such error no longer depends on the fast variable and it can be corrected adding to the asymptotic expansion a new term that depends only on the slow variable.

We continue to define the terms of the expansion, this time trying to cancel out the error due to the jump of the expression in (36) on Γ. Consider u^1 the solution of

$$\begin{gathered} -\Delta\, u^1 = 0 \quad \text{in } \Omega_s, \\ u^1 = -d_0\ \partial_{\boldsymbol{n}}\, u^0 + Z^{1,0} \quad \text{on } \Gamma, \qquad u^1 = 0 \quad \text{on } \partial\Omega_s \backslash \Gamma, \qquad u^1 = 0 \quad \text{on } \Omega_r^\varepsilon. \end{gathered} \tag{37}$$

Remark 2. Although (37) looks different from (8), it is not. In fact, $Z^{1,0} = z^{0,0}\ \partial_{\boldsymbol{n}}\, u^0$, but that will become clear later.

Adding $\varepsilon\, u^1$ to the expansion corrects the previous error, but results in a jump in the normal derivative across Γ. Mimicking what we did before, we add $\varepsilon^2\, \zeta^2$ to the expansion, where

$$\zeta^2(\mathbf{x}) = -\varepsilon^{-1} \rho\, \chi_r^\varepsilon(\rho)\ \partial_{\boldsymbol{n}}\, u^1(\theta, d_0\varepsilon).$$

Ideally the next contribution would be $\varepsilon^2\big[W^2(\varepsilon) + \chi_r^\varepsilon Z^2(\varepsilon)\big]$, where

$$-\Delta\, W^2(\varepsilon) = \chi_r^\varepsilon\, \Delta\big[\zeta^2 + Z^2(\varepsilon)\big] \quad \text{in } \Omega^\varepsilon, \tag{38}$$

$$W^2(\varepsilon) = -\zeta^2 - Z^2(\varepsilon) \quad \text{on } \Gamma_r^\varepsilon. \tag{39}$$

As in (34), (35),

$$W^2(\varepsilon) \sim W^{2,0} + \varepsilon\, W^{2,1} + \varepsilon^2\, W^{2,2} + \cdots, \tag{40}$$

$$Z^2(\varepsilon) \sim Z^{2,0} + \varepsilon\, Z^{2,1} + \varepsilon^2\, Z^{2,2} + \cdots. \tag{41}$$

On Γ_r^ε we shall have $W^{2,0} = -\zeta^2 - Z^{2,0}$, and $W^{2,j} = -Z^{2,j}$ for $j \neq 0$. Then, we simply add $\varepsilon^2\, \big[W^{1,1} + W^{2,0} + \chi_r^\varepsilon\big(Z^{1,1} + Z^{2,0}\big)\big]$ to our asymptotic expansion. Note that terms in ε^2 corresponding to the expansions for $W^1(\varepsilon)$, $Z^1(\varepsilon)$ are included now.

At this point, the asymptotic reads as

$$\begin{cases} \varepsilon\,\zeta^1 + \varepsilon^2\,\zeta^2 + \varepsilon\,W^{1,0} + \varepsilon\,Z^{1,0} + \varepsilon^2\left(W^{1,1} + W^{2,0}\right) + \varepsilon^2\left(Z^{1,1} + Z^{2,0}\right) & \text{in } \Omega_r^\varepsilon, \\ u^0 + \varepsilon\,u^1 + \varepsilon\,W^{1,0} + \varepsilon^2\left(W^{1,1} + W^{2,0}\right) & \text{in } \Omega_s. \end{cases} \tag{42}$$

The expansion pattern should be clear by now, and the successive terms are defined in similar manner. In general, after the kth step, the asymptotic expansion reads as

$$\begin{cases} \zeta_{k,\varepsilon} + W_{k,\varepsilon}^{BL} + Z_{k,\varepsilon} & \text{in } \Omega_r^\varepsilon, \\ u_{k-1,\varepsilon}^{smooth} + W_{k,\varepsilon}^{BL} & \text{in } \Omega_s, \end{cases} \tag{43}$$

where $\zeta_{k,\varepsilon}(\theta,\rho) = \varepsilon\,\zeta^1 + \cdots + \varepsilon^k\,\zeta^k$, and $\zeta^i = -\varepsilon^{-1}\rho\,\chi_r^\varepsilon\;\partial_{\boldsymbol{n}}\,u^{i-1}(\theta, d_0\varepsilon)$. Also,

$$W_{k,\varepsilon}^{BL} = \varepsilon\,W^{1,0} + \varepsilon^2\left(W^{1,1} + W^{2,0}\right) + \cdots + \varepsilon^k\left(W^{1,k-1} + W^{2,k-2} + \cdots + W^{k,0}\right),$$
$$Z_{k,\varepsilon} = \chi_r^\varepsilon\left[\varepsilon\,Z^{1,0} + \varepsilon^2\left(Z^{1,1} + Z^{2,0}\right) + \cdots + \varepsilon^k\left(Z^{1,k-1} + Z^{2,k-2} + \cdots + Z^{k,0}\right)\right].$$

Here, although we did not fully define these functions yet, $W^{i,j}$, $Z^{i,j}$ depend only on u^{i-1}, $W^{i,j-1}, \cdots, W^{i,0}$ and $Z^{i,j-1}, \cdots, Z^{i,0}$. Also, we shall have on Γ_r^ε that

$$W^{i,0} = -\zeta^i - Z^{i,0}, \qquad W^{i,j} = -Z^{i,j} \quad \text{for } j \neq 0.$$

Finally, $u_{k-1,\varepsilon}^{smooth} = u^0 + \varepsilon\,u^1 + \cdots + \varepsilon^{k-1}\,u^{k-1}$, where u^0 is as in (3), and for i positive,

$$-\Delta\,u^i = 0 \quad \text{in } \Omega_s,$$
$$u^i = -d_0\;\partial_{\boldsymbol{n}}\,u^{i-1} + Z^{1,i-1} + Z^{2,i-2} + \cdots + Z^{i,0} \quad \text{on } \Gamma, \qquad u^i = 0 \quad \text{on } \partial\Omega_s\backslash\Gamma,$$
$$u^i = 0 \quad \text{in } \Omega_r^\varepsilon.$$

5. The boundary corrector problem. We now analyze the boundary corrector problem, as (32), (33) and (38), (39), in more detail. The presence of the curvature makes this problem cumbersome and a lot of insight can be gained by studying the zero curvature case first; see [5] and references therein.

Consider the problem

$$-\Delta\,w(\varepsilon) = \chi_r^\varepsilon\;\Delta\left[-\varepsilon^{-1}\rho\,\phi(\theta) + z(\varepsilon)\right] \quad \text{in } \Omega^\varepsilon, \tag{44}$$
$$w(\varepsilon) = \varepsilon^{-1}\rho\,\phi(\theta) - z(\varepsilon) \quad \text{on } \Gamma_r^\varepsilon. \tag{45}$$

Here, ϕ is a given function of θ only. The function $z(\varepsilon)$ is unknown a priori, but it is introduced to guarantee that $w(\varepsilon)$ decays exponentially to zero with ρ. It is desirable to have $z(\varepsilon)$ as simple as possible, and it suffices to assume $z(\varepsilon)$ independent of ρ.

Although ϕ is not necessarily periodic, we try to make use of the periodicity of the wrinkles, and recast the corrector problem as a sequence of problems in periodic domains. Using the stretched coordinates $(\hat\theta,\hat\rho) = (\varepsilon^{-1}\theta, \varepsilon^{-1}\rho)$, we seek solutions that are product of functions in the stretched coordinates with functions of θ only. With this in mind, we write the Laplacian of a function in the form $v(\mathbf{x}) = h(\hat\theta,\hat\rho)\,g(\theta)$ as

$$\begin{aligned} -\Delta\,v = {} & -\varepsilon^{-2}\left(\partial_{\hat\theta\hat\theta}h + \partial_{\hat\rho\hat\rho}h\right)g + \varepsilon^{-1}\left(\kappa\,\partial_{\hat\rho}\,h - 2\,\kappa\,\hat\rho\,\partial_{\hat\theta\hat\theta}\,h\right)g - \varepsilon^{-1}2\,\partial_{\hat\theta}\,h\,g' \\ & -\sum_{j=0}^{\infty}\varepsilon^j\,\hat\rho^j\left[\left(\hat\rho\;\;a_1^{j+1}\,\partial_{\hat\rho}\,h + a_3^{j+1}\,\hat\rho\,\partial_{\hat\theta}\,h + a_2^{j+2}\,\hat\rho^2\,\partial_{\hat\theta\hat\theta}\,h\right)g + \left(a_3^j\,h + 2\,a_2^{j+1}\,\hat\rho\,\partial_{\hat\theta}\,h\right)g' \right. \\ & \left. + a_2^j\,h\,g''\right], \end{aligned} \tag{46}$$

where

$$a_1^j = -[\kappa(\theta)]^{j+1}, \quad a_2^j = (j+1)\,[\kappa(\theta)]^j, \quad a_3^j = \frac{j(j+1)}{2}\,[\kappa(\theta)]^{j-1}\,\kappa'(\theta),$$

and we recall that κ is the curvature of Γ.

From (46), and using that $z(\varepsilon)$ is independent of ρ,

$$\tag{47} \begin{gathered} -\Delta[\varepsilon^{-1}\rho\,\phi(\theta)] = \varepsilon^{-1}\kappa\,\phi - \sum_{j=0}^{\infty}\varepsilon^j\,\hat\rho^j\,\big[\hat\rho\,a_1^{j+1}\,\phi + \hat\rho(a_3^j\,\phi' + a_2^j\,\phi'')\big],\\ \Delta\,z(\varepsilon) = \sum_{j=0}^{\infty}\varepsilon^j\,\hat\rho^j\,\big[a_3^j\,\partial_\theta z(\varepsilon) + a_2^j\,\partial_{\theta\theta}\,z(\varepsilon)\big]. \end{gathered}$$

Assuming the expansion

$$\tag{48} z(\varepsilon) \sim z^0 + \varepsilon\,z^1 + \varepsilon^2\,z^2 + \cdots$$

and using the formal identity

$$\tag{49} \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\varepsilon^{i+j}\,c_j\,d_i = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\varepsilon^j\,c_k\,d_{j-k},$$

we gather from (47), (48), (49) the identity

$$\tag{50} \begin{aligned} \Delta\big[-\varepsilon^{-1}\rho\,\phi + z(\varepsilon)\big] = \varepsilon^{-1}\,\kappa\,\phi + \sum_{j=0}^{\infty}\varepsilon^j\,\Big\{&-\hat\rho^j\,\big[\hat\rho\,a_1^{j+1}\,\phi + \hat\rho\,(a_3^j\,\phi' + a_2^j\,\phi'')\big]\\ &+\sum_{k=0}^{j}\hat\rho^k\,\big(a_3^k\,\partial_\theta z^{j-k} + a_2^k\,\partial_{\theta\theta}\,z^{j-k}\big)\Big\}. \end{aligned}$$

Assuming the expansion

$$\tag{51} w(\varepsilon) \sim w^0 + \varepsilon\,w^1 + \varepsilon^2\,w^2 + \cdots$$

and that $w^0(\theta,\rho) = w^{0,0}(\hat\theta,\hat\rho)\,\phi(\theta)$ and $z^0(\theta) = z^{0,0}\,\phi(\theta)$, where $z^{0,0}$ is a constant, we gather that (44), (45), (46), and (50) lead to (5).

It is clear that w^0, z^0 do not satisfy (44) exactly, but only the highest order (with power ε^{-2}) term. In fact, if ϕ is smooth, it follows from usual regularity estimates and a scaling argument that

$$\big\|\Delta\big(\Upsilon w^0\big) - \varepsilon^{-1}\chi_r^\varepsilon\,\Delta\big(\rho\,\phi - z^0\big)\big\|_{L^2(\Omega^\varepsilon)} \le c\,\varepsilon^{-1/2}.$$

Here, as in page 1455, we need the cutoff function Υ, since w^0 is not well defined all over Ω^ε.

The remainder shall be corrected by the equations defining w^1, w^2, etc. Note also that defining w^0 as a product between $w^{0,0}$ and ϕ allows us to impose periodic boundary conditions in the PDE defining $w^{0,0}$. This trick reduces the original boundary corrector problem (44), (45) to a much easier to solve *sequence of cell problems*.

Continuing the procedure with the aid of (46), we set

$$w^1(\theta,\rho) = w^{1,0}(\hat\theta,\hat\rho)\,\kappa(\theta)\,\phi(\theta) + w^{1,1}(\hat\theta,\hat\rho)\,\phi'(\theta),$$

and $z^1(\theta) = z^{1,0}\,\kappa(\theta)\,\phi(\theta) + z^{1,1}\,\phi'(\theta)$, where $w^{1,0}$ and $w^{1,1} \in S(\Omega_r)$, and $z^{1,0}$ and $z^{1,1}$ are constants such that (19), (20) holds.

Now, $w^0 + \varepsilon\, w^1$ is a better approximate solution to (44) since,

$$\left\|\Delta\left[\Upsilon\left(w^0 + \varepsilon\, w^1\right)\right] - \varepsilon^{-1}\chi_r^\varepsilon\ \Delta\left(\rho\,\phi - z^0\right)\right\|_{L^2(\Omega^\varepsilon)} \le c\,\varepsilon^{1/2}.$$

It is easy to see that the right-hand sides of the equations become more involved as we proceed. The crucial point is to note that in the above cases, the equations do not involve the nonperiodic terms ϕ, κ or their derivatives.

Proceeding in a similar manner, we define w^2, w^3, etc., and

$$\left\|\Delta\left[\Upsilon\left(w^0 + \varepsilon\, w^1 + \cdots + \varepsilon^k\, w^k\right)\right] - \varepsilon^{-1}\chi_r^\varepsilon\ \Delta\left(\rho\,\phi - z^0\right)\right\|_{L^2(\Omega^\varepsilon)} \le c\,\varepsilon^{k-1/2}. \tag{52}$$

Finally, it follows from our computations that

$$w \sim w^{0,0}\,\phi + \varepsilon\,(w^{1,0}\,\kappa\,\phi + w^{1,1}\,\phi') + \varepsilon^2 \cdots, \tag{53}$$

$$z \sim z^{0,0}\,\phi + \varepsilon\,(z^{1,0}\,\kappa\,\phi + z^{1,1}\,\phi') + \varepsilon^2 \cdots. \tag{54}$$

In terms of the expansions for the boundary corrector for our original problem, see (32)–(35), we define

$$\begin{gathered}
W^{1,0}(\mathbf{x}) = \Upsilon(\rho)\,w^{0,0}(\hat\theta,\hat\rho)\ \partial_{\boldsymbol{n}}\,u^0(\theta, d_0\,\varepsilon),\\
W^{1,1}(\mathbf{x}) = \Upsilon(\rho)\,[w^{1,0}(\hat\theta,\hat\rho)\,\kappa(\theta)\ \partial_{\boldsymbol{n}}\,u^0(\theta, d_0\,\varepsilon) + w^{1,1}\,(\hat\theta,\hat\rho)\,\partial_\theta\,\partial_{\boldsymbol{n}}\,u^0(\theta, d_0\,\varepsilon)],\\
Z^{1,0}(\theta) = z^{0,0}\ \partial_{\boldsymbol{n}}\,u^0(\theta, d_0\,\varepsilon),\\
Z^{1,1}(\theta) = z^{1,0}\,\kappa(\theta)\ \partial_{\boldsymbol{n}}\,u^0(\theta, d_0\,\varepsilon) + z^{1,1}\,\partial_\theta\,\partial_{\boldsymbol{n}}\,u^0(\theta, d_0\,\varepsilon),
\end{gathered} \tag{55}$$

etc. Similarly, from (38)–(41), we define

$$\begin{gathered}
W^{2,0}(\mathbf{x}) = \Upsilon(\rho)\,w^{0,0}(\hat\rho,\hat\theta)\ \partial_{\boldsymbol{n}}\,u^1(\theta, d_0\,\varepsilon),\\
W^{2,1}(\mathbf{x}) = \Upsilon(\rho)\,[w^{1,0}(\hat\theta,\hat\rho)\,\kappa(\theta)\ \partial_{\boldsymbol{n}}\,u^1(\theta, d_0\,\varepsilon) + w^{1,1}\,(\hat\theta,\hat\rho)\,\partial_\theta\ \partial_{\boldsymbol{n}}\,u^1(\theta, d_0\,\varepsilon)],\\
Z^{2,0}(\theta) = z^{0,0}\ \partial_{\boldsymbol{n}}\,u^1(\theta, d_0\,\varepsilon),\\
Z^{2,1}(\theta) = z^{1,0}\,\kappa(\theta)\ \partial_{\boldsymbol{n}}\,u^1(\theta, d_0\,\varepsilon) + z^{1,1}\,\partial_\theta\,\partial_{\boldsymbol{n}}\,u^1(\theta, d_0\,\varepsilon),
\end{gathered}$$

and so on.

6. Convergence estimate. In this section we estimate the difference between a truncated asymptotic expansion and the exact solution. To bound such difference, some a priori estimates are necessary, thus the regularity of the terms in the expansion is worthy of consideration. The results below are based on standard regularity estimates [15].

The boundary layer terms $w^{i,j}$ solve Poisson problems of the form (4), and we assume first that Ω_r is a convex polygon. Then $w^{0,0} \in H^2(\Omega_r)$. For $i > 1$, it follows from (46) that the right-hand side of the Poisson problem for $w^{i,j}$ depends on a linear combination of $w^{k,l}$, $\partial_{\hat\rho} w^{k,l}$, $\partial_{\hat\theta} w^{k,l}$, $\partial_{\hat\theta\hat\theta} w^{k,l}$, where $k < i$. Thus, $w^{i,j} \in H^2(\Omega_r)$. Similarly, if Ω_r is a nonconvex polygon with largest angle equal to ω, then $w^{i,j} \in H^s(\Omega_r)$ for all $s < 1 + \pi/\omega$. Note that the regularity results above depend *only* on the geometry of Ω_r and not on f. On the other hand, the regularity of $W^{i,j}$ depends on f, since it also depends on the u^i; see, e.g., (55).

Concerning the regularity of u^i, we rely on smoothness of Ω_s to conclude from (3) that $\|u_0\|_{H^{m+2}(\Omega_s)} \le c\|f\|_{H^m(\Omega_s)}$ for all real m, where c depends only on Ω_s. Since

u^i is harmonic for i positive, its regularity is determined by its Dirichlet boundary condition on Γ. Using (46), (50), we gather that the boundary condition for u^i depends, among other more regular terms, on $\partial_\theta^j\,\partial_{\boldsymbol{n}}\,u^{i-j-1}$, for $j=0,\dots,i-1$. Thus, an induction argument leads to the existence of a constant c depending only on Ω_s and m such that $\|u^i\|_{H^{m+2-i}(\Omega_s)} \le c\|f\|_{H^m(\Omega_s)}$, for all real m.

Standard scaling arguments lead to the result below.

LEMMA 3. *Assume that f is a smooth function with support in Ω_s. Then, for every integers i, j, there exists a constant c such that*

$$\|W^{i,j}\|_{L^2(\Omega^\varepsilon)} \le c\,\varepsilon^{1/2}, \qquad \|W^{i,j}\|_{H^1(\Omega^\varepsilon)} \le c\,\varepsilon^{-1/2},$$
$$\|Z^{i,j}\|_{H^1(\Omega_r^\varepsilon)} + \|\zeta^i\|_{H^1(\Omega_r^\varepsilon)} \le c\,\varepsilon^{1/2}, \qquad \|u^i\|_{H^1(\Omega_s)} \le c.$$

The constant c might depend on f, Ω_s, and Ω_r, but it is independent of ε.

Considering now the truncated expansion as in (43), we define the error

$$e_k = \begin{cases} u^\varepsilon - \zeta_{k,\varepsilon} - W_{k,\varepsilon}^{BL} - Z_{k,\varepsilon} & \text{in } \Omega_r^\varepsilon, \\ u^\varepsilon - u_{k-1,\varepsilon}^{smooth} - W_{k,\varepsilon}^{BL} & \text{in } \Omega_s. \end{cases} \tag{56}$$

Aiming to use Lemma 2, we first note that e_k vanishes on $\partial\Omega^\varepsilon$. Also, the jumps across Γ are such that

$$[\![e_k]\!] = 0, \qquad [\![\partial_{\boldsymbol{n}}\,e_k]\!] = \varepsilon^k |\,\partial_{\boldsymbol{n}}\,u^k|. \tag{57}$$

Estimating $\Delta\,e_k$ is nontrivial since e_k is not harmonic in general. Indeed,

$$-\Delta\,e_k = \begin{cases} \Delta\big[\zeta_{k,\varepsilon} + W_{k,\varepsilon}^{BL} + Z_{k,\varepsilon}\big] & \text{in } \Omega_r^\varepsilon, \\ \Delta\,W_{k,\varepsilon}^{BL} & \text{in } \Omega_s. \end{cases}$$

It follows from the construction of $W_{k,\varepsilon}^{BL}$ in section 5, (52), and Lemma 3, that

$$\|\,\Delta\,e_k\|_{L^2(\Omega^\varepsilon)} \le c\,\varepsilon^{k-1/2}. \tag{58}$$

With the above estimates it is not hard to prove the following result, which shows the rate of convergence in ε of the asymptotic expansion.

THEOREM 4. *For any positive integer k there exists a constant c such that the difference between the truncated asymptotic expansion and the original solution measured in the original domain is bounded as follows:*

$$\|e_k\|_{H^1(\Omega_s)} + \|e_k\|_{H^1(\Omega_r^\varepsilon)} \le c\,\varepsilon^{k+1/2}. \tag{59}$$

Proof. From Lemmas 2 and 3 and estimates (57), (58), we have that

$$\|e_k\|_{H^1(\Omega_s)} + \|e_k\|_{H^1(\Omega_r^\varepsilon)} \le c\,\varepsilon^{k-1/2}. \tag{60}$$

Although the above estimate is not sharp, it is not hard to improve it. In fact,

$$\begin{aligned} \|e_k\|_{H^1(\Omega_s)} + \|e_k\|_{H^1(\Omega_r^\varepsilon)} &\le \|e_{k+1}\|_{H^1(\Omega_s)} + \|e_{k+1}\|_{H^1(\Omega_r^\varepsilon)} \\ &\quad + \|e_{k+1} - e_k\|_{H^1(\Omega_s)} + \|e_{k+1} - e_k\|_{H^1(\Omega_r^\varepsilon)} \le c\,\varepsilon^{k+1/2}, \end{aligned}$$

where we used (60) and Lemma 3 to obtain the last inequality. $\square$

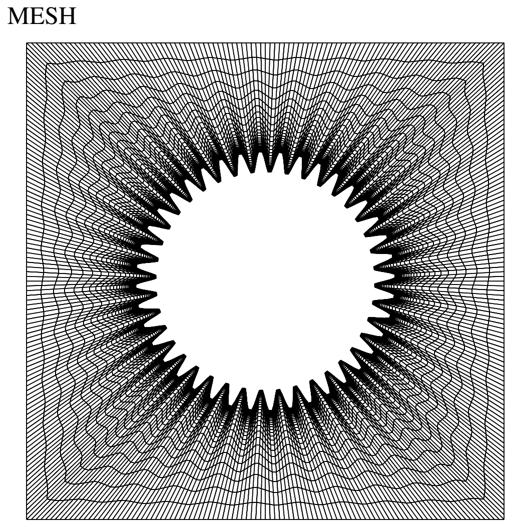

FIG. 3. *Mesh of rough domain.*

7. Numerical validation: Rough cylinder. We consider $\Omega_s \subset \mathbb{R}^2$ as the two-dimensional region having as outer boundary a square of size 4 and as inner boundary a circle of radius 1.15. Formally we have

$$\Omega_s = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \,:\, |\mathbf{x}| > 1.15,\, |x_i| < 2,\, i = 1, 2\}.$$

We define the rough domain as having the same square as outer boundary and a "perturbed" circle as the inner boundary. We lay upon a circle of unitary radius 20 periodic wrinkles of height 0.1. Thus $d_0\varepsilon = 0.15$. The test considered is a variation of (1), where $f = 0$, and $u = 1$ at the outer boundary. We obtain an "exact" solution by fully discretizing the rough domain with a refined mesh shown in Figures 3 and 4. We remark that the polygonal appearance of the boundary in Figure 4 is deceiving and results from approximation a smooth domain using polygonal meshes. Equations (25) yield the first order solution, and (26) yield the second order solution. Figures 5 and 6 show the isolines and profiles of the solutions for the first and second order cell problems (5) and (19). Note that we plot only $w^{1,0}$, since $z^{1,1} = 1.0 \times 10^{-5}$ and can be disregarded in the computations. The computed effective constants are $z^{0,0} = 0.77$ and $z^{1,0} = 0.27$. In Figure 7 we plot the level curves for the exact solution and the second order approximation. In Figures 8, 9, and 10 we compare the profiles of the exact solution with the first and second order approximations, at different heights above the wrinkles. It is possible to see that the second order approximation yields the best results, as predicted by the theory.

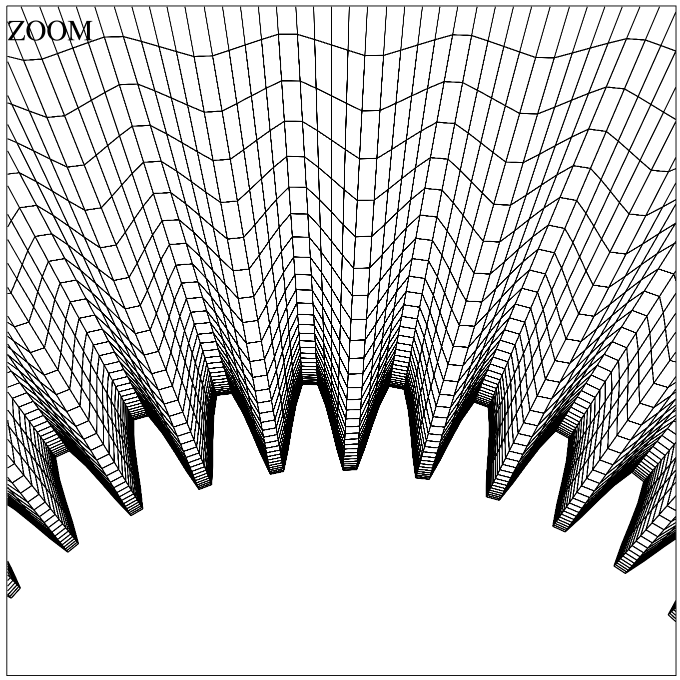

Fig. 4. *Zoom of the wrinkles.*

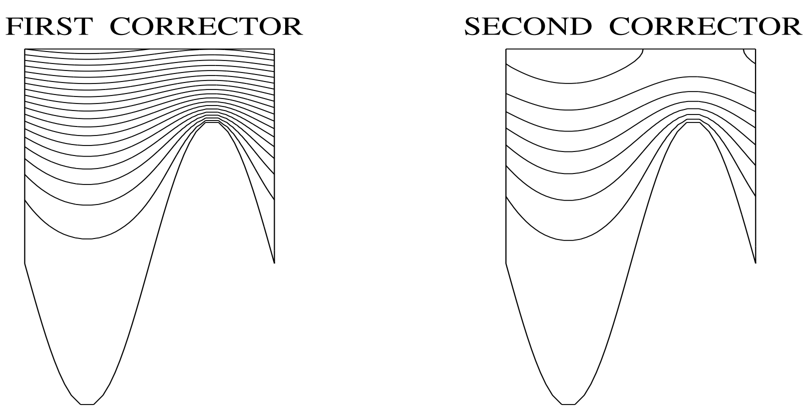

Fig. 5. *Isovalues of corrector* $\hat{\rho} - (w^{0,0} + z^{0,0})$ *(left) and* $w^{1,0} + z^{1,0}$ *(right) corresponding to the first and second order cell problems.*

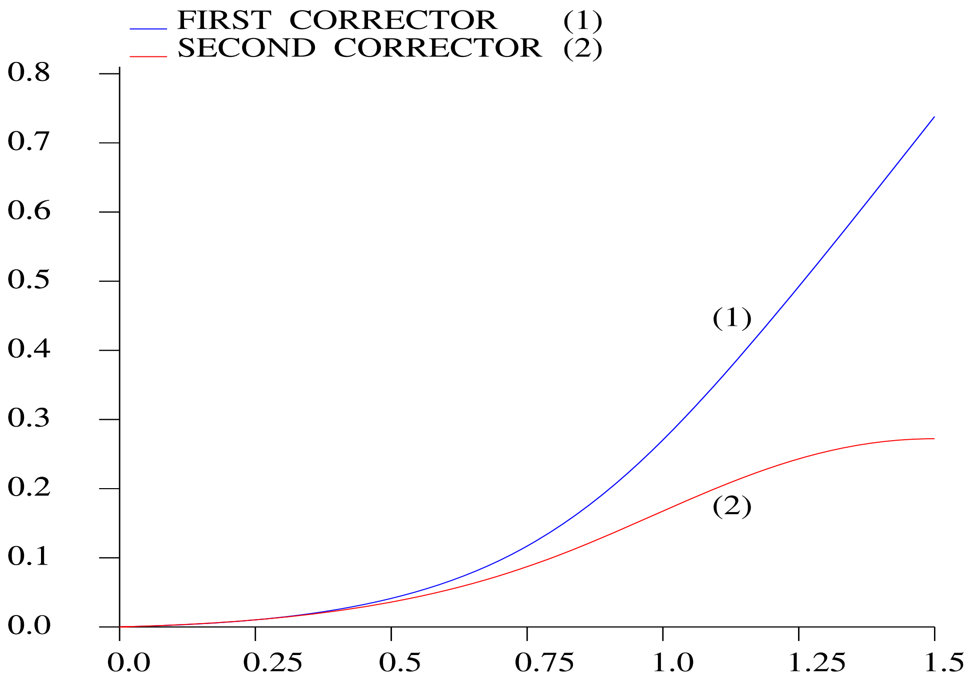

FIG. 6. *Profiles at* $\hat{\theta} = 0.78$ *of* $\hat{\rho} - (w^{0,0} + z^{0,0})$ *and* $w^{1,0} + z^{1,0}$.

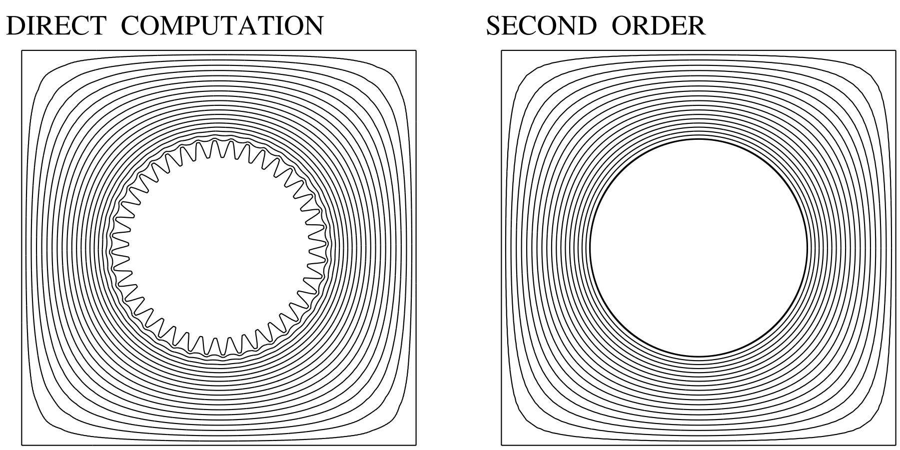

FIG. 7. *Second order approximation solves accurately the original problem.*

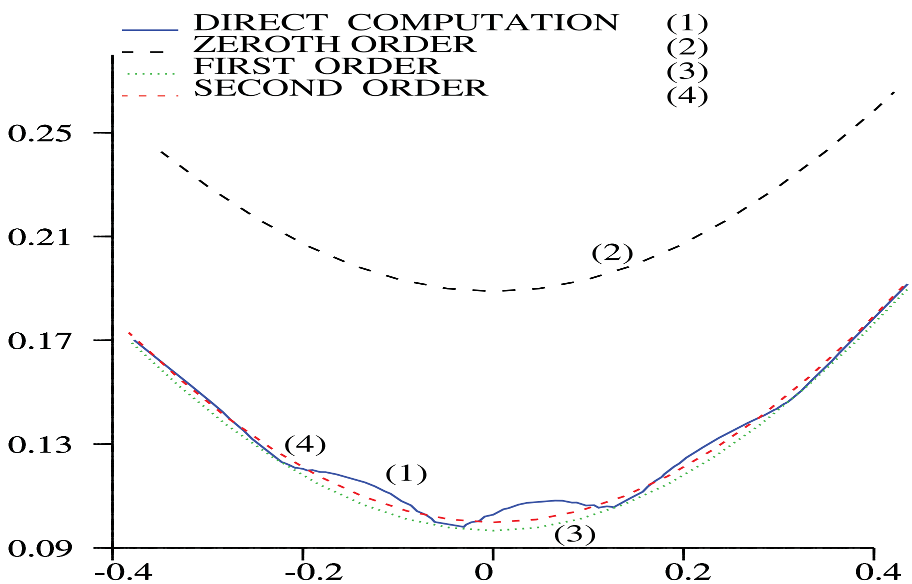

FIG. 8. *Profile of solutions at* $x_1 = 1.15$.

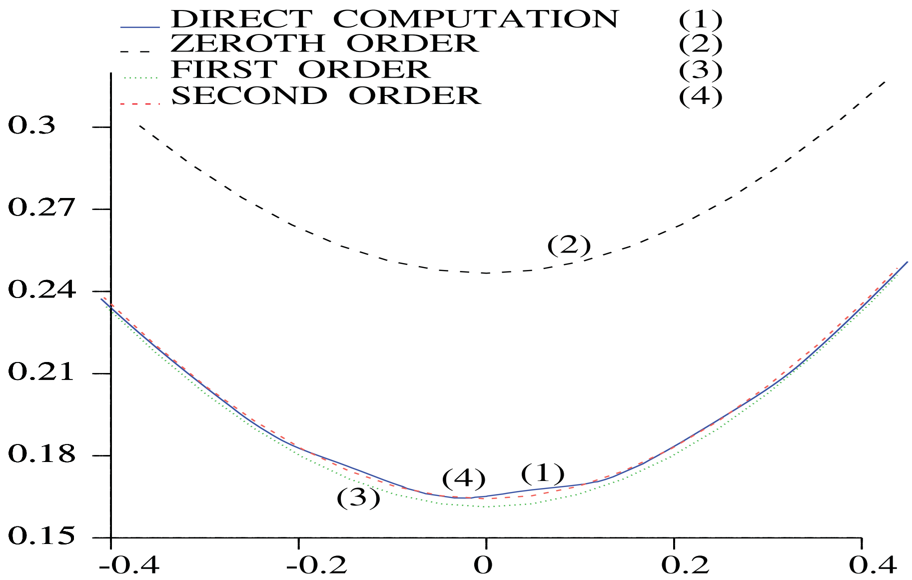

Fig. 9. *Profile of solutions at* $x_1 = 1.2$.

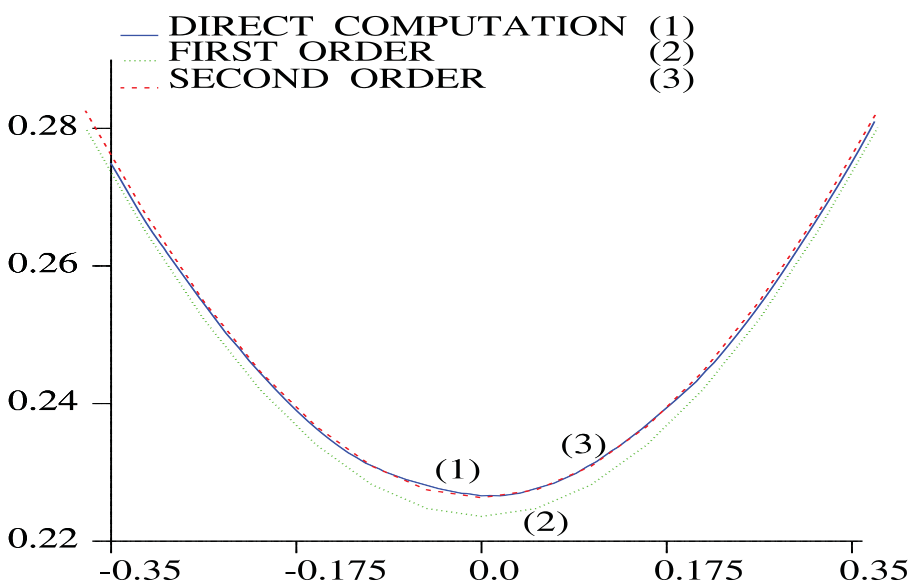

Fig. 10. *Profile of solutions at* $x_1 = 1.25$.

8. Conclusions. We investigated in this paper the problem of developing and estimating wall laws for problems defined in domains with rough and curved boundaries. For the sake of simplicity, the Poisson problem was considered. We developed a general methodology consisting of a two-scale expansion technique based on a domain decomposition result and obtained high order effective boundary conditions. Numerical tests accompanied the several sharp error estimates presented for first and second order approximations. In particular, this work proves that to obtain accurate numerical results, the curvature must be considered.

Our approach can be carried over to more sophisticated operators and to higher dimensions, yielding then a general procedure to develop and estimate effective boundary conditions.

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