ON THE RANGE OF APPLICABILITY OF THE REISSNER–MINDLIN AND KIRCHHOFF–LOVE PLATE BENDING MODELS

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Abstract. We show that the Reissner–Mindlin plate bending model has a wider range of applicability than the Kirchhoff–Love model for the approximation of clamped linearly elastic plates. Under the assumption that the body force density is constant in the transverse direction, the Reissner–Mindlin model solution converges to the three-dimensional linear elasticity solution in the relative energy norm for the full range of surface loads. However, for loads with a significant transverse shear effect, the Kirchhoff–Love model fails.

1. Introduction

The Kirchhoff–Love and Reissner–Mindlin models are the two most common dimensionally reduced models of a thin linearly elastic plate. It is often remarked in the engineering literature, based mostly on computational evidence, that the Reissner–Mindlin model is more accurate, particularly for moderately thin plates and when transverse shear plays a significant role, see [6]. However, as far as we know, all theoretical studies of the asymptotic behavior of the error in the plate models thus far fail to distinguish between the accuracy of the two models. In the words of Ciarlet [4], "While it is generally agreed in computational mechanics circles that the Reissner–Mindlin theory is 'better' than the Kirchhoff–Love theory, especially for 'moderately thin plates,' this assertion is not yet fully substantiated." It is the purpose of this note to show that in the asymptotic regime usually assumed in asymptotic analyses, the Reissner–Mindlin approximation is provably accurate over a wider range of loadings than the Kirchhoff–Love approximation for bending of clamped plates. In fact, under the assumption that the body force density is constant transversely, we shall show that the former is convergent for the full range of surface loads while the latter is divergent if the surface loads induce a significant transverse shear. This is surely not the whole answer. Reissner–Mindlin theory is also preferred because it better represents boundary conditions (it can distinguish bewteen hard and soft simply support) [8], because it offers at least some approximation of the boundary layer, and because it offers some advantages for numerical approximation.

Let Ω be a plane domain. For small but positive ϵ, consider the plate domain $P^\epsilon = \Omega \times (-\epsilon, \epsilon)$. We suppose that for each ϵ we are given surface tractions on the top and bottom faces of P^ϵ and volume forces in P^ϵ. If we then consider a linearly elastic plate with given elastic constants occupying P^ϵ and we impose appropriate boundary conditions on the lateral boundary, we obtain a well-posed boundary value problem for linear elasticity and so determine uniquely the displacement field $\underline{u}_*^\epsilon$. We are concerned with how well this displacement field is approximated by the displacement fields $\underline{u}_K^\epsilon$ and $\underline{u}_R^\epsilon$ on P^ϵ obtained from the Kirchhoff–Love and Reissner–Mindlin models, respectively. In order to quantify

The first author was supported by NSF grant DMS-0107233.

the quality of approximation we choose, for each ϵ, some norm or seminorm $\|\cdot\|_{P^\epsilon}$ on $\underline{H}^1(P^\epsilon)$. Since the true solution $\underline{u}_*^\epsilon$ is varying with ϵ, we consider the relative errors

$$\frac{\|\underline{u}_*^\epsilon - \underline{u}_K^\epsilon\|_{P^\epsilon}}{\|\underline{u}_*^\epsilon\|_{P^\epsilon}}, \quad \frac{\|\underline{u}_*^\epsilon - \underline{u}_R^\epsilon\|_{P^\epsilon}}{\|\underline{u}_*^\epsilon\|_{P^\epsilon}}$$

in the plate model approximations. We then say that the plate model is convergent (or convergence with order p) with respect to this sequence of loads and this norm or seminorm, if this corresponding relative error quantity tends to zero with ϵ (with order p). In this paper we prove that for totally clamped plates, and under the usual assumption on the dependence of the loading functions on the plate thickness, the Reissner–Mindlin model has a wider range of validity than Kirchhoff–Love. For some problems, the relative error of the Reissner–Mindlin model converges to zero, while the Kirchhoff–Love does not. The problems exhibiting this behavior will be quite simple. We shall take the plate to be isotropic, homogeneous, and linearly elastic with elastic moduli independent of the plate thickness, and clamped along the entire lateral boundary. The volume forces will be taken to be vertical and constant in the transverse direction. The tangential components of the imposed surface tractions will taken to be opposite each other on the top and bottom surfaces (which induces transverse shearing of the plate), and vertical components will be taken to be the same on top and bottom. This loading leads to pure bending of the plate in that the displacement transverse to the midsurface is even with respect to x_3 and tangential displacement is odd. We shall further assume a delicate balance between the imposed forces. See (15) below. It will follow that the Kirchhoff–Love solution in fact vanishes for each ϵ, so is not convergent in any norm. If the tangential surface forces are not zero, the three-dimensional displacement field will exhibits a significant transverse shear, which the Reissner–Mindlin model is able to capture, and we shall prove that relative energy norm convergence with order ϵ or $\epsilon^{1/2}$ depending on the situation. If the tangential surface forces vanish, the three-dimensional displacement is of higher order, which can also be captured by the Reissner–Mindlin model with a convergence of order $\epsilon^{1/2}$ in the relative energy norm.

The above discussion is reflected in the asymptotic expansions of the solutions of the elasticity equations and Reissner–Mindlin equations with respect the plate thickness. The standard expansion for linear elasticity on a thin plate begins with a term coming from the Kirchhoff–Love model. See [5] and [4]. The same is true of the asymptotic expansion of the Reissner–Mindlin solution [2]. In the case of the shear loads described above, this leading term vanishes, and the convergence of the Reissner–Mindlin model reflects the fact that the first nonvanishing terms of the asymptotic expansions agree.

Throughout the paper, as has already been seen, we indicate tensors in three variables with underbars. A first-order tensor (or 3-vector) is written with one underbar, a second-order tensor (or 3×3 matrix) with two underbars, etc. For tensors in two variables we use undertildes in the same way. By way of illustration, any 3-vector may be expressed in terms of a 2-vector giving its in-plane components and a scalar giving its transverse components, and any 3×3 symmetric matrix may be expressed in terms of a 2×2 symmetric matrix, a 2-vector, and a scalar thus:

$$\underline{v} = \begin{pmatrix} \underset{\sim}{v} \\ v_3 \end{pmatrix}, \quad \underline{\underline{\tau}} = \begin{pmatrix} \underset{\approx}{\tau} & \underset{\sim}{\tau} \\ \underset{\sim}{\tau}^T & \tau_{33} \end{pmatrix}.$$

Underbars and undertildes will be used for tensor-valued functions, operators yielding such functions, and spaces of such functions, as well. Even without explicit mention, all second-order tensors arising in this paper will be assumed symmetric. Thus, for example, the notation $\underset{\approx}{H}^s(\Omega)$ denotes the Sobolev space of order s which consists of all functions on Ω with values in $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ whose partial derivatives of order at most s are square integrable. We denote the space of square integrable functions on a domain Q by $L^2(Q)$. In the case when $Q = \Omega$ we just write L^2, etc.

The paper is arranged as follows. In Section 2, we describe the three-dimensional elastic boundary value problem for a plate and its Kirchhoff–Love and Reissner–Mindlin approximations. In Section 3, we recall the known results on the convergence of the two lower dimensional models. We identify the case in which the Kirchhoff–Love fails while the Reissner–Mindlin is still convergent and prove the convergence in the relative energy norm. In the last section, we give some examples which exhibit the superiority of Reissner–Mindlin over Kirchhoff–Love.

2. The elastic plate and the two approximations

First we describe precisely the boundary value problems defining $\underline{u}^\epsilon_*$, $\underline{u}^\epsilon_K$, and $\underline{u}^\epsilon_R$. Let $\partial P^\epsilon_L = \partial\Omega \times (-\epsilon, \epsilon)$, $\partial P^\epsilon_+ = \Omega \times \{\epsilon\}$, and $\partial P^\epsilon_- = \Omega \times \{-\epsilon\}$ denote the lateral portion and the top and bottom portions of the plate boundary, respectively, and set $\partial P^\epsilon_\pm = \partial P^\epsilon_+ \cup \partial P^\epsilon_-$. Let $\underline{g}^\epsilon : \partial P^\epsilon_\pm \to \mathbb{R}^3$ denote the given surface traction on the upper and lower surfaces of the plate and $\underline{f}^\epsilon : P^\epsilon \to \mathbb{R}^3$ the body force. As usual we define the elasticity tensor as $\underline{\underline{\underline{C}}}\,\underline{\underline{\tau}} = 2\mu\underline{\underline{\tau}} + \lambda\,\mathrm{tr}(\underline{\underline{\tau}})\underline{\underline{\delta}}$ where μ and λ are the Lamé coefficients and $\underline{\underline{\delta}}$ is the 3×3 identity map. By $\underline{\underline{e}}(\underline{u})$ we denote the infinitesimal strain tensor, i.e., the symmetric part of the gradient of $\underline{u}$. Thus the displacement vector $\underline{u}^\epsilon_* : P^\epsilon \to \mathbb{R}^3$ satisfies the boundary value problem

$$-\underline{\mathrm{div}}\,\underline{\underline{\underline{C}}}\,\underline{\underline{e}}(\underline{u}^\epsilon_*) = \underline{f}^\epsilon \text{ in } P^\epsilon, \tag{1}$$

$$[\underline{\underline{\underline{C}}}\,\underline{\underline{e}}(\underline{u}^\epsilon_*)]\underline{n} = \underline{g}^\epsilon \text{ on } \partial P^\epsilon_\pm, \quad \underline{u}^\epsilon_* = 0 \text{ on } \partial P^\epsilon_L. \tag{2}$$

We shall assume throughout this paper that we are in the situation of *plate bending* rather than *plate stretching*, and that the body force is constant in the transverse direction. In other words, we shall assume that $\underset{\sim}{f}^\epsilon = 0$, $f^\epsilon_3 = f^\epsilon_3(\underset{\sim}{x})$, $\underset{\sim}{g}^\epsilon(\underset{\sim}{x}, \epsilon) = -\underset{\sim}{g}^\epsilon(\underset{\sim}{x}, -\epsilon) = \underset{\sim}{g}^\epsilon(\underset{\sim}{x})$, $g^\epsilon_3(\underset{\sim}{x}, \epsilon) = g^\epsilon_3(\underset{\sim}{x}, -\epsilon) = g^\epsilon_3(\underset{\sim}{x})$. (If the body force density is not constant in the transverse direction, we can use certain transverse moments to define $\underset{\sim}{f}^\epsilon$ and f^ϵ_3. However, the convergence of the model solution to the three-dimensional solution cannot be proved without some restrictions on the transverse variation of the body force density. We shall not pursue this issue further because volume loads which vary significantly across the thickness of a thin plate are very uncommon.)

The Kirchhoff–Love approximation to the plate bending problem is given by

$$\underline{u}^\epsilon_K(\underset{\sim}{x}, x_3) = \left(-\underset{\sim}{\nabla}\zeta^\epsilon(\underset{\sim}{x})x_3,\ \zeta^\epsilon(\underset{\sim}{x}) + \frac{\lambda}{2(2\mu+\lambda)}x_3^2\Delta\zeta^\epsilon(\underset{\sim}{x})\right), \tag{3}$$

where $\zeta^\epsilon : \Omega \to \mathbb{R}$ is determined by the biharmonic equation

$$D\,\epsilon^2\,\Delta^2\zeta^\epsilon = F_K^\epsilon \text{ in } \Omega, \tag{4}$$

$$\zeta^\epsilon = \frac{\partial \zeta^\epsilon}{\partial n} = 0 \text{ on } \partial\Omega, \tag{5}$$

with $D = 4\mu(\mu+\lambda)/[3(2\mu+\lambda)]$ and the loading function F_K^ϵ is given by

$$F_K^\epsilon(\underset{\sim}{x}) = \epsilon^{-1}\, g_3^\epsilon(\underset{\sim}{x}) + f_3^\epsilon(\underset{\sim}{x}) + \operatorname{div} \underset{\sim}{g}^\epsilon(\underset{\sim}{x}). \tag{6}$$

Frequently, the term involving $\Delta\zeta^\epsilon$ is dropped from the third component of $\underline{u}_K^\epsilon$, but, as was first noticed by Morgenstern [7] and we comment on below, without it the Kirchhoff–Love solution does not converge to the three-dimensional solution in the energy norm.

The Reissner–Mindlin approximation to (1) and (2) is

$$\underline{u}_R^\epsilon(\underset{\sim}{x}, x_3) = \Big(-x_3\underset{\sim}{\theta}^\epsilon(\underset{\sim}{x}), w^\epsilon(\underset{\sim}{x}) + (x_3^2 - \epsilon^2/5)y^\epsilon(\underset{\sim}{x})\Big), \tag{7}$$

where w^ϵ and $\underset{\sim}{\theta}^\epsilon$ are determined by the boundary value problem

$$-\epsilon^2\frac{1}{3}\underset{\sim}{\operatorname{div}}\underset{\approx}{C}^*\underset{\approx}{e}(\underset{\sim}{\theta}^\epsilon) + \frac{5}{6}\mu(\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\nabla} w^\epsilon) = \underset{\sim}{G}_R^\epsilon, \tag{8}$$

$$\frac{5}{6}\mu\operatorname{div}(\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\nabla} w^\epsilon) = F_R^\epsilon, \tag{9}$$

$$\underset{\sim}{\theta}^\epsilon = 0, \quad w^\epsilon = 0 \text{ on } \partial\Omega. \tag{10}$$

Here $\underset{\approx}{C}^*\underset{\approx}{\tau} = 2\mu\underset{\approx}{\tau} + \lambda^*\operatorname{tr}(\underset{\approx}{\tau})\underset{\approx}{\delta}$, $\lambda^* = 2\mu\lambda/(2\mu+\lambda)$. The loading functions are given by

$$\begin{aligned}\underset{\sim}{G}_R^\epsilon &= -\frac{5}{6}\underset{\sim}{g}^\epsilon - \frac{1}{3}\frac{\lambda}{2\mu+\lambda}\epsilon^2[\epsilon^{-1}\,\underset{\sim}{\nabla} g_3^\epsilon + \frac{1}{5}\underset{\sim}{\nabla} F_K^\epsilon(\underset{\sim}{x})],\\ F_R^\epsilon &= -\frac{5}{6}\operatorname{div}\underset{\sim}{g}^\epsilon + F_K^\epsilon(\underset{\sim}{x}).\end{aligned} \tag{11}$$

Finally,

$$y^\epsilon = \frac{1}{2(2\mu+\lambda)}(\lambda\operatorname{div}\underset{\sim}{\theta} + \epsilon^{-1}\,g_3^\epsilon) + cF_K^\epsilon(\underset{\sim}{x}), \tag{12}$$

where $c = [10\mu(2\mu+3\lambda) + 3\lambda^2]/[70\mu(2\mu+\lambda)(2\mu+3\lambda)]$. Before closing this section we remark that many slight variants of the Kirchhoff–Love and Reissner–Mindlin models can be found in the literature. The versions of the models presented here are the results of systematic mathematical derivations. Namely, the Kirchhoff–Love model presented arises from an asymptotic analysis in which certain assumptions are imposed on the dependence of the loads on ϵ, the plate is scaled to one of unit half-thickness, the limit as ϵ tends to zero is determined there, and the limit solution is scaled back to the physical domain. See [4] for the detailed derivation. The Reissner–Mindlin model presented arises from a variational argument in which the true three-dimensional displacement and stress fields are characterized as a saddle point of the Hellinger–Reissner variational principle, and then approximate displacement and stress fields are determined as the unique saddle point of the same function but restricted to functions having a specified polynomial dependence on x_3. See [1] for the detailed derivation.

3. Convergence of the two approximations

We specify the dependence of the loads on ϵ by supposing that

$$g^\epsilon(x) = g(x), \quad g_3^\epsilon(x) = \epsilon g_3(x), \quad f_3^\epsilon(x) = f_3(x). \tag{13}$$

for some functions g, g_3, and f_3 independent of ϵ. Since we are only concerned with relative error, we could instead assume that $g^\epsilon = \epsilon^p g$, $g_3^\epsilon = \epsilon^{p+1} g_3$, and $f_3^\epsilon = \epsilon^p f_3$ for any value of p, without any change in the results. However the relation between the powers, that is, the assumption that the transverse surface load is one order of ϵ higher than the tangential surface load and body load is important. This is the same assumption that is generally made in the derivation and justification of the Kirchhoff–Love model by asymptotic analysis [4], [5].

Simply put our results are as follows. If g, g_3, and f_3 are any H^1 functions on Ω, and if the condition

$$\operatorname{div} g + g_3 + f_3 \neq 0 \tag{14}$$

is satisfied, then for the load sequence (13) both the Kirchhoff–Love model and the Reissner–Mindlin model are convergent (and the rates of convergence of the two models are identical). However, if

$$\operatorname{div} g + g_3 + f_3 = 0, \tag{15}$$

then the Kirchhoff–Love model does not converge. In this case, the Reissner–Mindlin model does converge at the convergence rate of $\epsilon^{1/2}$ in the relative energy norm. In fact, the non-convergence of the Kirchhoff–Love model in the case of (15) is immediate: from (6) we see that $u_K^\epsilon \equiv 0$ for all ϵ.

Before going on to state the results precisely and prove them, let us comment on their practical significance. In reality one does not confront a sequence of plates of decreasing thickness, but rather one particular plate of fixed thickness ϵ and one particular loading $\underline{g}^\epsilon$ and $\underline{f}^\epsilon$. In view of (13), the unfavorable case, (15), for the Kirchhoff–Love model may be written in terms of the physical loads as

$$\epsilon \operatorname{div} g^\epsilon + g_3^\epsilon + f_3^\epsilon = 0. \tag{16}$$

When this condition is exactly satisfied, then the Kirchhoff–Love model tells us nothing about the three-dimensional solution, no matter how thin the plate is. If, on the other hand, the two quantities $\epsilon \operatorname{div} g^\epsilon$ and $g_3^\epsilon + f_3^\epsilon$ do not cancel exactly, but nearly so, then we can expect that the Kirchhoff–Love solution will not be accurate. For example, if the three-dimensional solution is the sum of two parts, for one of which the loads satisfy (16), and if this part is not much smaller than the complementary part, then the Kirchhoff–Love solution, by virtue of missing the first part completely, cannot accurately model the whole. See the example in Section 4.

First we discuss the case (14), for which the convergence theory is summarized in Theorem 1. In this case rather complete results are known, including the relative convergence rates for the L^2 norms of each of the displacement components and for each of their first derivatives, and the relative convergence rate for the energy norm $\|\underline{u}\|_{E^\epsilon} := \left(\int_{P^\epsilon} [\underline{\underline{C}}\,\underline{\underline{e}}(\underline{u})] : \underline{\underline{e}}(\underline{u})\, d\underline{x}\right)^{1/2}$.

TABLE 1. Relative error convergence rates in various seminorms for the Kirchhoff–Love and Reissner–Mindlin models assuming (14).

$\|\underline{u}\|_{P^\epsilon}$	$\|\underline{u}^\epsilon_* - \underline{u}^\epsilon_M\|_{P^\epsilon}/\|\underline{u}^\epsilon_*\|_{P^\epsilon}$
$\|\underset{\sim}{u}\|_{L^2(P^\epsilon)}$	$O(\epsilon)$
$\|u_3\|_{L^2(P^\epsilon)}$	$O(\epsilon)$
$\|\underset{\approx}{\nabla}\underset{\sim}{u}\|_{L^2(P^\epsilon)}$	$O(\epsilon^{1/2})$
$\|\dfrac{\partial \underset{\sim}{u}}{\partial x_3}\|_{L^2(P^\epsilon)}$	$O(\epsilon)$
$\|\underset{\sim}{\nabla} u_3\|_{L^2(P^\epsilon)}$	$O(\epsilon)$
$\|\dfrac{\partial u_3}{\partial x_3}\|_{L^2(P^\epsilon)}$	$O(\epsilon^{1/2})$
$\|\underline{u}\|_{E^{\epsilon}}$	$O(\epsilon^{1/2})$

Theorem 1. *Suppose that $\partial\Omega$ and the loads $\underline{g}$ and f_3 are sufficiently smooth, and that (14) holds. For each ϵ define $\underline{u}^\epsilon_*$ by (1), (2) and let $\underline{u}^\epsilon_M$ denote either $\underline{u}^\epsilon_K$ or $\underline{u}^\epsilon_R$, defined by (3)–(5) or (7)–(12), respectively. Let $\|\cdot\|_{P^\epsilon}$ denote one of the seminorms in Table 1. Then there exists a constant C depending only on Ω and the loads such that*

$$\frac{\|\underline{u}^\epsilon_* - \underline{u}^\epsilon_M\|_{P^\epsilon}}{\|\underline{u}^\epsilon_*\|_{P^\epsilon}} \le C\epsilon^p,$$

where the rate of convergence, p, is given in the table.

We remark that the convergence estimates given in the first five lines of Table 1 remain unchanged if the the term involving $\Delta\zeta^\epsilon$ is dropped from u^ϵ_{K3} or the term involving y^ϵ is dropped from u^ϵ_{R3}. However, without these terms, $\partial u_{M3}/\partial x_3$ vanishes, and so the convergence estimates in the last two lines of the table would not hold.

The convergence results asserted in Theorem 1 are simple consequences of known results. Indeed, from [5] we get estimates such as

$$\|\underset{\sim}{u}^\epsilon_* - \underset{\sim}{u}^\epsilon_K\| \le C\epsilon^{1/2}, \quad \|u^\epsilon_{*3} - u^\epsilon_{K3}\| \le C\epsilon^{-1/2}, \tag{17}$$

where the norms are those of $L^2(P^\epsilon)$ and C is a constant depending on $\underline{g}$ and f_3 but independent of ϵ. (In [5] a volume load is considered, but this requires only a minor modification of the analysis.) In fact, these estimates don't require (14), but in order to extract from them convergence to zero of the relative error, we need to bound the quantities $\|\underset{\sim}{u}^\epsilon_*\|$ and $\|u^\epsilon_{*3}\|$ from below. In view of (6) and (14) we find that $F^\epsilon_K = F_K$ where F_K is independent of ϵ, and, hence, $\zeta^\epsilon = \epsilon^{-2}\zeta$ where ζ is independent of ϵ. It follows immediately that

$$\|\underset{\sim}{u}^\epsilon_K\| = c_1\epsilon^{-1/2}, \quad \|u^\epsilon_{K3}\| = c_2\epsilon^{-3/2},$$

for some nonzero constants c_1 and c_2. Using the triangle inequality and (17), we find that

$$\|\underset{\sim}{u}^\epsilon_*\| \ge (c_1/2)\epsilon^{-1/2}, \quad \|u^\epsilon_{*3}\| \ge (c_2/2)\epsilon^{-3/2}, \tag{18}$$

for ϵ sufficiently small. Then from (17) and (18) we find that

$$\frac{\|\underset{\sim}{u}^\epsilon_* - \underset{\sim}{u}^\epsilon_K\|}{\|\underset{\sim}{u}^\epsilon_*\|} = O(\epsilon), \quad \frac{\|u^\epsilon_{*3} - u^\epsilon_{K3}\|}{\|u^\epsilon_{*3}\|} = O(\epsilon).$$

This establishes first order convergence with respect to the $L^2(P^\epsilon)$ norm for both the tangential and transverse components of the Kirchhoff–Love solution.

In a similar way, the relative L^2 convergence rates asserted in the table for the first derivatives of the displacement components can be established. (For the estimate on the transverse derivative of the transverse component of the displacement, the terms involving x_3^2 in (3) and (7) are needed.) For the estimate of the relative energy norm error, see [1].

With the estimates for the Kirchhoff–Love model established, it is easy to obtain them for the Reissner–Mindlin model as well (still under the hypothesis (14)). In [2] rigorous error estimates are established for the difference between the two model solutions. It is then simply a matter of using the triangle inequality to extend the error bounds for the Kirchhoff–Love model to the Reissner–Mindlin model.

Now we turn to the case (15). In this case we have the following theorem, which establishes convergence in relative energy norm for the Reissner–Mindlin model. Of course no such convergence holds for the Kirchhoff–Love solution since in the case (15), $\underline{u}^\epsilon_K$ is identically zero.

Theorem 2. *Let $\underline{g} = (\underset{\sim}{g}, g_3) \in \underline{H}^1(\Omega)$ and $f_3 \in L^2(\Omega)$. Suppose that the relation* (15) *holds. For each ϵ define $\underline{u}^\epsilon_*$ by* (1), (2) *and $\underline{u}^\epsilon_R$ by* (7)–(10) *where the loads are given by* (11). *Then there exists a constant C depending only on Ω and $\underline{g}$ such that*

$$\frac{\|\underline{u}^\epsilon_* - \underline{u}^\epsilon_R\|_{E^\epsilon}}{\|\underline{u}^\epsilon_*\|_{E^\epsilon}} \le C\epsilon^{1/2}.$$

The proof of Theorem 2 will be given below. It is based on the two energies principle, which we recall in the next theorem. In it we use the notation $\|\underline{\underline{\sigma}}\|_{C^\epsilon} = [\int_{P^\epsilon} (\underline{\underline{\underline{A}}}\,\underline{\underline{\sigma}}) : \underline{\underline{\sigma}}\, d\underline{x}]^{1/2}$ with $\underline{\underline{\underline{A}}}$ the compliance tensor (the inverse of the elasticity tensor $\underline{\underline{\underline{C}}}$) for the complementary energy norm of a stress tensor field $\underline{\underline{\sigma}}$, which is equivalent to its $\underline{\underline{L}}^2(P^\epsilon)$ norm.

Theorem 3. *(The two energies principle.) Suppose that $\underline{\underline{\sigma}} \in \underline{\underline{H}}(\underline{\mathrm{div}}, P^\epsilon)$, the space of tensor valued functions whose components and row divergences are square integrable, is statically admissible, i.e.*

$$\underline{\mathrm{div}}\underline{\underline{\sigma}} + \underline{f}^\epsilon = 0 \quad in \quad P^\epsilon, \qquad \underline{\underline{\sigma}}\,\underline{n} = \underline{g}^\epsilon \quad on \quad \partial P^\epsilon_\pm,$$

where $\underline{n}$ is the unit outer normal to the surface, and suppose $\underline{u} \in \underline{H}^1(P^\epsilon)$ is kinematically admissible, i.e.

$$\underline{u} = 0 \quad on \quad \partial P^\epsilon_L.$$

Then

$$\|\underline{u} - \underline{u}^\epsilon_*\|^2_{E^\epsilon} + \|\underline{\underline{\sigma}} - \underline{\underline{\sigma}}^\epsilon_*\|^2_{C^\epsilon} = \|\underline{\underline{\sigma}} - \underline{\underline{\underline{C}}}\,\underline{\underline{e}}(\underline{u})\|^2_{C^\epsilon}.$$

The right hand side of the identity is equivalent to the $\underline{\underline{L}}^2(P^\epsilon)$ norm of the constitutive residual $\underline{\underline{\rho}} = \underline{\underline{\underline{A}}}\,\underline{\underline{\sigma}} - \underline{\underline{e}}(\underline{u})$.

We will construct a statically admissible stress field $\underline{\underline{\sigma}}$ and a kinematically admissible displacement field $\underline{u}$ from the solution of the Reissner–Mindlin model, and get the bound of the errors of $\underline{\underline{\sigma}}$ and $\underline{u}$ by estimating the constitutive residual.

In view of the assumption (15) on the loads, the forcing functions given in (11) reduce to

$$\underset{\sim}{G}_R^\epsilon = -\frac{5}{6}\underset{\sim}{g} - \epsilon^2 \frac{1}{3}\frac{\lambda}{2\mu+\lambda}\underset{\sim}{\nabla} g_3, \quad F_R^\epsilon = -\frac{5}{6}\operatorname{div}\underset{\sim}{g}.$$

The derivation of the Reissner–Mindlin model given in [1] naturally provides a stress field which is statically admissible. In the present case, the membrane, transverse shear, and normal components of this field are

$$\begin{aligned} \underset{\approx}{\sigma} &= -x_3 \underset{\approx}{C}^* \underset{\approx}{e}(\underset{\sim}{\theta}^\epsilon) + x_3 \frac{\lambda}{2\mu+\lambda}\underset{\approx}{\delta} g_3, \\ \underset{\sim}{\sigma} &= \underset{\sim}{g} - (1 - \frac{x_3^2}{\epsilon^2})\frac{5}{4}[\mu(\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\nabla} w^\epsilon) + \underset{\sim}{g}], \\ \sigma_{33} &= x_3 g_3. \end{aligned} \tag{19}$$

Invoking the model equations (8) and (9), it is readily checked that this field is indeed statically admissible.

In view of the boundary conditions (10), the displacement field defined by

$$\underline{u}(\underset{\sim}{x}, x_3) = \Big(-x_3 \underset{\sim}{\theta}^\epsilon(\underset{\sim}{x}), w^\epsilon(\underset{\sim}{x})\Big) \tag{20}$$

is kinematically admissible, and the constitutive residual $\underline{\underline{\rho}} = \underline{\underline{\underline{\underline{A}}}}\,\underline{\underline{\sigma}} - \underline{\underline{e}}(\underline{u})$ between the displacement (20) and the stress (19) is given by

$$\underset{\approx}{\rho} = 0, \quad \underset{\sim}{\rho} = \frac{5}{8\mu}(\frac{x_3^2}{\epsilon^2} - \frac{1}{5})[\mu(\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\nabla} w^\epsilon) + \underset{\sim}{g}], \quad \rho_{33} = \frac{x_3}{2\mu+\lambda}(\lambda \operatorname{div}\underset{\sim}{\theta}^\epsilon + g_3). \tag{21}$$

To get rigorous bounds on the constitutive residual, we need some *a priori* estimates of the Reissner–Mindlin solution $\underset{\sim}{\theta}^\epsilon$ and w^ϵ, which are given in the lemma below.

Lemma 4. Let $\underset{\sim}{\gamma}^\epsilon = \mu(\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\nabla} w^\epsilon) + \underset{\sim}{g}$. Then there exists a constant C only dependent on Ω such that

$$\|\underset{\sim}{\theta}^\epsilon\|_{\underset{\sim}{H}^1} + \|w^\epsilon\|_{H^1} + \epsilon^{-1}\|\underset{\sim}{\gamma}^\epsilon\|_{\underset{\sim}{L}^2} \le C(\|g_3\|_{L^2} + \epsilon^{-1/2}\|\underset{\sim}{g}\|_{\underset{\sim}{H}^1}). \tag{22}$$

Proof. For the more general mixed Reissner–Mindlin system

$$\begin{aligned} -\frac{1}{3}\underset{\sim}{\operatorname{div}} \underset{\approx}{C}^* \underset{\approx}{e}(\underset{\sim}{\theta}) + \underset{\sim}{\zeta} = \underset{\sim}{G}, \quad \operatorname{div}\underset{\sim}{\zeta} = F, \\ -\underset{\sim}{\theta} + \underset{\sim}{\nabla} w + \frac{6}{5}\mu^{-1}\epsilon^2 \underset{\sim}{\zeta} = \underset{\sim}{J}, \end{aligned}$$

together with clamping lateral boundary conditions, the following regularity result was proven in [3]:

$$\|\underset{\sim}{\theta}\|_{\underset{\sim}{H}^1} + \|w\|_{H^1} + \|\underset{\sim}{\zeta}\|_{\underset{\sim}{H}^{-1}(\operatorname{div})\cap \epsilon\cdot \underset{\sim}{L}^2} \cong \|\underset{\sim}{G}\|_{\underset{\sim}{H}^{-1}} + \|F\|_{H^{-1}} + \|\underset{\sim}{J}\|_{\underset{\sim}{\mathring{H}}(\operatorname{rot}) + \epsilon^{-1}\cdot \underset{\sim}{L}^2}. \tag{23}$$

This result is an equivalence of norms, with constants uniform in ϵ. The spaces $\underset{\sim}{H}^{-1}(\mathrm{div}) \cap \epsilon \cdot \underset{\sim}{L}^2$ and $\mathring{\underset{\sim}{H}}(\mathrm{rot}) + \epsilon^{-1} \cdot \underset{\sim}{L}^2$ both coincide with $\underset{\sim}{L}^2$ as sets for every $\epsilon > 0$ and their norms are equivalent to the $\underset{\sim}{L}^2$ norm, but not uniformly in ϵ. Specifically

$$\|\underset{\sim}{\zeta}\|_{\underset{\sim}{H}^{-1}(\mathrm{div})\cap\epsilon\cdot\underset{\sim}{L}^2} = \|\underset{\sim}{\zeta}\|_{H^{-1}} + \|\operatorname{div}\underset{\sim}{\zeta}\|_{H^{-1}} + \epsilon\|\underset{\sim}{\zeta}\|_{L^2},$$

and

$$\|\underset{\sim}{J}\|_{\mathring{\underset{\sim}{H}}(\mathrm{rot})+\epsilon^{-1}\cdot\underset{\sim}{L}^2} = \inf(\|\underset{\sim}{J}_1\|_{L^2} + \|\operatorname{rot}\underset{\sim}{J}_1\|_{L^2} + \epsilon^{-1}\|\underset{\sim}{J}_2\|_{L^2}),$$

where the infimum is taken over all sums $\underset{\sim}{J}_1 + \underset{\sim}{J}_2 = \underset{\sim}{J}$ with $\underset{\sim}{J}_1 \in \mathring{\underset{\sim}{H}}(\mathrm{rot})$ (the space of $\underset{\sim}{L}^2$ vectorfields with L^2 rotation and whose tangential component vanishes on the boundary), and $\underset{\sim}{J}_2 \in \underset{\sim}{L}^2$.

We consider a special case in which $\underset{\sim}{J} \equiv 0$ and $F = \operatorname{div}\underset{\sim}{G}$ with $\underset{\sim}{G} \in \underset{\sim}{L}^2$, and let $\bar{\underset{\sim}{\zeta}} = \underset{\sim}{\zeta} - \underset{\sim}{G}$. Then

$$-\frac{1}{3}\underset{\sim}{\operatorname{div}}\underset{\approx}{C}^*\underset{\approx}{e}\underset{\sim}{\theta} + \bar{\underset{\sim}{\zeta}} = 0, \quad \operatorname{div}\bar{\underset{\sim}{\zeta}} = 0,$$
$$-\underset{\sim}{\theta} + \underset{\sim}{\nabla} w + \frac{6}{5}\mu^{-1}\epsilon^2\bar{\underset{\sim}{\zeta}} = -\frac{6}{5}\mu^{-1}\epsilon^2\underset{\sim}{G},$$

and

$$\|\underset{\sim}{\theta}\|_{H^1} + \|w\|_{H^1} + \|\underset{\sim}{\zeta} - \underset{\sim}{G}\|_{\underset{\sim}{H}^{-1}(\mathrm{div})\cap\epsilon\cdot\underset{\sim}{L}^2} \le C\epsilon^2\|\underset{\sim}{G}\|_{\mathring{\underset{\sim}{H}}(\mathrm{rot})+\epsilon^{-1}\cdot\underset{\sim}{L}^2}. \tag{24}$$

follows from (23).

We decompose the solution $\underset{\sim}{\theta}^\epsilon$ and w^ϵ of the Reissner–Mindlin equation (8)–(10) as

$$\underset{\sim}{\theta}^\epsilon = \underset{\sim}{\theta}_0 + \underset{\sim}{\theta}_1, \quad w^\epsilon = w_0 + w_1,$$

where

$$\begin{gathered}
-\epsilon^2\frac{1}{3}\underset{\sim}{\operatorname{div}}\underset{\approx}{C}^*\underset{\approx}{e}(\underset{\sim}{\theta}_0) + \frac{5}{6}\mu(\underset{\sim}{\theta}_0 - \underset{\sim}{\nabla} w_0) = -\frac{5}{6}\underset{\sim}{g}, \\
\frac{5}{6}\mu\operatorname{div}(\underset{\sim}{\theta}_0 - \underset{\sim}{\nabla} w_0) = -\frac{5}{6}\operatorname{div}\underset{\sim}{g}, \\
\underset{\sim}{\theta}_0 = 0, \quad w_0 = 0 \text{ on } \partial\Omega.
\end{gathered} \tag{25}$$

and

$$\begin{gathered}
-\epsilon^2\frac{1}{3}\underset{\sim}{\operatorname{div}}\underset{\approx}{C}^*\underset{\approx}{e}(\underset{\sim}{\theta}_1) + \frac{5}{6}\mu(\underset{\sim}{\theta}_1 - \underset{\sim}{\nabla} w_1) = -\epsilon^2\frac{1}{3}\frac{\lambda}{2\mu+\lambda}\underset{\sim}{\nabla} g_3, \\
\frac{5}{6}\mu\operatorname{div}(\underset{\sim}{\theta}_1 - \underset{\sim}{\nabla} w_1) = 0, \\
\underset{\sim}{\theta}_1 = 0, \quad w_1 = 0 \text{ on } \partial\Omega.
\end{gathered} \tag{26}$$

From (23) with $\underset{\sim}{J} = 0$, $\underset{\sim}{G} = -\lambda/[3(2\mu+\lambda)]\underset{\sim}{\nabla} g_3$, and $F = 0$, we get

$$\|\underset{\sim}{\theta}_1\|_{H^1} + \|w_1\|_{H^1} + \epsilon^{-1}\|\underset{\sim}{\theta}_1 - \underset{\sim}{\nabla} w_1\|_{L^2} \le C\|g_3\|_{L^2}.$$

From (24) with $\underset{\sim}{J} = 0$, $\underset{\sim}{G} = -\epsilon^{-2}\dfrac{5}{6}\underset{\sim}{g}$, and $F = -\epsilon^{-2}\dfrac{5}{6}\operatorname{div}\underset{\sim}{g}$, we get

$$\|\underset{\sim}{\theta}_0\|_{H^1} + \|w_0\|_{H^1} + \epsilon^{-1}\|(\underset{\sim}{\theta}_0 - \underset{\sim}{\nabla} w_0) + \underset{\sim}{g}\|_{L^2} \le C\|\underset{\sim}{g}\|_{\mathring{\underset{\sim}{H}}(\mathrm{rot})+\epsilon^{-1}\cdot\underset{\sim}{L}^2}. \tag{27}$$

By a standard cut-off argument, we see $\|\underset{\sim}{g}\|_{\overset{\circ}{\underset{\sim}{H}}(\mathrm{rot})+\epsilon^{-1}\cdot\underset{\sim}{L}^2} \le C\,\epsilon^{-1/2}\,\|\underset{\sim}{g}\|_{\underset{\sim}{H}^1}$ The lemma then follows from the sum of these inequalities. □

Proof of Theorem 2. We give a detailed proof under the further assumption that $\underset{\sim}{g} \neq 0$. This is the case where the applied loads induce a significant transverse shear. We will briefly discuss the case of $\underset{\sim}{g} = 0$ at the end of the proof. First, We estimate the error between $\underline{u}_*^\epsilon$ the displacement field $\underline{u}$ defined by (20). By the two energies principle, we have the upper bound

$$\|\underline{u}_*^\epsilon - \underline{u}\|_{E^\epsilon} \le C(\|\underset{\sim}{\rho}\|_{\underset{\sim}{L}^2(P^\epsilon)} + \|\rho_{33}\|_{L^2(P^\epsilon)}).$$

From the lemma, we get

$$\|\underset{\sim}{\theta}^\epsilon\|_{\underset{\sim}{H}^1} + \|w^\epsilon\|_{H^1} \le C(\|g_3\|_{L^2} + \epsilon^{-1/2}\|\underset{\sim}{g}\|_{H^1}), \tag{28}$$

$$\|\mu(\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\nabla} w) + \underset{\sim}{g}\|_{\underset{\sim}{L}^2} \le C(\epsilon\,\|g_3\|_{L^2} + \epsilon^{1/2}\|\underset{\sim}{g}\|_{H^1}). \tag{29}$$

By the expression of the constitutive residual (21), we immediately get the upper bound

$$\|\underline{u}_*^\epsilon - \underline{u}\|_{E^\epsilon} \le C\epsilon. \tag{30}$$

The triangle inequality gives

$$\|\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\nabla} w\|_{\underset{\sim}{L}^2} \ge \frac{1}{\mu}\|\underset{\sim}{g}\|_{\underset{\sim}{L}^2} - C(\epsilon\,\|g_3\|_{L^2} + \epsilon^{1/2}\|\underset{\sim}{g}\|_{H^1}).$$

Since $\|\underset{\sim}{g}\|_{\underset{\sim}{L}^2} \neq 0$, we obtain in this way a positive lower bound on $\|\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\nabla} w\|_{\underset{\sim}{L}^2}$, and so we have

$$\|\underline{u}\|_{E^\epsilon} \ge C\epsilon^{1/2}. \tag{31}$$

Now we estimate the difference between $\underline{u}$ and $\underline{u}_R^\epsilon$. The correction factor y^ϵ in this case is

$$y^\epsilon = \frac{1}{2(2\mu+\lambda)}(\lambda\,\mathrm{div}\,\underset{\sim}{\theta}^\epsilon + g_3).$$

The $L^2(P^\epsilon)$ norms of the normal and shear strains corresponding to the difference $\underline{u}_R^\epsilon - \underline{u} = \big(0, 0, (x_3^2 - \epsilon^2/5)y^\epsilon\big)$ can be bounded by

$$\epsilon^{3/2}(\|\underset{\sim}{\theta}^\epsilon\|_{H^1} + \|g_3\|_{L^2}) \quad \text{and} \quad \epsilon^{5/2}(\|\underset{\sim}{\theta}^\epsilon\|_{\underset{\sim}{H}^2} + \|g_3\|_{H^1}), \tag{32}$$

respectively, and the corresponding membrane strain is zero. From the Reissner–Mindlin equation (8), we see

$$-\epsilon^2\frac{2}{3}\underset{\sim}{\mathrm{div}}\,\underset{\approx}{C}^*\underset{\approx}{e}(\underset{\sim}{\theta}^\epsilon) = -\frac{5}{3}(\mu(\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\nabla} w^\epsilon) + \underset{\sim}{g}) - \epsilon^2\frac{2}{3}\frac{\lambda}{2\mu+\lambda}\underset{\sim}{\nabla} g_3.$$

By the regularity theorem of two-dimensional elasticity, the $\underset{\sim}{H}^2$ norm of $\underset{\sim}{\theta}^\epsilon$ is bounded by the $\underset{\sim}{L}^2$ norm of the right hand side. We get

$$\|\underset{\sim}{\theta}^\epsilon\|_{H^2} \le C(\epsilon^{-\frac{3}{2}}\|\underset{\sim}{g}\|_{\underset{\sim}{H}^1} + \epsilon^{-1}\,\|g_3\|_{L^2} + \|g_3\|_{H^1}). \tag{33}$$

Combining the bounds in (32), (28), and (33), we obtain

$$\|\underline{u} - \underline{u}_R^\epsilon\|_{E^\epsilon} \le C\epsilon. \tag{34}$$

The theorem follows easily from (30), (34), and (31). From (27) it is easy to see that if $\underset{\sim}{g} \in \overset{\circ}{\underset{\sim}{H}}(\mathrm{rot})$, then the convergence rate in the theorem is $O(\epsilon)$.

In the above arguments, the assumption $\underset{\sim}{g} \neq 0$ was essential, since our lower bound on the energy norm of $\underline{u}_*^\epsilon$ was based on it. If $\underset{\sim}{g} = 0$, the condition (15) reduces to $g_3 + f_3 = 0$, i.e., the surface forces are vertical and exactly balance the body force. Since the solution of (25) is identically equal to zero, the Reissner–Mindlin solution is solely given by the solution of (26). We still use the two energies principle to prove the convergence. The statically admissible stress field is again defined by (19), while the kinematically admissible displacement will be defined by

$$\underline{U} = \left(-x_3 \underset{\sim}{\theta}^\epsilon(\underset{\sim}{x}), w^\epsilon(\underset{\sim}{x}) + (x_3^2 - \epsilon^2/5) Y^\epsilon(\underset{\sim}{x})\right),$$

in which $Y^\epsilon \in H_0^1$ is obtained by cutting off the edge of y^ϵ, and it is defined as the solution of

$$\epsilon^2 (\underset{\sim}{\nabla} Y^\epsilon, \underset{\sim}{\nabla} v)_{\underset{\sim}{L}^2} + (Y^\epsilon, v)_{L^2} = (y^\epsilon, v)_{L^2}, \quad \forall\ v \ \in \ H_0^1.$$

By arguments similar to what we used above or that of [1], depending on whether $\Delta g_3 = 0$ or not, a convergence rate $\epsilon^{1/2}$ of $\underline{U}$ to $\underline{u}_*^\epsilon$ in the relative energy norm can be established. The same convergence of $\underline{u}_R^\epsilon$ to $\underline{u}_*^\epsilon$ then can be proved by estimating the energy norm of $\underline{u}_R^\epsilon - \underline{U}$. □

4. Examples

Here we give two simple examples for which the relation (15) holds, and one example for which the condition (15) is not exactly satisfied but nearly so. In these examples, the plate extends infinitely in the x_2-direction, and the displacement is orthogonal to this direction and independent of x_2. Thus the elasticity problem reduces to a two-dimensional problem (plane strain), and the Reissner–Mindlin and Kirchhoff–Love equations reduce to ordinary differential equations.

For the first example, the plate domain is $(0,1) \times \mathbb{R} \times (-\epsilon, \epsilon)$ and we impose the traction boundary condition $[\underline{\underline{\underline{C}}}\,\underline{\underline{e}}(\underline{u})]\underline{n} = (1,0,0)$ on the top surface and $[\underline{\underline{\underline{C}}}\,\underline{\underline{e}}(\underline{u})]\underline{n} = (-1,0,0)$ on the bottom surface. The solution is also clamped to zero on the lateral boundary $\{0,1\} \times \mathbb{R} \times (-\epsilon, \epsilon)$. For this simple problem, we can compute the Reissner–Mindlin solution exactly, namely

$$\underset{\sim}{\theta}^\epsilon = c^\epsilon (x_1^2 - x_1, 0), \quad w^\epsilon = \frac{c^\epsilon}{6}(2x_1^3 - 3x_1^2 + x_1),$$

where $c^\epsilon = (\mu/6 + c_1 \epsilon^2)^{-1}$ with $c_1 = [16\mu(\mu+\lambda)]/[5(2\mu+\lambda)]$. Notice that $\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\nabla} w^\epsilon = c^\epsilon(\mu/6, 0)$, and so does not converge to zero. Thus the Kirchhoff–Love hypothesis is violated for this problem, even in the limit. We see that for this problem the transverse displacement converges to a finite nonzero limit, $\mu^{-1}(2x_1^3 - 3x_1^2 + x_1)$.

The second example is even simpler, but disposes with the clamping of the lateral boundary. Rather we consider a biperiodic problem, so the plate domain is $\mathbb{R}^2 \times (-\epsilon, \epsilon)$ and in place of lateral boundary conditions we require that the solution be 1-periodic with respect to x_1 and x_2. We impose the same shearing loads on the top and bottom surfaces as in the previous example. This time we find that the elasticity solution and the Reissner–Mindlin solution coincide. They are both the simple shears: $\underline{u}_*^\epsilon = \underline{u}_R^\epsilon = (x_3/\mu, 0, 0)^T$. This is an extreme case: the Reissner–Mindlin solution captures the elasticity solution exactly, while the Kirchhoff–Love solution, which is identically zero, misses it entirely. Again, the Kirchhoff hypothesis is violated.

Finally we present an example in which (15) does not hold, but the superior accuracy of the Reissner–Mindlin model is apparent. As in the first example, we take a plate occupying the region $\{0, 1\} \times \mathbb{R} \times (-\epsilon, \epsilon)$, clamped to zero on the lateral boundary, and loaded by constant tractions on the top and bottom surfaces. This time we impose the traction boundary conditions $[\underline{\underline{\underline{C}}}\,\underline{\underline{e}}(\underline{u})]\underline{n} = (1, 0, 10^{-3})$ on the top and $[\underline{\underline{\underline{C}}}\,\underline{\underline{e}}(\underline{u})]\underline{n} = (-1, 0, 10^{-3})$ on the bottom. As half-thickness we take $\epsilon = 1/40$ and we take both Lamé coefficients equal to unity. Figure 1 shows the deformed plate cross-section as modeled by two-dimensional plain strain linear elasticity (computed numerically via an adaptive finite element solver) in the middle, and by the Reissner–Mindlin and Kirchhoff–Love approximations (computed analytically) on the left and right of the figure. We see that the Reissner–Mindlin model captures the deformation very well, while the Kirchhoff–Love model misses essential features of the solution and is highly inaccurate.

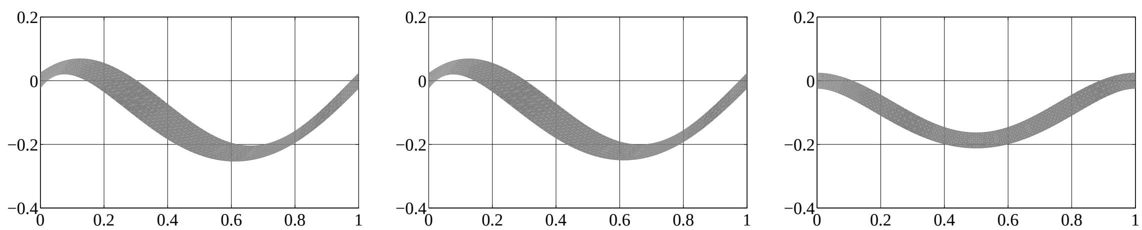

FIGURE 1. Deformation of the cross-section of a plate in plane strain as determined by the Reissner–Mindlin, elastic, and Kirchhoff–Love models, respectively

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