THE CONVERGENCE OF A NUMERICAL SCHEME FOR THE FULLY NONLINEAR WATER WAVE PROBLEM

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Abstract. In this paper, the analysis of a numerical method for the water wave problem is performed. This problem is governed by a system of nonlinear pseudo-differential equations. To solve the integral equation present in the problem, we split the integral kernel in its singular, Hilbert-like transform part, and its smooth parts. We deal with the singular part of the integrals explicitly, and it is computed with $n\log(n)$ computational cost. A spectrally accurate fast integral method is proposed for the smooth integral. Our numerical scheme does not need the inclusion of smoothing techniques, in opposition to previous methods. After making a spatial discretization, we find a system of ordinary differential equations for the numerical error and we prove that at any given time, the error goes to zero as the number of nodal points increase.

1. Introduction.

In this work, we analyse and prove the convergence of a numerical scheme for the water wave problem. It is well known that careless discretizations for this problem lead to numerical instabilities, and some kind of smoothing is usually employed to control spurious oscillations. These oscillations are not present in the continuous case, and [BHL-Cont], [Mad] proved that perturbations linearized around an exact solution have bounded growth rate.

Using a boundary integral method, the problem's dimension is reduced by one, at the price of having to solve an integral equation. Also, the wave velocity is given by an integral with singular kernel. At the discrete level, the quadrature used can introduce undesirable high frequency modes. In their numerical method, to deal with the singular integral, [BHL-Num] use an alternating trapezoidal rule. This quadrature is spectrally accurate but introduces aliasing errors, wich were eliminated with smoothing techniques, and makes the analysis more complicated, in oposition to the relative simplicity of the work in [BHL-Cont].

The key point is that at the continuous level, an integral operator with smooth kernel is a smoothing operator. The alternating trapezoidal rule, due to aliasing errors, does not smooths out the high frequency modes. Applying smoothing

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techniques to the integrand, this alternating trapezoidal quadrature regains the smoothing property. This is essencial for the wave estability.

On the other hand, given a smooth kernel, the trapezoidal rule have the same smoothing property of the continuous integral. We use this as we split our kernel in a simple but singular part and a more complicated but smooth part. The resulting singular integral is computed explicitly using Fourier coefficients. In fact, the singular term in the integral is computed exactly for trigonometric polynomials. The smooth integral is computed using the standart trapezoidal rule, originating no aliasing errors. Doing this, we could avoid unnecessary smoothing on the wave profile. Another advantage of our approach is that the analysis is considerably simplified.

In the rest of this section, we set up both the continuous problem and its spatial discretization. Some useful definitions are made. In Section 2, some technical lemmas that we use future are stated. We also present a consistency result. In Section 3, we perform a stability analysis for the wave velocity, which is decomposed in its most singular terms and bounded terms. Section 4 concerns the actual convergence of the method. A stable fast integration method is proposed in Section 5 and some comments are made in Section 6. Finally, Section 7 contain proofs for results stated along the paper.

We assume the wave to be periodic, with no surface tension and with infinite depth. The wave profile is parametrized in the complex plane by $z(\alpha,t) = x(\alpha,t) + iy(\alpha,t)$, where $\alpha \in [-\pi,\pi)$. The wave potential is given by $\phi(\alpha,t)$ and $\gamma(\alpha,t)$ is the double layer density. Both the wave's potential and the double layer potential are real functions.

Troughtout this paper, we will not make the dependence on the time variable explicity, in order to simplify the notation. So, for instance, $z(\alpha,t)$ will be denoted simply by $z(\alpha)$. This will cause no confusion. The derivatives with respect to α will be denoted as in $z_\alpha(\alpha)$. Finally, $\overline{z}(\alpha)$ will denote the complex conjugation.

The water wave problem can be described by two evolution equations (one for the wave profile, another for the wave potential), coupled with an integral equation of the second kind that relates the double layer density and the potential, at each fixed time. For this model derivation and an analysis of the continuous case, we refer the reader to [Mad]. In what follows, the problem's equations are presented and they are then analyzed.

The continuous problem can be written as

$$\begin{aligned}
\frac{\partial \overline{z}}{\partial t} &= e^{-iz}\mathcal{K}\left(z, \frac{\gamma}{iz_\alpha e^{-iz}}\right) + \frac{\gamma}{2z_\alpha}\,, \\
\frac{\partial \phi}{\partial t} &= \frac{|w|^2}{2} - gy\,, \\
\phi_\alpha &= \Im\left(iz_\alpha e^{-iz}\mathcal{K}\left(z, \frac{\gamma}{iz_\alpha e^{-iz}}\right)\right) + \frac{\gamma}{2}\,,
\end{aligned} \tag{1.1}$$

where

$$(\mathcal{K}(z,\rho))(\alpha_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(e^{-iz(\alpha_0)} - e^{-iz(\alpha)}\right)\rho_\alpha(\alpha)\,d\alpha\,, \tag{1.2}$$

and the dependence of $\mathcal{K}$ on the wave profile is clear.

Denote the complex velocity as

$$w = u - iv = \frac{\partial \overline{z}}{\partial t}\,. \tag{1.3}$$

Definition. Let the operator $\mathcal{H} : L^2(-\pi,\pi) \to L^2(-\pi,\pi)$ be defined by

$$\mathcal{H}(\rho) = \frac{1}{\pi}\int_{-\pi}^{\pi} \log\left|2\sin\frac{\alpha_0-\alpha}{2}\right| \rho_\alpha(\alpha)\,d\alpha\,.$$

Remark 1.1. The operator $\mathcal{H}$ can be computed using Fourier coefficients. In fact, for $\rho \in L^2(-\pi,\pi)$, we have:

$$\widehat{(\mathcal{H}(\rho))}_k = -i\ sgn(k)\ \hat{\rho}_k\ .$$

Remark 1.2. It is immediate from the definition of $\mathcal{K}$ in (1.2) that

$$(\mathcal{K}(z,\rho))(\alpha_0) = \frac{\mathcal{H}(\rho)}{2} + \frac{1}{2\pi}\int_{-\pi}^{\pi} S(z,\alpha_0,\alpha)\,\rho_\alpha(\alpha)\,d\alpha\,,$$

where the smooth kernel S is defined by:

$$S(z,\alpha_0,\alpha) = \begin{cases} \log \dfrac{e^{-iz(\alpha_0)} - e^{-iz(\alpha)}}{\left|2\sin\frac{\alpha_0-\alpha}{2}\right|} & \text{for } \alpha \neq \alpha_0\,, \text{ and} \\ \log\left(-iz_\alpha(z_0)\,e^{-iz(\alpha_0)}\right) & \text{for } \alpha = \alpha_0\,. \end{cases} \tag{1.4}$$

To approximate equations (1.1), we discretize the parameter domain $[-\pi,\pi)$ by:

$$\alpha_j = jh\ \text{, for } j = -\frac{N}{2},\ldots,\frac{N}{2}-1\ ,$$

where $h = 2\pi/N$ and N is an even positive number.

Definitions.
Given the nodal values f_j, for $j = -N/2,\ldots,N/2-1$, the discrete Fourier coefficients are denoted by $\tilde{f}_k$ and are given by:

$$\tilde{f}_k = \frac{1}{N}\sum_{j=-\frac{N}{2}}^{\frac{N}{2}-1} f_j e^{-ik\alpha_j}, \quad \text{for } k = -\frac{N}{2},\ldots,\frac{N}{2}-1\ .$$

Let $V_h = span\left\{e^{ik\alpha}\ ,\ -\frac{N}{2} \le k \le \frac{N}{2}-1\right\}$. For $\rho^h \in V_h$, we will denote its nodal values by ρ^h_j.

We can also define $I_h : C^0[-\pi,\pi] \to V_h$, such that if $\rho \in C^0[-\pi,\pi]$, then $I_h\rho$ is the element of V_h interpolating the nodal values $\rho(\alpha_j)$. We compute $I_h\rho$ using discrete Fourier coefficients:

$$\widetilde{(I_h\rho)}_k = \widetilde{\rho}_k \quad \text{for } k = -\frac{N}{2},\ldots,\frac{N}{2}-1.$$

Let D_h be a discrete derivative operator such that the action of D_h is computed by

$$\widetilde{(D_h\rho)}_k = ik\theta(kh)\widetilde{\rho}_k \quad \text{for } k = -\frac{N}{2}, \dots, \frac{N}{2} - 1, \tag{1.5}$$

where $0 \leq \theta(kh) \leq 1$ satisfies the following:

$$\begin{aligned} &\theta(-x) = \theta(x) \ , \ \theta(x) \geq 0 \text{ and } \theta(\pi) = 0 \ , \\ &\theta(x) = 1 \text{ for } |x| \leq \lambda\pi \text{ where } 0 \leq \lambda \leq 1, \\ &\theta(\cdot) \in C^2[-\pi, \pi] \ . \end{aligned} \tag{1.6}$$

The spatial discretization for (1.1) is as shown below. For $j = -N/2, \dots, N/2-1$ we have:

$$\begin{aligned} &\frac{\partial \overline{z}_j^h}{\partial t} = e^{-iz_j^h}\mathcal{K}_h\left(z_j^h, \frac{\gamma_j^h}{iD_h z_j^h}e^{iz_j^h}\right) + \frac{\gamma_j^h}{2D_h z_j^h}, \\ &\frac{\partial \phi_j^h}{\partial t} = \frac{\left|w_j^h\right|^2}{2} - gy_j^h \quad \text{and} \\ &D_h\phi_j^h = \Im\left(iD_h z_j^h e^{-iz_j^h}\mathcal{K}_h\left(z_j^h, \frac{\gamma_j^h}{iD_h z_j^h}e^{iz_j^h}\right)\right) + \frac{\gamma_j^h}{2}. \end{aligned} \tag{1.7}$$

The operator $\mathcal{K}$ approximation is based on remark (1.2). Denote the value of $S(z, \alpha_j, \alpha_k)$ by $S_{j,k}(z)$, for $j \neq k$, and let $S_{j,j}(z) = \log\left(-iD_h z e^{-iz(\alpha_j)}\right)$. We can then define the jth component of $\mathcal{K}_h$ as follows:

$$\mathcal{K}_h\left(z_j^h, \rho_j^h\right) = \frac{\mathcal{H}_h\left(\rho_j^h\right)}{2} + h\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} S_{j,k}\left(z^h\right)D_h\rho_k^h \ , \text{ for } j = -\frac{N}{2}, \dots, \frac{N}{2} - 1\,, \tag{1.8}$$

where we define $\mathcal{H}_h = \mathcal{H}I_h$.

Definitions. Similarly to (1.3), we write

$$w_j^h = u_j^h - iv_j^h = \frac{\partial \overline{z}_j^h}{\partial t}. \tag{1.9}$$

Also the jth component of the numerical approximation error is denoted by:

$$\begin{aligned} &\dot{z}_j^h = z_j^h - z(\alpha_j)\,, \\ &\dot{\phi}_j^h = \phi_j^h - \phi(\alpha_j) \quad \text{and} \\ &\dot{\gamma}_j^h = \gamma_j^h - \gamma(\alpha_j)\,, \end{aligned} \tag{1.10}$$

where z, ϕ and γ are solutions of (1.1).

We will decompose the evolution equations in its most singular terms and its bounded terms. We call A_{-m} any operator such that

$$\left\|A_{-m}\left(\rho^h\right)\right\|_{H^m} \leq C\left\|\rho^h\right\|_{H^0} \quad \text{for } 0 \leq k \leq m \text{ and } \rho^h \in V_h.$$

The operator A_{-m} might depend on the underlying solutions z, ϕ and γ, and on the number of nodal points N. The constant C is independent N, but may depends on z, ϕ and γ. A particular case of above definition is that any bounded operator in H^0 will be denoted by A_0.

2. Technical Lemmas.

In this section, we will state some technical lemmas that will be useful troughout the paper. All proofs are presented in section 7.

The following lemma describes the behaviour of the discrete derivative operator defined in (1.5).

Lemma 2.1. *Assume θ is an even $C^2(-\pi,\pi)$ function, with $\theta(\pi)=0$. If f is a 2π-periodic function and $f\in\mathbb{H}^3(\mathbb{R})$ then*

$$D_h\left(f(\alpha_j)z_j^h\right)=f(\alpha_j)D_hz_j^h+A_h(f,z^h)(\alpha_j)\,.$$

For an independent constant C, we have $\left\|A_h(f,z^h)\right\|_{\mathbb{H}^0}\le C\left\|\theta\right\|_{C^2}\left\|f\right\|_{\mathbb{H}^{\frac{5}{2}}}\left\|z^h\right\|_{\mathbb{H}^0}$.

Lemma 2.3. *Assume $g(\alpha,\beta)\in\mathbb{H}^{m+1}(-\pi,\pi)^2$ is 2π-periodic in both variables, and let D_h be as in (1.5). If $\left\|\theta\right\|_\infty\le 1$ and if*

$$\chi_{j_1}^h:=h\sum_{j_2=-\frac{N}{2}}^{\frac{N}{2}-1}g(\alpha_{j_1},\alpha_{j_2})D_h\rho_{j_2}^h\,,$$

then $\left\|\chi^h\right\|_{\mathbb{H}^0}\le 2\pi\left\|g\right\|_{\mathbb{H}^2(-\pi,\pi)^2}\left\|\rho^h\right\|_{\mathbb{H}^{-1}}$.

In order to estimate the linear variation of the integral operator, i.e., to prove Lemma (3.1), we will need a technical lemma that computes the action of the singular part of the perturbed integral operator on the trigonometric polynomials. This following original result, of surprisingly simplicity, will guarantee that wave perturbations do not upset the boundedness of the integral operator.

Lemma 2.4. *If N is an even positive integer, $h:=2\pi/N$ and $\alpha_j:=jh$, then the following identity holds:*

$$h\sum_{\substack{k=-\frac{N}{2}\\k\neq j}}^{\frac{N}{2}-1}\left.\frac{d}{d\alpha}\right|_{\alpha=\alpha_k}\log\left|2\sin\frac{\alpha_j-\alpha}{2}\right|e^{il\alpha_k}=ie^{il\alpha_j}\Sigma(l)\,, \tag{2.1}$$

where

$$\Sigma(l)=\begin{cases}-\pi-lh & \text{for}\quad l=-\dfrac{N}{2},\dots,-1\,,\\ 0 & \text{for}\quad l=0\,,\\ \pi-lh & \text{for}\quad l=1,\dots,\dfrac{N}{2}-1\,.\end{cases} \tag{2.2}$$

Remark 2.5. Note that $|g(l)|\le\pi$ for $l=-N/2,\dots,N/2-1$ and all even positive integer N. This implies a uniform bound on the summation (2.1).

The following result will guarantee the commutator Lemma (2.9).

Lemma 2.7. *Let $f\in\mathbb{H}^{\frac{5}{2}}(-\pi,\pi)$ be a 2π-periodic function and let θ be an even $C^2[-\pi,\pi]$ function with $\theta(-\pi)=\theta(\pi)=0$. Let the derivative operator be defined by (1.5). Then, if $P_h:L^2(-\pi,\pi)\to V_h$ denotes the L^2 projection, we have:*

$$\left\|I_h\left(fD_h\rho^h\right)-P_h\left(fD_h\rho^h\right)\right\|_{\mathbb{H}^0}\le\left\|\theta\right\|_{C^1}\left\|f\right\|_{\mathbb{H}^2}\left\|\rho^h\right\|_{\mathbb{H}^0}\ ,\ \textit{for }\rho^h\in V_h\,.$$

Lemma 2.9. *Let $f \in \mathbb{H}^{\frac{5}{2}}$ be a 2π-periodic function and let the discrete derivative operator be as in (1.5). Then, for any $\rho \in V_h$, the following holds:*

$$\mathcal{H}_h\left(f D_h \rho^h\right) - I_h\left(f \mathcal{H}_h\left(D_h \rho^h\right)\right) = B_{com}\left(f, \rho^h\right), \text{ where}$$

$$\left\|B_{com}\left(f, \rho^h\right)\right\|_{\mathbb{H}^0} \le \left(2C_2(\theta)\left\|f\right\|_{\mathbb{H}^{\frac{5}{2}}} + C_3\right)\left\|\rho^h\right\|_{\mathbb{H}^0}.$$

Remark 2.10. The result in Lemma (2.9) does not hold for spectral derivative without Fourier filtering, i.e., the $\theta = 1$ case.

It is somewhat straightforward to prove the following consistency result.

Lemma 2.11. *The exact solution of (1.1) satisfy the numerical scheme (1.7) with an $O(h^r)$ error, where r is the order of the discrete derivative operator.*

3. Stability Analysis.

In this section, the behaviour of the right hand side of (1.7-i) is analyzed when applied to the error function and written down in terms of simple, linear operators. The nonlinear part is bounded and then cause no trouble. The computations presented in this section allow us to write the evolution equation in a different form, suitable for an energy estimate, in section 4. The way the numerical solution behaves is based on the lemmas stated bellow. The proof of Lemma (3.1) can be found in section 7.

Lemma 3.1. *The operator $\mathcal{K}_h$ applied to an element ρ^h of V_h can be written as*

$$\mathcal{K}_h\left(z^h, \rho^h\right) = \mathcal{K}_h(z, \rho) + \frac{\mathcal{H}_h\left(\dot{\rho}^h\right)}{2} + B_1\left(z, \rho, \dot{z}^h\right) + B_2\left(z, \rho, \dot{\rho}^h\right) + NL(\dot{z}^h, \dot{\rho}^h).$$

where $\dot{\rho}_j^h := \rho_j^h - \rho(\alpha_j)$ and $NL(\dot{z}^h, \dot{\rho}^h)$ contains the nonlinear terms in $\dot{z}^h$ and $\dot{\rho}^h$. The operators B_1, B_2 can be bounded as follows:

$$\left\|B_1\left(z, \rho, \dot{z}^h\right)\right\|_{\mathbb{H}^0} \le C_4(z, \rho)\left\|\dot{z}^h\right\|_{\mathbb{H}^0}$$

$$\left\|B_2\left(z, \rho, \dot{\rho}^h\right)\right\|_{\mathbb{H}^0} \le \left\|S(z, \cdot, \cdot)\right\|_{\mathbb{H}^1(-\pi,\pi)^2}\left\|\dot{\rho}^h\right\|_{\mathbb{H}^{-1}}$$

and $C_4(z, \rho) := \bar{C}_1(z, \rho) + \bar{C}_2(z, \rho) + \bar{C}_3(z) + O(h^r)$. The $\bar{C}_j$'s are defined in the lemma's proof.

Equation (1.7-i) gives the wave velocity, and its approximation error can be found using Lemmas (3.1), (2.11) and some straightforward computations. If $\dot{w}_j^h := w_j^h - w(\alpha_j)$, then:

(3.1)
$$\begin{aligned}
\dot{w}_j^h &= e^{-iz(\alpha_j)}\frac{\mathcal{H}_h}{2i}\left[\left(\frac{\dot{\gamma}_j^h}{D_h z(\alpha_j)} - \frac{\gamma(\alpha_j) D_h \dot{z}_j^h}{\left(D_h z(\alpha_j)\right)^2}\right)e^{iz(\alpha_j)}\right] + \frac{\dot{\gamma}_j^h}{2D_h z(\alpha_j)} \\
&\quad - \frac{\gamma(\alpha_j) D_h \dot{z}_j^h}{2\left(D_h z(\alpha_j)\right)^2} + B_1\left(z, \frac{\gamma e^{iz}}{iD_h z}, \dot{z}^h\right) + B_2\left(z, \frac{\gamma e^{iz}}{iD_h z}, \dot{\gamma}^h\right) + NL(\dot{z}_j^h, \dot{\gamma}_j^h) \\
&= e^{-iz(\alpha_j)}\left(-i\mathcal{H}_h + I_h\right)\left[\frac{e^{iz(\alpha_j)}}{2D_h z(\alpha_j)}\left(\dot{\gamma}_j^h - \frac{\gamma(\alpha_j) D_h \dot{z}_j^h}{D_h z(\alpha_j)}\right)\right] \\
&\quad + B_1\left(z, \frac{\gamma e^{iz}}{iD_h z}, \dot{z}^h\right) + B_2\left(z, \frac{\gamma e^{iz}}{iD_h z}, \dot{\gamma}^h\right) + NL(\dot{z}_j^h, \dot{\gamma}_j^h).
\end{aligned}$$

The value of $\dot{\gamma}_j^h$ is given in Lemma (3.3) and is computed using the variation of the discrete integral equation (1.7-iii) and Lemma (3.1). The proof of the lemma is given in section 7.

Lemma 3.3. *The double layer density error can be written as*

$$\dot{\gamma}_j^h = 2\ D_h\dot{\phi}_j^h - 2\Im\left(iD_h\dot{z}_j^h e^{-iz(\alpha_j)}w_0(\alpha_j)\right) + \Im\mathcal{H}_h\left(\frac{\gamma(\alpha_j)}{D_hz(\alpha_j)}D_h\dot{z}_j^h\right)$$
$$+ A_0\left(\dot{\phi}^h\right) + A_0\left(\dot{z}^h\right) + NL(\dot{z}^h,\dot{\gamma}^h) + O\left(h^r\right)\ ,$$

where

$$w_0(\alpha) := \mathcal{K}_h\left(z, \frac{\gamma}{iD_hze^{-iz}}\right)(\alpha)\ . \tag{.2}$$

Applying Lemmas (3.3) and (2.9) in equation (3.1), we get that the wave velocity is given by:

$$\dot{w}_j^h = \frac{1}{2D_hz(\alpha_j)}\left(-i\mathcal{H}_h + I_h\right)\Big[2D_h\dot{\phi}_j^h - 2\Im\left(iD_h\dot{z}_j^h e^{-iz(\alpha_j)}w_0(\alpha_j)\right)$$
$$+ \Im\mathcal{H}_h\left(\frac{\gamma(\alpha_j)}{D_hz(\alpha_j)}D_h\dot{z}_j^h\right) - \frac{\gamma(\alpha_j)}{D_hz(\alpha_j)}D_h\dot{z}_j^h\Big] + A\left(\dot{\phi}_j^h, \dot{z}_j^h, \dot{\gamma}_j^h\right)\ ,$$

where

$$A\left(\dot{\phi}^h, \dot{z}^h, \dot{\gamma}^h\right) := A_0\left(\dot{\phi}^h\right) + A_0\left(\dot{z}^h\right) + NL(\dot{z}^h,\dot{\gamma}^h) + O\left(h^r\right)\ . \tag{3.3}$$

Let

$$T^* := \sup\{\,t\,;\ t \le T,\ \|\dot{z}^h\|_{H^0},\ \|\dot{\phi}^h\|_{H^0} \le h^{\frac{5}{2}}\ ,\ \|\dot{\gamma}^h\|_{H^0} \le h^{\frac{3}{2}}\}\,. \tag{3.4}$$

For $t \le T^*$ we will assume that

$$\left\|NL(\dot{z}^h,\dot{\gamma}^h)\right\|_{H^0} \le C\,\|\dot{z}^h\|_{H^0}. \tag{3.5}$$

Using the identity $\left(-i\mathcal{H}_h + I_h\right)\Im\mathcal{H}_h = i\left(-i\mathcal{H}_h + I_h\right)\Im$, and

$$-\Im\left(iD_h\dot{z}_j^h e^{-iz(\alpha_j)}w_0(\alpha_j)\right) + i\Im\left(\frac{\gamma(\alpha_j)}{2D_hz(\alpha_j)}D_h\dot{z}_j^h\right) - \frac{\gamma(\alpha_j)}{2D_hz(\alpha_j)}D_h\dot{z}_j^h$$
$$= -\Re\left(D_h\dot{z}_j^h\left(e^{-iz(\alpha_j)}w_0(\alpha_j) + \frac{\gamma(\alpha_j)}{2D_hz(\alpha_j)}\right)\right)$$
$$= -\Re\left(D_h\dot{z}_j^h w(\alpha_j)\right) + O\left(h^r\right)\ ,$$

where definitions (1.3), (.2) and Consistency Lemma (2.11) were used, we just proved the following result.

Proposition 3.4. *The wave profile velocity can be written as*

$$\dot{w}_j^h = \frac{1}{D_hz(\alpha_j)}\left(-i\mathcal{H}_h + I_h\right)\left(D_h\dot{\phi}_j^h - \Re\left(D_h\dot{z}_j^h w(\alpha_j)\right)\right) + A\left(\dot{\phi}_j^h, \dot{z}_j^h, \dot{\gamma}_j^h\right)\ , \tag{3.6}$$

where

$$\left\|A\left(\dot{\phi}^h, \dot{z}^h, \dot{\gamma}^h\right)\right\|_{H^0} \le C\left(\|\dot{\phi}^h\|_{H^0} + \|\dot{z}^h\|_{H^0} + O\left(h^r\right)\right)\ . \tag{3.7}$$

Remark 3.5. Relation (3.6) is a crucial one for the energy estimates of section 4. This identity was first obtained by [BHL-Num], but they had to use smoothing techniques to control high frequency modes introduced by their numerical scheme.

4. Energy Estimate and Numerical Convergence.

Using the results of section 3, we are able to write down the evolution equation for the numerical error in a simplified manner, and then we make some estimatives.

Analogously to the continuous case in [BHL-Num], define

$$\dot{F}_j^h = \dot{\phi}_j^h - \Re\left(w\left(\alpha_j\right)\dot{z}_j^h\right) \tag{4.1}$$

and then, substituting in (3.6) and applying Lemma (2.1), the following identity holds:

$$\dot{w}_j^h = \frac{1}{D_h z\left(\alpha_j\right)}\left(-i\mathcal{H}_h + I_h\right)D_h\dot{F}_j^h + A\left(\dot{\phi}_j^h, \dot{z}_j^h, \dot{\gamma}_j^h\right) . \tag{4.2}$$

Definitions. Let $\sigma\left(\alpha\right) = \left|x_\alpha^2\left(\alpha\right) + y_\alpha^2\left(\alpha\right)\right|^{-1/2}$. Decomposing the wave profile in its normal and tangential directions, we define:

$$\begin{aligned} \dot{z}_j^{Th} &= \sigma\left(\alpha_j\right)\left(D_h x\left(\alpha_j\right)\dot{x}_j^h + D_h y\left(\alpha_j\right)\dot{y}_j^h\right) , \\ \dot{z}_j^{Nh} &= \sigma\left(\alpha_j\right)\left(-D_h y\left(\alpha_j\right)\dot{x}_j^h + D_h x\left(\alpha_j\right)\dot{y}_j^h\right) . \end{aligned} \tag{4.3}$$

Differentiating (4.3) in time, and using (4.2), we get

$$\frac{\partial(\dot{z}_j^{Th})}{\partial t} - i\frac{\partial(\dot{z}_j^{Nh})}{\partial t} = \sigma\left(\alpha_j\right)\left(-i\mathcal{H}_h + I_h\right)D_h\dot{F}_j^h + A\left(\dot{\phi}_j^h, \dot{z}_j^h, \dot{\gamma}_j^h\right) \tag{4.4}$$

Noting that the evolution equations for z_j^{Th} and z_j^{Nh} are similar, we define

$$\dot{\delta}^h = \dot{z}^{Th} + \mathcal{H}_h\dot{z}^{Nh} , \tag{4.5}$$

and then, using Lemma (2.9) we have

$$\frac{\partial(\dot{\delta}^h)}{\partial t} = A\left(\dot{\phi}^h, \dot{z}^h, \dot{\gamma}^h\right) . \tag{4.6}$$

Similarly to (3.3), define

$$A\left(\dot{F}^h, \dot{z}^{Nh}, \dot{\delta}^h\right) = A_0\left(\dot{F}^h\right) + A_0\left(\dot{z}^{Nh}\right) + A_0\left(\dot{\delta}^h\right) + O\left(h^r\right) . \tag{4.7}$$

The following proposition gives the error's evolution equation in the new variables:

Proposition 4.1. *Defining the new variables, we can write the evolution equations as:*

$$\begin{aligned} \frac{\partial\left(\dot{z}_j^{Nh}\right)}{\partial t} &= \sigma(\alpha_j)\Lambda_h\dot{F}_j^h + A\left(\dot{F}_j^h, \dot{z}_j^{Nh}, \dot{\delta}_j^h\right) , \\ \frac{\partial\left(\dot{\delta}_j^h\right)}{\partial t} &= A\left(\dot{F}_j^h, \dot{z}_j^{Nh}, \dot{\delta}_j^h\right) , \\ \frac{\partial\left(\dot{F}_j^h\right)}{\partial t} &= -c\left(\alpha_j\right)\dot{z}_j^{Nh} + \frac{1}{2}\left(\left(\dot{u}_j^h\right)^2 + \left(\dot{v}_j^h\right)^2\right) \end{aligned} \tag{4.8}$$

where we denote $\Lambda_h := \mathcal{H}_h D_h$ and $c\left(\alpha, t\right) := -\nabla p\cdot\vec{n}$ is the inner product between the pressure gradient and the surface normal vector.

We will assume that:

$$c\left(\alpha, t\right) \geq c_0 > 0 \tag{4.9}$$

With these definitions, we can bound the growth of the new variables in the following way:

Proposition 4.2. *The following inequality holds for* $t \le T^*$*:*

$$\left(\left\| \dot{z}^{Nh}(t) \right\|_{H^0}^2 + \left\| \dot{\delta}^h(t) \right\|_{H^0}^2 + \left\| \dot{F}^h(t) \right\|_{H^{\frac{1}{2}}}^2 \right)^{\frac{1}{2}} \le C(T) h^r \ \text{for}\ t \le T^* . \tag{4.10}$$

i.e., in the new variables and for any fixed time less then T^**, the numerical solution converges to the exact solution in the above discrete norm.*

In the original variables, we can estimate

$$\begin{aligned} &\left\| \dot{z}^h(t) \right\|_{H^0} + \left\| \dot{\phi}^h(t) \right\|_{H^0} \le B(T) h^r , \\ &\left\| \dot{\gamma}^h(t) \right\|_{H^0} \le B(T) h^{r-1} \end{aligned} \qquad \text{for } t \le T^* . \tag{4.11}$$

The above estimate (4.11-i) follows from Proposition (4.2) and from the new variables definitions (4.1), (4.3) and (4.5). The estimate for $\dot{\gamma}^h$ comes from Lemma (3.3).

If $r \ge 3$ and h is small enough, then

$$\begin{aligned} &\left\| \dot{z}^h(t) \right\|_{H^0} + \left\| \dot{\phi}^h(t) \right\|_{H^0} \le h^{\frac{5}{2}} \text{ and} \\ &\left\| \dot{\gamma}^h(t) \right\|_{H^0} \le h^{\frac{3}{2}} . \end{aligned}$$

From the definition of T^* in (3.4) we conclude that

$$T = T^* .$$

and then the estimates are valid for the whole time interval. We then proved the following convergence result.

Theorem 4.3. *Assuming that the flow satisfies*

$$\begin{aligned} &\left| z(\alpha_0, t) - z(\alpha, t) \right| \ge c_1 \left| \alpha_0 - \alpha \right| \ \text{for}\ \alpha_0 \ne \alpha, \\ &z ,\ \phi \in C^{m+2}\left[-\pi, \pi \right] , \\ &\gamma \in C^{m+1}\left[-\pi, \pi \right] ,\ \text{and} \\ &z_\alpha(\alpha, t) \ne 0 , \end{aligned}$$

with $m \ge 3$ *and* $c_1 > 0$ *independent of* t*. Also assume* $(u_t, v_t + g) \cdot \vec{n} \ge c_0 > 0$.

Then, for $0 \le t \le T$*, and* $h < h_0(T)$ *the numerical solution for the water wave equations satisfy:*

$$\begin{aligned} &\left\| z^h(\cdot, t) - z(\cdot, t) \right\|_{H^0} + \left\| \phi^h(\cdot, t) - \phi(\cdot, t) \right\|_{H^0} \le C(T) h^r . \\ &\left\| \gamma^h(\cdot, t) - \gamma(\cdot, t) \right\|_{H^0} \le C(T) h^{r-1} , \end{aligned}$$

where r *is the order of the derivative operator.*

5. Stability of a Fast Numerical Integration Method.

It is easy to see that the most costly step in our numerical method is to solve the integral equation (1.7-iii). Using iterative methods, it is possible to achieve convergence in few steps. Accurate initial guesses, given by extrapolating solutions of previous time steps reduce even further the necessary number of steps for the iterative solver to convergence.

The problem is that each step involves a numerical integration, an $O(N^2)$ process, where N is the number of nodal points. In this section, we present a fast integral method that, while preserving the spectral accuracy of the trapezoidal rule, computes the numerical integral in a much faster way.

The issue here is the computation of the operator $\mathcal{K}_h$. To make the notation more clear, in this section only denote the $N \times N$ matrix by

$$S_{j,k}^{N}(z) := S_{j,k}(z) \ ,$$

where $S_{j,k}(z)$ was defined just above (1.8). Let

$$\left(S^N(z)\,\rho^h\right)_j := h \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} S_{j,k}^N(z)\,\rho_k^h \ , \text{ for } j = -\frac{N}{2}, \dots, \frac{N}{2} - 1 \,. \tag{5.1}$$

Then, we can rewrite definition (1.8) as

$$\mathcal{K}_h\left(z^h, \rho^h\right) = \frac{\mathcal{H}_h\left(\rho^h\right)}{2} + S^N\left(z^h\right) D_h \rho^h \,.$$

The action of $\mathcal{H}_h$ in above sum is an $O(N \log N)$ operation, if one uses the Fast Fourier Transform (FFT). The action of $S^N\left(z^h\right)$ (defined in (5.1)) is more expensive ($O(N^2)$), and in this section, we will restrict our attention on how to compute (5.1) in a more efficient way.

The main idea is to compute the matrix multiplication in a "smaller" space. Let's, for the sake of simplicity, consider the case where the small space is V_{2h}.

First we project an element of V_h into V_{2h}, then we compute the matrix multiplication using $N/2$ nodal points, and finally, we project the result back to V_h.

To make the restriction from V_h to V_{2h}, we compute an element's N discrete Fourier Coefficients, consider only the $N/2$ first coefficients and compute the inverse Fourier transform, finding finally a vector of size $\frac{N}{2} \times \frac{N}{2}$. This whole operation can be carried on by:

$$\Pi_h^{2h} := F_{\frac{N}{2}}^{-1} \left[I_{\frac{N}{2}} \ 0\right] F_N \,,$$

where F_N denotes the FFT of size N, and F_N^{-1} denotes its inverse. The operator $I_{\frac{N}{2}}$ is just the identity matrix of size $N/2$. This operator can be seen as a "discrete L^2projection", as $\left(\Pi_h^{2h}\rho^h, \chi\right) = \left(\rho^h, \chi\right)$, $\forall \chi \in V_h$.

The inclusion operator is simply:

$$\widetilde{\Pi}_{2h}^h := F_N^{-1} \left[I_{\frac{N}{2}} \ 0\right]^T F_{\frac{N}{2}} \,.$$

Remark 5.1. These same ideas appear in Lothar Reichel paper [Rei], with a slightly different point of view. He estimates the accuracy of the fast summation by looking at the magnitude of the two dimensional discrete Fourier coefficients of the original matrix $S^N(z)$.

A stability analysis can be carried as follows. First define

$$z^{2h} := \Pi_h^{2h} z^h \quad \text{and} \quad \rho^{2h} := \Pi_h^{2h} \rho^h .$$

We will use analogous notation for $\dot{z}^{2h}$ and $\dot{\rho}^{2h}$.

Similarly to Lemma (3.1), we write

$$\begin{aligned} S^{\frac{N}{2}}\left(z^{2h}\right)\rho^{2h} = {} & S^{\frac{N}{2}}\left(\Pi_h^{2h} z\right)\Pi_h^{2h}\rho + \dot{S}^{\frac{N}{2}}\left(\dot{z}^{2h}\right)\Pi_h^{2h}\rho + S^{\frac{N}{2}}\left(\Pi_h^{2h} z\right)\dot{\rho}^{2h} \\ & + NL\left(\dot{z}^h, \dot{\rho}^h\right) , \end{aligned}$$

where $\dot{S}^{\frac{N}{2}}\left(\dot{z}^{2h}\right)$ denotes the linear part of $S^{\frac{N}{2}}\left(\Pi_h^{2h}\left(\dot{z}^h + z\right)\right) - S^{\frac{N}{2}}\left(\Pi_h^{2h} z\right)$. We will consider the nonlinear term $NL\left(\dot{z}^h, \dot{\rho}^h\right)$ to be a small "contribution", as was done before in (3.5). Our goal is to prove two properties:

i) That the term $\widetilde{\Pi}_{2h}^h \dot{S}^{\frac{N}{2}}\left(\dot{z}^{2h}\right)\Pi_h^{2h}(\rho)$ is a bounded operator for $\dot{z}^h$. But working as in the proof of Lemma (3.1), we have:

$$\left\|\widetilde{\Pi}_{2h}^h \dot{S}^{\frac{N}{2}}\left(\dot{z}^{2h}\right)\Pi_h^{2h}\rho\right\|_{H^0} \leq \left\|\dot{S}^{\frac{N}{2}}\left(\dot{z}^{2h}\right)\Pi_h^{2h}\rho\right\|_{H^0} \leq C\left\|\dot{z}^{2h}\right\|_{H^0} \leq C\left\|\dot{z}^h\right\|_{H^0} .$$

ii) That $\widetilde{\Pi}_{2h}^h S^{\frac{N}{2}}\left(\Pi_h^{2h} z\right)\Pi_h^{2h}\left(\dot{\rho}^h\right)$ smooths out the high frequency components of $\dot{\rho}^h$:

$$\left\|\widetilde{\Pi}_{2h}^h S^{\frac{N}{2}}\left(\Pi_h^{2h} z\right)\dot{\rho}^{2h}\right\|_{H^{m-1}} \leq \left\|S^{\frac{N}{2}}\left(\Pi_h^{2h} z\right)\dot{\rho}^{2h}\right\|_{H^{m-1}} \leq C\left\|\dot{\rho}^{2h}\right\|_{H^0} \leq C\left\|\dot{\rho}^h\right\|_{H^0} ,$$

where Lemma (2.3) was used.

The above analysis show us that the use of this fast integral will not upset the stability analysis of section 3. The cost of the fast computation would be $O(N^2/4 + Nlog(N))$. Of course, in general we can easily divide the number of nodal points by any power of two. To appropriately choose the size of the coarse mesh, one would have to compromise between accuracy of the method (what becomes more critical for not so smooth wave profiles) and the velocity of the matrix multiplication. The question of how accurate this fast method is will be addressed in future work.

6. Further Comments.

The main difference between the present work and [BHL-Num] is in the stability analysis. Their alternating trapezoidal quadrature introduced aliasing errors and made the stability result more difficult to obtain. Also they were forced to use smoothing techniques to control these aliasing errors. In our analysis, it was possible to remove the smoothing. To our knowledge, we have the first numerical scheme for the water waves problem that does not require smoothing.

Another advantage of the present analysis is that the singularity present in the integral is dealt with explicitly. Each integration is performed in two steps. First the singular part is computed with $O(N\log(N))$ cost, via FFT. The smooth part is then then computed with $O(N^2)$ cost, but fast integration techniques can be applied, as long as they do not introduce spurious high frequency modes. If a stability analysis is performed and proves that a new fast quadrature is still a smoothing operator, then it can be incorporated to the numerical scheme with no problems.

7. Results' proofs.

Lemma 2.1. *Assume θ is an even $C^2(-\pi,\pi)$ function, with $\theta(\pi)=0$. If f is a 2π-periodic function and $f \in \mathbb{H}^3(\mathbb{R})$ then*

$$D_h\left(f(\alpha_j)z_j^h\right) = f(\alpha_j)D_h z_j^h + A_h(f,z^h)(\alpha_j)\,.$$

For an independent constant C, we have $\left\|A_h(f,z^h)\right\|_{\mathbb{H}^0} \le C\left\|\theta\right\|_{C^2}\left\|f\right\|_{\mathbb{H}^{\frac{5}{2}}}\left\|z^h\right\|_{\mathbb{H}^0}$.

Proof. Define $e_j^h = D_h\left(f(\alpha_j)z_j^h\right) - f(\alpha_j)D_h z_j^h - f_\alpha(\alpha_j)Q\left(z_j^h\right)z_j^h$, where

$$Q\left(z_j^h\right) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1}\widetilde{\left(z^h\right)}_k\frac{d}{dx}g(kh)\,e^{ikz_j}\,.$$

Then, using techniques similar to [BHL-Num], one can show that for $g(x) = x\theta(x)$, the following estimate holds:

$$\begin{aligned}\left|\tilde{e}_k^h\right| &\le \frac{3h}{2}\left\|g''\right\|_\infty\left(\sum_{k_1=-\frac{N}{2}}^{\frac{N}{2}-1}\left|\tilde{f}_{k_1}\right|^2 k_1^4\right)^{\frac{1}{2}}\left(\sum_{k_2=-\frac{N}{2}}^{\frac{N}{2}-1}\left|\widetilde{\left(z^h\right)}_k\right|^2\right)^{\frac{1}{2}}\\ &\le C\sqrt{h}\left\|\theta\right\|_{C^2}\left\|I_h f\right\|_{\mathbb{H}^{\frac{3}{2}}}\left\|z^h\right\|_{\mathbb{H}^0} \le C\sqrt{h}\left\|\theta\right\|_{C^2}\left\|f\right\|_{\mathbb{H}^{\frac{5}{2}}}\left\|z^h\right\|_{\mathbb{H}^0}\end{aligned}$$

Using that $\left\|Q\left(z_j^h\right)\right\|_{\mathbb{H}^0} \le \left\|g'\right\|_\infty\left\|z^h\right\|_{\mathbb{H}^0} \le \left\|\theta\right\|_{C^1}\left\|z^h\right\|_{\mathbb{H}^0}$, the result follows. □

Lemma 2.3. *Assume $g(\alpha,\beta) \in \mathbb{H}^{m+1}(-\pi,\pi)^2$ is 2π-periodic in both variables, and let D_h be as in (1.5). If $\left\|\theta\right\|_\infty \le 1$ and if*

$$\chi_{j_1}^h := h\sum_{j_2=-\frac{N}{2}}^{\frac{N}{2}-1} g(\alpha_{j_1},\alpha_{j_2})D_h\rho_{j_2}^h\,,$$

then $\left\|\chi^h\right\|_{\mathbb{H}^0} \le 2\pi\left\|g\right\|_{\mathbb{H}^2(-\pi,\pi)^2}\left\|\rho^h\right\|_{\mathbb{H}^{-1}}$.

Proof. It is easy to see that

$$\widetilde{\left(\chi^h\right)}_k = 2\pi\sum_{l=-\frac{N}{2}}^{\frac{N}{2}-1}\tilde{g}_{l,k}\widetilde{\left(\rho^h\right)}_{-l}\left(-il\theta\left(-lh\right)\right)$$

Then, using the Cauchy-Schwarz inequality, the following estimate holds:

$$\begin{aligned}\left\|\chi^h\right\|_{\mathbb{H}^0}^2 &= \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1}\left|\widetilde{\left(\chi^h\right)}_k\right|^2\\ &\le (2\pi)^2\left\|\rho^h\right\|_{\mathbb{H}^{-1}}^2\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1}\sum_{l=-\frac{N}{2}}^{\frac{N}{2}-1}(1+|l|)^4\left|\tilde{g}_{l,k}\right|^2\\ &\le 2(2\pi)^2\left\|\rho^h\right\|_{\mathbb{H}^{-1}}^2\left\|g\right\|_{\mathbb{H}^2(-\pi,\pi)^2}^2\end{aligned}$$

and then the Lemma follows. □

Lemma 2.4. *If N is an even positive integer, $h := 2\pi/N$ and $\alpha_j := jh$, then the following identity holds:*

$$h \sum_{\substack{k=-\frac{N}{2} \\ k\neq j}}^{\frac{N}{2}-1} \frac{d}{d\alpha}\bigg|_{\alpha=\alpha_k} \log\left|2\sin\frac{\alpha_j-\alpha}{2}\right| e^{il\alpha_k} = ie^{il\alpha_j}\Sigma(l)\ , \tag{2.1}$$

where

$$\Sigma(l) = \begin{cases} -\pi - lh & \textit{for} \quad l = -\frac{N}{2}, \ldots, -1\,, \\ 0 & \textit{for} \quad l = 0\,, \\ \pi - lh & \textit{for} \quad l = 1, \ldots, \frac{N}{2} - 1\,. \end{cases} \tag{2.2}$$

Proof. The summation (2.1) can be simplified as follows:

$$\begin{aligned} -h \sum_{\substack{k=-\frac{N}{2} \\ k\neq j}}^{\frac{N}{2}-1} \frac{1}{2}\cot\frac{\alpha_j-\alpha_k}{2} e^{il\alpha_k} &= -e^{il\alpha_j}\frac{h}{2} \sum_{\substack{m=-\frac{N}{2} \\ m\neq 0}}^{\frac{N}{2}-1} \cot\frac{\alpha_m}{2} e^{-il\alpha_m} \\ = ie^{il\alpha_j} h \sum_{m=1}^{\frac{N}{2}-1} \cot\frac{\alpha_m}{2}\sin l\alpha_m &= ie^{il\alpha_j}\Sigma(l) \end{aligned}$$

where

$$\Sigma(l) = h \sum_{m=1}^{\frac{N}{2}-1} (1+\cos\alpha_m)\frac{\sin l\alpha_m}{\sin\alpha_m}.$$

It is easy to see that $\Sigma(1) = \pi - h$ and $\Sigma(2) = \pi - 2h$. Now suppose that $l > 1$. Then

$$\begin{aligned} \Sigma(l+1) &= h \sum_{m=1}^{\frac{N}{2}-1} (1+\cos\alpha_m)\frac{\sin(l\alpha_m+\alpha_m)}{\sin\alpha_m} \\ &= h \sum_{m=1}^{\frac{N}{2}-1} (1+\cos\alpha_m)\frac{\sin l\alpha_m\cos\alpha_m + \sin\alpha_m\cos l\alpha_m}{\sin\alpha_m} \\ &= h \sum_{m=1}^{\frac{N}{2}-1} (1+\cos\alpha_m)\frac{\sin l\alpha_m}{\sin\alpha_m}\cos\alpha_m + h \sum_{m=1}^{\frac{N}{2}-1} (1+\cos\alpha_m)\cos l\alpha_m\,. \end{aligned}$$

This expression can be computed with the help of the following identities:

$$h \sum_{m=1}^{\frac{N}{2}-1} (1+\cos\alpha_m)\cos l\alpha_m = -h\,,$$

and

$$h \sum_{m=1}^{\frac{N}{2}-1} (1+\cos\alpha_m)\frac{\sin l\alpha_m}{\sin\alpha_m}\cos\alpha_m = \sum_{m=1}^{\frac{N}{2}-1} (1+\cos\alpha_m)\frac{\sin l\alpha_m}{\sin\alpha_m} = \Sigma(l)\,.$$

This last identity follows from the fact that, for $l>1$, we have:

$$h\sum_{m=1}^{\frac{N}{2}-1}\sin l\alpha_m\sin\alpha_m=0\Rightarrow h\sum_{m=1}^{\frac{N}{2}-1}\left(1-\cos^2\alpha_m\right)\frac{\sin l\alpha_m}{\sin\alpha_m}=0$$
$$\Rightarrow h\sum_{m=1}^{\frac{N}{2}-1}\left(1+\cos\alpha_m\right)\left(1-\cos\alpha_m\right)\frac{\sin l\alpha_m}{\sin\alpha_m}=0\,.$$

Putting all these facts together we get $\Sigma(l+1)=\Sigma(l)-h\,,\quad l=1,\dots,N/2-1$, or equivalently

$$\Sigma(l)=\pi-lh\,.$$

The $l=0$ case is trivial and for $-1\geq l\geq -N/2$ it is enough to realize that $\Sigma(l)$ is an odd function. Then the result follows □

Lemma 2.7. *Let $f\in\mathbb{H}^{\frac{5}{2}}(-\pi,\pi)$ be a 2π-periodic function and let θ be an even $C^2[-\pi,\pi]$ function with $\theta(-\pi)=\theta(\pi)=0$. Let the derivative operator be defined by (1.5). Then, if $P_h:L^2(-\pi,\pi)\to V_h$ denotes the L^2 projection, we have:*

$$\left\|I_h\left(fD_h\rho^h\right)-P_h\left(fD_h\rho^h\right)\right\|_{\mathbb{H}^0}\leq\|\theta\|_{C^1}\|f\|_{\mathbb{H}^2}\left\|\rho^h\right\|_{\mathbb{H}^0}\text{ , for }\rho^h\in V_h.$$

Proof. Define $e^h=I_h\left(fD_h\rho^h\right)-P_h\left(fD_h\rho^h\right)\ \in\ V_h$. Then

$$\widetilde{\left(e^h\right)}_k=\sum_{\substack{k_1,k_2=-\frac{N}{2}\\k_1+k_2=k\pm N}}^{\frac{N}{2}-1}ik_2\theta\left(k_2h\right)\tilde f_{k_1}\tilde\rho^h_{k_2}$$

First assume $k<0$. In fact, as $\widetilde{\left(e^h\right)}_{-1}=0$, we can assume $-1<k\leq-\frac{N}{2}$. Then

$$\widetilde{\left(e^h\right)}_k=\sum_{\substack{k_1,k_2=1\\k_1+k_2=k+N}}^{\frac{N}{2}-1}ik_2\theta\left(k_2h\right)\tilde f_{k_1}\tilde\rho^h_{k_2}=\sum_{k_2=k+\frac{N}{2}+1}^{\frac{N}{2}-1}ik_2\theta\left(k_2h\right)\tilde f_{k+N-k_2}\tilde\rho^h_{k_2}$$
$$=\sum_{l=k+1}^{-1}\tilde f_{k+\frac{N}{2}-l}\,i\left(\frac{N}{2}+l\right)\theta\left(\pi+lh\right)\tilde\rho^h_{l+\frac{N}{2}}$$

Expanding θ in its Taylor series, we can estimate:

$$\left|\widetilde{\left(e^h\right)}_k\right|\leq\frac{\|\theta\|_{C^1}h}{\pi}\sum_{l=k+1}^{-1}\left|\tilde f_{k+\frac{N}{2}-l}\right|\left|\frac{N}{2}+l\right||l|\left|\tilde\rho^h_{l+\frac{N}{2}}\right|$$

Using the fact that $\left|k+\frac{N}{2}-l\right|=k+\frac{N}{2}-l\geq\min\left\{-l,k+\frac{N}{2}\right\}$ for $l=k+1,\dots,-1$ and then using Cauchy-Schwarz inequality, we have:

$$\left|\widetilde{\left(e^h\right)}_k\right|\leq\frac{\|\theta\|_{C^1}h}{\pi}\frac{N}{2\left|k+\frac{N}{2}\right|}\sum_{l=k+1}^{-1}\left|\tilde f_{k+\frac{N}{2}-l}\right|\left|k+\frac{N}{2}+l\right|^2\left|\tilde\rho^h_{l+\frac{N}{2}}\right|$$
$$\leq\frac{\|\theta\|_{C^1}}{\left|k+\frac{N}{2}\right|}\|f\|_{\mathbb{H}^2}\left\|\rho^h\right\|_{\mathbb{H}^0}\ .$$

The same kind of estimate holds for $k>0$, *mutatis mutandis.* As $\widetilde{\left(e^h\right)}_0=0$, we have:

$$\left\|e^h\right\|_{\mathbb{H}^0}\leq4\|\theta\|_{C^1}\|f\|_{\mathbb{H}^2}\left\|\rho^h\right\|_{\mathbb{H}^0}$$

□

Lemma 2.9. *Let $f \in \mathbb{H}^{\frac{5}{2}}$ be a 2π-periodic function and let the discrete derivative operator be as in (1.5). Then, for any $\rho \in V_h$, the following holds:*

$$\mathcal{H}_h\left(fD_h\rho^h\right) - I_h\left(f\mathcal{H}_h\left(D_h\rho^h\right)\right) = B_{com}\left(f,\rho^h\right)\text{ , where}$$

$$\left\|B_{com}\left(f,\rho^h\right)\right\|_{\mathbb{H}^0} \leq \left(2C_2\left(\theta\right)\|f\|_{\mathbb{H}^{\frac{5}{2}}} + C_3\right)\left\|\rho^h\right\|_{\mathbb{H}^0}\,.$$

Proof. The discrete commutator error is given by:

$$\begin{aligned}&\mathcal{H}_h\left(fD_h\rho^h\right) - I_h\left(f\mathcal{H}\left(D_h\rho^h\right)\right) = \left[\mathcal{H}I_h\left(fD_h\rho^h\right) - \mathcal{H}P_h\left(fD_h\rho^h\right)\right]\\ &+\left[\mathcal{H}P_h\left(fD_h\rho^h\right) - P_h\left(f\mathcal{H}\left(D_h\rho^h\right)\right)\right] + \left[P_h\left(f\mathcal{H}\left(D_h\rho^h\right)\right) - I_h\left(f\mathcal{H}\left(D_h\rho^h\right)\right)\right]\\ &= I + II + III\,.\end{aligned}$$

The above expression will be estimated as follows:

i) The estimate bellow uses that the Hilbert transform has norm one and also uses Lemma (2.7):

$$\begin{aligned}\left\|\mathcal{H}I_h\left(fD_h\rho^h\right) - \mathcal{H}P_h\left(fD_h\rho^h\right)\right\|_{\mathbb{H}^0} &\leq \left\|I_h\left(fD_h\rho^h\right) - P_h\left(fD_h\rho^h\right)\right\|_{\mathbb{H}^0}\\ &\leq C\|\theta\|_{C^1}\|f\|_{\mathbb{H}^{\frac{5}{2}}}\left\|\rho^h\right\|_{\mathbb{H}^0}\end{aligned}$$

ii) Using that $\mathcal{H}P_h = P_h\mathcal{H}$, that the norm of P_h is one and the commutator Lemma for the continuous Hilbert transform, we have

$$\begin{aligned}\left\|\mathcal{H}P_h\left(fD_h\rho^h\right) - P_h\left(f\mathcal{H}\left(D_h\rho^h\right)\right)\right\|_{\mathbb{H}^0} &\leq \left\|\mathcal{H}\left(fD_h\rho^h\right) - \left(f\mathcal{H}\left(D_h\rho^h\right)\right)\right\|_{\mathbb{H}^0}\\ &\leq C\left\|\rho^h\right\|_{\mathbb{H}^0}\end{aligned}$$

iii) First we use the fact that D_h and $\mathcal{H}$ commutes when applied to a function in V_h, combined with Lemma (2.7). Finally, we use once again the bound on the Hilbert transform norm.

$$\begin{aligned}\left\|P_h\left(f\mathcal{H}\left(D_h\rho^h\right)\right) - I_h\left(f\mathcal{H}\left(D_h\rho^h\right)\right)\right\|_{\mathbb{H}^0} &\leq C\|\theta\|_{C^1}\|f\|_{\mathbb{H}^{\frac{5}{2}}}\left\|\mathcal{H}\rho^h\right\|_{\mathbb{H}^0}\\ &\leq C\|\theta\|_{C^1}\|f\|_{\mathbb{H}^{\frac{5}{2}}}\left\|\rho^h\right\|_{\mathbb{H}^0}\end{aligned}$$

A triangle inequality combining the above estimates yields the desired final result. □

Lemma 3.1. *The operator $\mathcal{K}_h$ applied to an element ρ^h of V_h can be written as*

$$\mathcal{K}_h\left(z^h,\rho^h\right) = \mathcal{K}_h\left(z,\rho\right) + \frac{\mathcal{H}_h\left(\dot{\rho}^h\right)}{2} + B_1\left(z,\rho,\dot{z}^h\right) + B_2\left(z,\rho,\dot{\rho}^h\right) + NL(\dot{z}^h,\dot{\rho}^h)\,.$$

where $\dot{\rho}^h_j := \rho^h_j - \rho\left(\alpha_j\right)$ and $NL(\dot{z}^h,\dot{\rho}^h)$ contains the nonlinear terms in $\dot{z}^h$ and $\dot{\rho}^h$. The operators B_1, B_2 can be bounded as follows:

$$\left\|B_1\left(z,\rho,\dot{z}^h\right)\right\|_{\mathbb{H}^0} \leq C_4\left(z,\rho\right)\left\|\dot{z}^h\right\|_{\mathbb{H}^0}$$

$$\left\|B_2\left(z,\rho,\dot{\rho}^h\right)\right\|_{\mathbb{H}^0} \leq \left\|S\left(z,\cdot,\cdot\right)\right\|_{\mathbb{H}^1(-\pi,\pi)^2}\left\|\dot{\rho}^h\right\|_{\mathbb{H}^{-1}}$$

and $C_4(z,\rho) := \bar{C}_1(z,\rho) + \bar{C}_2(z,\rho) + \bar{C}_3(z) + O(h^r)$*. The* $\bar{C}_j$*'s are defined in the lemma's proof.*

Proof. By definition of $\mathcal{K}_h$ and a straighforward computation, we have

(7.1)
$$\mathcal{K}_h\left(z_j^h, \rho_j^h\right) = \mathcal{K}_h\left(z\left(\alpha_j\right), \rho\left(\alpha_j\right)\right) + \frac{\mathcal{H}_h\left(\dot{\rho}_j^h\right)}{2}$$
$$+ h \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \dot{S}_{j,k}\left(z_j^h\right) D_h\rho\left(\alpha_k\right) + h \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} S_{j,k}(z)\, D_h\dot{\rho}_k^h + NL\left(\dot{z}_j^h, \dot{\rho}_j^h\right),$$

where

$$\dot{S}_{j,k} = \frac{-i\dot{z}_j^h e^{-iz(\alpha_j)} + i\dot{z}_k^h e^{-iz(\alpha_k)}}{e^{-iz(\alpha_j)} - e^{-iz(\alpha_k)}}, \text{ for } j \neq k \text{, and}$$
$$\dot{S}_{j,j} = \frac{D_h\dot{z}_j^h - iD_h z\left(\alpha_j\right)\dot{z}_j^h}{D_h z\left(\alpha_j\right)}.$$

Assume that $z \in \mathbb{H}^p(-\pi,\pi)$ with p large enough such that $S(z,\cdot,\cdot) \in \mathbb{H}^2(-\pi,\pi)^2$, where $S(z,\alpha,\beta)$ was defined at (1.4). Then, by Lemma (2.3), the following estimate holds:

$$\left\| h \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} S_{j,k}(z)\, D_h\dot{\rho}_k^h \right\|_{\mathbb{H}^0} \leq 2\pi \left\|S(z,\cdot,\cdot)\right\|_{\mathbb{H}^1(-\pi,\pi)^2} \left\|\dot{\rho}^h\right\|_{\mathbb{H}^{-1}}. \tag{7.2}$$

So, it remains to prove that the summation on $\dot{S}$ at (7.1) is bounded by the norm of $\dot{z}_j^h$.

We can write

$$g(z,\alpha,\beta) = \frac{-i}{e^{-iz(\alpha)} - e^{-iz(\beta)}} = \frac{1}{z_\alpha(\alpha)\, e^{-iz(\alpha)}} \left(\frac{d}{d\alpha} \log\left(e^{-iz(\alpha_j)} - e^{-iz(\alpha)}\right)\right)$$
$$= g_1(z,\alpha,\beta) + g_2(z,\alpha,\beta),$$

where

$$g_1(z,\alpha,\beta) = \frac{1}{z_\alpha(\alpha)\, e^{-iz(\alpha)}} \frac{d}{d\alpha} \log \frac{e^{-iz(\alpha)} - e^{-iz(\beta)}}{\left|2\sin\frac{\alpha-\beta}{2}\right|} \quad \text{for } \alpha \neq \beta,$$

and $g_1(z,\alpha,\alpha) = \lim_{\beta\to\alpha} g_1(z,\alpha,\beta)$. Also, for $\alpha \neq \beta$,

$$g_2(z,\alpha,\beta) = \frac{1}{z_\alpha(\alpha)\, e^{-iz(\alpha)}} \frac{d}{d\alpha} \log\left|2\sin\frac{\alpha-\beta}{2}\right|.$$

Using this notation, we have

$$
\begin{aligned}
h\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} & \dot{S}_{j,k}\left(z_j^h\right) D_h\rho\left(\alpha_k\right) \\
&= h\sum_{\substack{k=-\frac{N}{2}\\ k\neq j}}^{\frac{N}{2}-1} g_1\left(z,\alpha_j,\alpha_k\right) D_h\rho\left(\alpha_k\right)\left(\dot{z}_j^h e^{-iz(\alpha_j)} - \dot{z}_k e^{-iz(\alpha_k)}\right) \\
&+ h\sum_{\substack{k=-\frac{N}{2}\\ k\neq j}}^{\frac{N}{2}-1} g_2\left(z,\alpha_j,\alpha_k\right) D_h\rho\left(\alpha_k\right)\left(\dot{z}_j^h e^{-iz(\alpha_j)} - \dot{z}_k e^{-iz(\alpha_k)}\right) \\
&+ h\frac{D_h\dot{z}_j^h - iD_h z\left(\alpha_j\right)\dot{z}_j^h}{D_h z\left(\alpha_j\right)} = I + II + III\,.
\end{aligned}
$$

i) Estimating I:
Using Lemma (2.3) and some sort of consistency argument, we have that

$$
\|I\|_{\mathbb{H}^0} \leq \bar{C}_1\left(z,\rho\right)\left\|\dot{z}^h\right\|_{\mathbb{H}^0} + O\left(h^r\right)
$$

where we define

$$
\bar{C}_1\left(z,\rho\right) = 2\pi\left\|g_1\left(z,\cdot,\cdot\right)\right\|_{C^0}\left\|\rho_\alpha\right\|_\infty + \left\|\rho_\alpha e^{-iz}\right\|_{C^0} \tag{7.3}
$$

ii) Estimating II:
Expanding the following functions in its discrete Fourier series:

$$
\frac{D_h\rho\left(\alpha_k\right)}{z_\alpha\left(\alpha_k\right)e^{-iz(\alpha_k)}} = \sum_{l=-\frac{N}{2}}^{\frac{N}{2}-1}\left(\widetilde{\frac{D_h\rho}{z_\alpha e^{-iz}}}\right)_l e^{il\alpha_k} \quad \text{and}
$$

$$
\frac{\dot{z}_k D_h\rho\left(\alpha_k\right)}{z_\alpha\left(\alpha_k\right)} = \sum_{l=-\frac{N}{2}}^{\frac{N}{2}-1}\left(\widetilde{\frac{\dot{z}_j^h D_h\rho}{z_\alpha}}\right)_l e^{il\alpha_k}\,,
$$

then

$$
\begin{aligned}
II = &-\frac{1}{2}h\dot{z}_j^h e^{-iz(\alpha_j)}\sum_{\substack{k=-\frac{N}{2}\\ k\neq j}}^{\frac{N}{2}-1}\sum_{l=-\frac{N}{2}}^{\frac{N}{2}-1}\left.\frac{d}{d\alpha}\right|_{\alpha=\alpha_k}\log\left|2\sin\frac{\alpha_j-\alpha}{2}\right|\left(\widetilde{\frac{D_h\rho}{z_\alpha e^{-iz}}}\right)_l e^{il\alpha_k} \\
&+\frac{1}{2}h\sum_{\substack{k=-\frac{N}{2}\\ k\neq j}}^{\frac{N}{2}-1}\sum_{l=-\frac{N}{2}}^{\frac{N}{2}-1}\left.\frac{d}{d\alpha}\right|_{\alpha=\alpha_k}\log\left|2\sin\frac{\alpha_j-\alpha}{2}\right|\left(\widetilde{\frac{\dot{z}_j^h D_h\rho}{z_\alpha}}\right)_l e^{il\alpha_k} \\
= &-\frac{i}{2}\dot{z}_j^h e^{-iz(\alpha_j)}\sum_{l=-\frac{N}{2}}^{\frac{N}{2}-1}\left(\widetilde{\frac{D_h\rho}{z_\alpha e^{-iz}}}\right)_l\Sigma(l)e^{il\alpha_j} + \frac{i}{2}\sum_{l=-\frac{N}{2}}^{\frac{N}{2}-1}\left(\widetilde{\frac{\dot{z}_j^h D_h\rho}{z_\alpha}}\right)_l\Sigma(l)e^{il\alpha_j}\,,
\end{aligned}
$$

where Lemma (2.4) was used and Σ was defined in (2.2). As $|g(l)| \leq \pi$ for $l = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$, note that:

$$\left\| \sum_{l=-\frac{N}{2}}^{\frac{N}{2}-1} \widetilde{\left(\frac{D_h \rho}{z_\alpha e^{-iz}} \right)}_l \Sigma(l) e^{il\alpha_j} \right\|_{\mathbb{H}^0} \leq \pi \left\| I_h \left(\frac{D_h \rho}{z_\alpha e^{-iz}} \right) \right\|_{\mathbb{H}^0} \leq 2\pi \left\| \frac{D_h \rho}{z_\alpha e^{-iz}} \right\|_{\mathbb{H}^0} .$$

Another estimate is given by

$$\left\| \sum_{l=-\frac{N}{2}}^{\frac{N}{2}-1} \widetilde{\left(\frac{\dot{z}_j^h D_h \rho}{z_\alpha} \right)}_l \Sigma(l) e^{il\alpha_j} \right\|_{\mathbb{H}^0} \leq 2\pi \left\| \frac{D_h \rho}{z_\alpha} \right\|_\infty \left\| \dot{z}^h \right\|_{\mathbb{H}^0}$$

Combining this two previous estimates, we can conclude that:

$$\|II\|_{\mathbb{H}^0} \leq \bar{C}_2 (z, \rho) \left\| \dot{z}^h \right\|_{\mathbb{H}^0} , \tag{7.4}$$

where $\bar{C}_2 (z, \rho) = \pi C_0 \left\| \frac{D_h \rho}{z_\alpha e^{-iz}} \right\|_{\mathbb{H}^0} + \pi C_0 \left\| \frac{D_h \rho}{z_\alpha} \right\|_{\mathbb{H}^0}$.

iii) For the last term, we have:

$$\|III\|_{\mathbb{H}_0} \leq h \left\| D_h \dot{z}^h \right\|_{\mathbb{H}^0} \left\| \frac{1}{D_h z} \right\|_\infty + h \left\| \dot{z}^h \right\|_{\mathbb{H}^0} \leq \bar{C}_3 (z) \left\| \dot{z}^h \right\|_{\mathbb{H}^0} \tag{7.5}$$

where $\bar{C}_3 (z) = 2\pi \left\| \frac{1}{D_h z} \right\|_\infty + 1$.

Finally, using estimate (7.2) and i), ii) and iii) we have the desired result.

To complete the proof, we just need to observe that $h\dot{f}_{j,j} D_h \rho (\alpha_k) = A_0 \left(\dot{z}^h \right)$, and then the result follows. □

Lemma 3.3. *The double layer density error can be written as*

$$\dot{\gamma}_j^h = 2\ D_h \dot{\phi}_j^h - 2\Im \left(i D_h \dot{z}_j^h e^{-iz(\alpha_j)} w_0 (\alpha_j) \right) + \Im\, \mathcal{H}_h \left(\frac{\gamma (\alpha_j)}{D_h z (\alpha_j)} D_h \dot{z}_j^h \right)$$
$$+ A_0 \left(\dot{\phi}^h \right) + A_0 \left(\dot{z}^h \right) + NL(\dot{z}^h, \dot{\gamma}^h) + O\left(h^r\right) ,$$

where

$$w_0 (\alpha) := \mathcal{K}_h \left(z, \frac{\gamma}{i D_h z e^{-iz}} \right) (\alpha) . \tag{.2}$$

Proof. From the discrete integral equation (1.7-iii), we compute

(7.6)

$$D_h \dot{\phi}_j^h = \Im \left(i D_h \dot{z}_j^h e^{-iz(\alpha_j)} w_0 (\alpha_j) \right)$$
$$+ \Im \left(i D_h z (\alpha_j) e^{-iz(\alpha_j)} \mathcal{K}_h \left(z (\alpha_j) , \frac{\dot{\gamma}_j^h e^{iz(\alpha_j)}}{i D_h z (\alpha_j)} - \frac{\gamma (\alpha_j) D_h \dot{z}_j^h e^{iz(\alpha_j)}}{i \left(D_h z (\alpha_j) \right)^2} \right) \right)$$
$$+ \frac{\dot{\gamma}_j^h}{2} + B_1 \left(z, \frac{\gamma}{i D_h z e^{-iz}}, \dot{z}_j^h \right) + NL(\dot{z}_j^h, \dot{\gamma}_j^h) ,$$

where w_0 was defined in (.2).

Define

$$\widetilde{\mathcal{K}_h}\left(\rho^h\right) = 2\,\Im\, I_h\left(D_h z e^{-iz}\mathcal{K}_h\left(z, \frac{\rho^h e^{iz}}{D_h z}\right)\right) \tag{7.7}$$

and

$$G_j^h = 2\dot{\phi}_j^h + \Im\left(-2i\dot{z}_j^h e^{-iz(\alpha_j)} w_0\left(\alpha_j\right) + \mathcal{H}_h\left(\frac{\gamma\left(\alpha_j\right)\dot{z}_j^h}{D_h z\left(\alpha_j\right)}\right)\right)$$

Then equation (7.6), with Lemmas (2.1), (2.9) and (3.1) give:

$$\left(I_h + \widetilde{\mathcal{K}_h}\right)\dot{\gamma}^h = D_h G^h + A_0\left(\dot{z}^h\right) + NL(\dot{z}^h, \dot{\gamma}^h) + O\left(h^r\right) \tag{7.8}$$

Assuming that the operator on the left hand side of (7.8) has bounded inverse, we get

$$\widetilde{\mathcal{K}_h}\left(\dot{\gamma}^h\right) = \left(I_h + \widetilde{\mathcal{K}_h}\right)^{-1}\widetilde{\mathcal{K}_h} D_h G^h + A_0\left(\dot{z}^h\right) + NL(\dot{z}^h, \dot{\gamma}^h) + O\left(h^r\right)$$

But using definition (7.7), Lemma (3.1), Lemma (2.9) and that G^h is a real function, we have

$$\begin{aligned}\widetilde{\mathcal{K}_h} D_h G_j^h &= \Im\left(D_h z\left(\alpha_j\right) e^{-iz(\alpha_j)}\mathcal{H}_h\left(\frac{D_h G_j^h e^{iz(\alpha_j)}}{D_h z\left(\alpha_j\right)}\right) + A_{-m+2}\left(\dot{G}_j^h\right)\right)\\ &= \Im\left(\mathcal{H}_h\left(D_h G_j^h\right) + A_0\left(G_j^h\right)\right) = A_0\left(G_j^h\right),\end{aligned}$$

and then $\widetilde{\mathcal{K}_h}\left(\dot{\gamma}^h\right) = A_0\left(\dot{\phi}^h\right) + A_0\left(\dot{z}^h\right) + NL(\dot{z}^h, \dot{\gamma}^h) + O\left(h^r\right)$.

So, substituting in (7.8) we conclude that

$$\begin{aligned}\dot{\gamma}_j^h &= D_h G_j^h + A_0\left(\dot{\phi}^h\right) + A_0\left(\dot{z}^h\right) + NL(\dot{z}^h, \dot{\gamma}^h) + O\left(h^r\right)\\ &= 2\ D_h\dot{\phi}_j^h - 2\Im\left(iD_h\dot{z}_j^h e^{-iz(\alpha_j)} w_0\left(\alpha_j\right)\right) + \Im\,\mathcal{H}_h\left(\frac{\gamma\left(\alpha_j\right)}{D_h z\left(\alpha_j\right)} D_h\dot{z}_j^h\right)\\ &+ A_0\left(\dot{\phi}^h\right) + A_0\left(\dot{z}^h\right) + NL(\dot{z}^h, \dot{\gamma}^h) + O\left(h^r\right),\end{aligned}$$

as desired. □

Proposition 4.1. *Defining the new variables, we can write the evolution equations as:*

$$\begin{aligned}\frac{\partial\left(\dot{z}_j^{Nh}\right)}{\partial t} &= \sigma(\alpha_j)\Lambda_h\dot{F}_j^h + A\left(\dot{F}_j^h, \dot{z}_j^{Nh}, \dot{\delta}_j^h\right),\\ \frac{\partial\left(\dot{\delta}_j^h\right)}{\partial t} &= A\left(\dot{F}_j^h, \dot{z}_j^{Nh}, \dot{\delta}_j^h\right),\\ \frac{\partial\left(\dot{F}_j^h\right)}{\partial t} &= -c\left(\alpha_j\right)\dot{z}_j^{Nh} + \frac{1}{2}\left(\left(\dot{u}_j^h\right)^2 + \left(\dot{v}_j^h\right)^2\right)\end{aligned} \tag{4.8}$$

where we denote $\Lambda_h := \mathcal{H}_h D_h$ and $c(\alpha,t) := -\nabla p \cdot \vec{n}$ is the inner product between the pressure gradient and the surface normal vector.

Proof. First note that by definitions (3.3), (4.7) and inequality (3.7) we have

$$A\left(\dot{F}^h, \dot{z}^{Nh}, \dot{\delta}^h\right) = A\left(\dot{\phi}^h, \dot{z}^h, \dot{\gamma}^h\right). \tag{7.9}$$

Then, the evolution equations for z_j^{Nh} and $\dot{\delta}_j^h$ follows from (4.4-ii), (4.7) and (7.9). The evolution equations for both and $\dot{\delta}_j^h$ were already given in (4.4-ii) and just below (4.5). Let us now write down the evolution equation for $\dot{F}^h$ in terms of the new variables.

From the definition of $\dot{F}^h$ in (4.1), we have

$$\frac{\partial\left(\dot{F}_j^h\right)}{\partial t} = \frac{\partial(\dot{\phi}_j^h)}{\partial t} - u(\alpha_j)\,\dot{u}_j^h - v(\alpha_j)\,\dot{v}_j^h - u_t(\alpha_j)\,\dot{x}_j^h - v_t(\alpha_j)\,\dot{y}_j^h,$$

where $\dot{u}_j^h = u_j^h - u(\alpha_j)$, and $\dot{v}_j^h = v_j^h - v(\alpha_j)$. The definitions of u_j^h, v_j^h and $u(\alpha_j)$, $v(\alpha_j)$ are given in (1.1-i) and (1.7-i).

The error on the Bernoulli's equation (1.7-ii) is given by

$$\begin{aligned}\frac{\partial\left(\dot{\phi}_j^h\right)}{\partial t} &= \left(\phi_j^h\right)_t - \phi_t(\alpha_j) = \frac{1}{2}\left(\left(u_j^h\right)^2 + \left(v_j^h\right)^2 - u^2(\alpha_j) - v^2(\alpha_j)\right) - g\dot{y}_j^h\\ &= u(\alpha_j)\,\dot{u}_j^h + v(\alpha_j)\,\dot{v}_j^h + \frac{1}{2}\left(\left(\dot{u}_j^h\right)^2 + \left(\dot{v}_j^h\right)^2\right) - g\dot{y}_j^h,\end{aligned}$$

and then we get $\frac{\partial(\dot{F}_j^h)}{\partial t} = -g\dot{y}_j^h - u_t(\alpha_j)\,\dot{x}_j^h - v_t(\alpha_j)\,\dot{y}_j^h + \frac{1}{2}\left(\left(\dot{u}_j^h\right)^2 + \left(\dot{v}_j^h\right)^2\right)$. Noting that the Lagrangian velocity (u,v) satisfies the Euler equations, i.e.,

$$u_t + p_x = 0 \quad \text{and} \quad v_t + p_y + g = 0,$$

we can conclude that $\frac{\partial(\dot{F}_j^h)}{\partial t} = \nabla p \cdot (\dot{x}, \dot{y}) + 1/2\left(\left(\dot{u}_j^h\right)^2 + \left(\dot{v}_j^h\right)^2\right)$. Assuming that $p = 0$ on the interface, then ∇p is normal to the wave profile, and then

$$\frac{\partial\left(\dot{F}_j^h\right)}{\partial t} = -c(\alpha_j)\,\dot{z}_j^{Nh} + \frac{1}{2}\left(\left(\dot{u}_j^h\right)^2 + \left(\dot{v}_j^h\right)^2\right),$$

where $c(\alpha,t)$, was defined below (4.8). $\square$

Proposition 4.2. *The following inequality holds for $t \le T^*$:*

$$\left(\left\|\dot{z}^{Nh}(t)\right\|_{H^0}^2 + \left\|\dot{\delta}^h(t)\right\|_{H^0}^2 + \left\|\dot{F}^h(t)\right\|_{H^{\frac{1}{2}}}^2\right)^{\frac{1}{2}} \le C(T)h^r \text{ for } t \le T^*. \tag{4.10}$$

i.e., in the new variables and for any fixed time less then T^*, the numerical solution converges to the exact solution in the above discrete norm.

Proof. Define the norm

$$\|f\|_{H_h^{\frac{1}{2}}} = \left((\Lambda_h + 1)\,f, f\right)_h^{\frac{1}{2}}.$$

Defining also

$$y_0^2(t) = \left\| \left(\frac{c}{\sigma}\right)^{\frac{1}{2}} \dot{z}^{Nh}(t) \right\|_{H_h^0}^2 + \left\| \dot{\delta}^h(t) \right\|_{H_h^0}^2 + \left\| \dot{F}^h \right\|_{H_h^{\frac{1}{2}}}^2 ,$$

we have by simple differentiation in time, and by equations (4.8) that:

$$\begin{aligned} \frac{1}{2}\frac{dy_0^2}{dt} &= \left(\Lambda_h \dot{F}^h, c\dot{z}^{Nh}\right)_h - \left(c\dot{z}^{Nh}, (\Lambda_h+1)\,\dot{F}^h\right)_h \\ &\quad + \frac{1}{2}\left((\Lambda_h+1)\,\dot{F}^h, \left(\dot{u}^h\right)^2 + \left(\dot{v}^h\right)^2\right)_h + \left(f_1, \dot{z}^{Nh}\right)_h + \left(f_2, \dot{\delta}^h\right)_h , \end{aligned}$$

where $\max\{ \|f_1\|_{H_h^0}, \|f_2\|_{H_h^0}\} \leq C\left(\|\dot{z}^{Nh}\|_{H_h^0} + \left\|\dot{\delta}^h\right\|_{H_h^0} + \left\|\dot{F}^h\right\|_{H_h^0} + h^r\right)$. The terms in Λ_h cancel out and we find that:

$$\begin{aligned} \frac{1}{2}\frac{dy_0^2}{dt} &= -\left(c\dot{z}^{Nh}, \dot{F}^h\right)_h + \frac{1}{2}\left((\Lambda_h+1)\,\dot{F}^h, \left(\dot{u}^h\right)^2 + \left(\dot{v}^h\right)^2\right)_h \\ &\quad + \left(f_1, \dot{z}^{Nh}\right)_h + \left(f_2, \dot{\delta}^h\right)_h . \end{aligned}$$

Using the fact that

$$\begin{aligned} &-\left(c\dot{z}^{Nh}, \dot{F}^h\right)_h + \left(f_1, \dot{z}^{Nh}\right)_h + \left(f_2, \dot{\delta}^h\right)_h \\ &\leq \left(\|c\dot{z}^{Nh}\|_{H_h^0} + \left\|\dot{\delta}^h\right\|_{H_h^0} + \left\|\dot{F}^h\right\|_{H_h^{\frac{1}{2}}}\right)\left(\|c\dot{z}^{Nh}\|_{H_h^0} + \left\|\dot{\delta}^h\right\|_{H_h^0} + \left\|\dot{F}^h\right\|_{H_h^{\frac{1}{2}}} + h^r\right) \\ &\leq C\left(y_0^2(t) + h^r y_0(t)\right) , \end{aligned}$$

and applying Lemma (7.1) below, we have:

$$\frac{1}{2}\frac{dy_0^2}{dt} \leq C\left(y_0^2(t) + h^r y_0(t)\right) .$$

By Gronwall inequality, we get:

$$y_0(t) \leq C(T)h^r \text{ for } t \leq T^*. \tag{4.10}$$

Also, as Λ_h has Fourier symbol $|k|\theta(kh)$, where θ is as in (1.6), then

$$\left\|\dot{f}^h\right\|_{H_h^{\frac{1}{2}}} \leq \left\|\dot{f}^h\right\|_{H^{\frac{1}{2}}} \leq C\left\|\dot{f}^h\right\|_{H_h^{\frac{1}{2}}} . \tag{4.10}$$

Then (4.10) follows. □

Lemma 7.1. *The following inequality holds for $t \leq T^*$:*

$$\left|\left((\Lambda_h+1)\,\dot{F}^h, \left(\dot{u}^h\right)^2 + \left(\dot{v}^h\right)^2\right)_h\right| \leq C\left(y_0^2(t) + h^r y_0(t)\right) .$$

Proof. First of all, note that

$$\left|\left((\Lambda_h+1)\,\dot{F}_j^h, \left(\dot{u}_j^h\right)^2 + \left(\dot{v}_j^h\right)^2\right)_h\right| \leq \left\|(\Lambda_h+1)\,\dot{F}_j^h\right\|_{H_h^0}\left\|\left(\dot{u}_j^h\right)^2 + \left(\dot{v}_j^h\right)^2\right\|_{H_h^0} . \tag{7.10}$$

Using (4.2) we have

$$\left\|\left(\dot{u}^h\right)^2+\left(\dot{v}^h\right)^2\right\|_{H_h^0}=\left\|\left(\dot{u}^h+i\dot{v}^h\right)\left(\dot{u}^h-i\dot{v}^h\right)\right\|_{H_h^0}$$
$$\leq \max_j\left|\dot{u}_j^h+i\dot{v}_j^h\right|\left\|\frac{1}{D_h z}\left(-i\mathcal{H}_h+I_h\right)D_h\dot{F}^h+A\left(\dot{\phi}^h,\dot{z}^h,\dot{\gamma}^h\right)\right\|_{H_h^0}$$

Note that

$$h\left|\dot{u}_j^h\right|^2\leq h\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1}\left|\dot{u}_j^h\right|^2=2\pi\left\|\dot{u}^h\right\|_{H_h^0}^2 \implies \max_j\left|\dot{u}_j^h\right|\leq\frac{C}{h^{\frac{1}{2}}}\left\|\dot{u}^h\right\|_{H_h^0}$$

and then

$$\left\|\left(\dot{u}^h\right)^2+\left(\dot{v}^h\right)^2\right\|_{H_h^0}\leq\frac{C}{h^{\frac{1}{2}}}\left(\left\|\dot{u}^h\right\|_{H_h^0}+\left\|\dot{v}^h\right\|_{H_h^0}\right)\left\|D_h\dot{F}^h+A\left(\dot{\phi}^h,\dot{z}^h,\dot{\gamma}^h\right)\right\|_{H_h^0} \tag{7.11}$$

But by (3.6) it is true that

$$\left\|\dot{u}^h\right\|_{H_h^0}\leq\left\|\frac{1}{D_h z}\left(-i\mathcal{H}_h+I_h\right)\left(D_h\dot{\phi}^h-\Re\left(D_h\dot{z}^h w\right)\right)\right\|_{H_h^0}$$
$$\leq C\left(\left\|D_h\dot{z}^h\right\|_{H_h^0}+\left\|D_h\dot{\phi}^h\right\|_{H_h^0}\right)$$

Using (3.4), we have

$$\left\|D_h\dot{z}^h\right\|_{H_h^0}^2=\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1}\left|k\theta\left(kh\right)\right|^2\left|\left(\widetilde{\dot{z}^h}\right)_k\right|^2\leq N^2\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1}\left|\left(\widetilde{\dot{z}^h}\right)_k\right|^2=N^2\left\|\dot{z}^h\right\|_{H_h^0}^2\leq Ch^3,$$

and analogously, $\left\|D_h\dot{\phi}^h\right\|_{H_h^0}^2\leq Ch^3$.

We have then $\left\|\dot{u}^h\right\|_{H_h^0}\leq Ch^{3/2}$. Using the same techniques for $\dot{v}^h$ and substituting in (7.11), we can conclude that:

$$\left\|\left(\dot{u}^h\right)^2+\left(\dot{v}^h\right)^2\right\|_{H_h^0}\leq Ch\left(\left\|D_h\dot{F}^h\right\|_{H_h^0}+\left\|A\left(\dot{\phi}^h,\dot{z}^h,\dot{\gamma}^h\right)\right\|_{H_h^0}\right)$$

It is straightforward to prove that

$$\max\left\{\left\|D_h\dot{F}^h\right\|_{H_h^0}\ ,\ \left\|\left(\Lambda_h+1\right)\dot{F}^h\right\|_{H_h^0}\right\}\leq\frac{C}{h^{\frac{1}{2}}}\left\|\dot{F}^h\right\|_{H_h^{\frac{1}{2}}}. \tag{7.12}$$

By (3.7) we have

$$\left\|A\left(\dot{\phi}^h,\dot{z}^h,\dot{\gamma}^h\right)\right\|_{H_h^0}\leq C\left(\|\dot{F}^h\|_{H^0}+\|\dot{z}^{Nh}\|_{H^0}+\|\dot{\delta}^h\|_{H^0}+O\left(h^r\right)\right)$$

and then

$$\left\| \left(\dot{u}^h\right)^2 + \left(\dot{v}^h\right)^2 \right\|_{H_h^0} \leq Ch \left(h^{-\frac{1}{2}} \left\| \dot{F}^h \right\|_{H_h^{\frac{1}{2}}} + \left\| \dot{\delta}^h \right\|_{H_h^0} + \left\| \dot{z}^{Nh} \right\|_{H_h^0} + O(h^r) \right) \tag{7.13}$$

Using (7.10), (7.12) and (7.13), we conclude that:

$$\begin{aligned}
&\left| \left((\Lambda_h + 1)\, \dot{F}^h, \left(\dot{u}^h\right)^2 + \left(\dot{v}^h\right)^2 \right)_h \right| \\
&\leq \left\| \dot{F}^h \right\|_{H_h^{\frac{1}{2}}} \left(\left\| \dot{F}^h \right\|_{H_h^{\frac{1}{2}}} + \left\| \dot{\delta}^h \right\|_{H_h^0} + \left\| \dot{z}^{Nh} \right\|_{H_h^0} + h^r \right) \\
&\leq \left(\left\| \dot{F}^h \right\|_{H_h^{\frac{1}{2}}} + \left\| \dot{\delta}^h \right\|_{H_h^0} + \left\| \dot{z}^{Nh} \right\|_{H_h^0} + h^r \right)^2 \\
&\leq C \left(y_0^2\,(t) + h^r y_0\,(t) \right) \,,
\end{aligned}$$

as desired. $\square$

References

[BHL-Num] J.T. Beale, T.Y. Hou and J. Lowengrub, *Convergence of Boundary Integral Methods for Water Waves*, SIAM Journal of Numerical Analysis **33 5** (1996), 1797–1843.

[BHL-Cont] J.T. Beale, T.Y. Hou and J. Lowengrub, *Growth Rates for the Linearized Motion of Fluids away from Equilibrium*, Communications on Pure and Applied Mathematics **XLVI** (1993), 1269–1301.

[Mad] A.L. Madureira, *Linear Convergence of the Water Wave Problem*, LNCC Preprint (2002).

[Rei] Lothar Reichel, *A matrix Problem with Application to Rapid solution of Integral Equations*, SIAM J. Sci. Stat. Comput. **11** (1990), 263–280.

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