Applied Mathematical Modelling 36 (2012) 3555–3569

Contents lists available at SciVerse ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Hierarchical modeling of piezoelectric plates

Alexandre L. Madureira

Departamento de Matemática Aplicada, Laboratório Nacional de Computação Científica, Av. Getúlio Vargas 333, 25651-070 Petrópolis, RJ, Brazil

ARTICLE INFO

Article history:
Received 19 February 2010
Received in revised form 26 September 2011
Accepted 29 September 2011
Available online 20 October 2011

Keywords:
Piezoelectric plate
Hierarchical modeling
Asymptotic consistency

ABSTRACT

We use variational techniques to derive a class of two-dimensional models for three-dimensional linearly elastic piezoelectric plates. The models result from a mixed formulation for the original problem within spaces of functions with polynomial dependence in the transverse direction. We show that the resulting system of equations is well-posed, and then discuss the asymptotic consistency of the simplest of such models.

1. Introduction

Dimension reduction is a powerful tool to model physical phenomena that occur in slender domains, and as more complex problems are considered, it is useful to have a mathematically sound technique to do so. Variational arguments yield just that, and our goal in this paper is to derive a simple model for piezoelectric plates. Our work was particularly motivated by the investigation developed by Sène [1], and Raoult and Sène [2], latter partially extended in [3].

In this paper we use variational principles to develop two-dimensional models for static piezoelectric plates. For the importance and applications of such problem, the reader can check [4,5].

The first assumption is that a piezoelectric material is occupying a plate domain given by $P = \Omega \times (-\varepsilon, \varepsilon)$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary $\partial\Omega$. The union of the plate's top and bottom surfaces are given by $\partial P_\pm = \Omega \times \{-\varepsilon, \varepsilon\}$, and $\partial P_L = \partial\Omega \times (-\varepsilon, \varepsilon)$ denotes the lateral surface of the plate. We denote a typical point in P by $\underline{x} = (\underset{\sim}{x}, x_3)$, where $\underset{\sim}{x} \in \Omega$ and $x_3 \in (-\varepsilon, \varepsilon)$.

The problem is to find the displacement $\underline{u}^\epsilon$, the electrical potential ϕ^ε, the stress tensor $\underline{\underline{\sigma}}^\varepsilon$, and the electrical displacement $\underline{D}^\varepsilon$ of the plate subject to prescribed internal force density $\underline{f} : P \to \mathbb{R}^3$, surface force density $\underline{g} : \partial P_\pm \to \mathbb{R}^3$, and electric potential $\phi_{bc} : \partial P_\pm \to \mathbb{R}$. The constitutive relations are

$$\underline{\underline{\sigma}}^\varepsilon = \underline{\underline{\underline{\underline{C}}}}\ \underline{\underline{e}}(\underline{u}^\varepsilon) + \underline{\nabla}\phi^\varepsilon \underline{\underline{\underline{Q}}}, \quad \underline{D}^\varepsilon = \underline{\underline{\underline{Q}}}\ \underline{\underline{e}}(\underline{u}^\varepsilon) - \underline{\underline{d}}\ \underline{\nabla}\phi^\varepsilon, \tag{1}$$

or componentwise,

$$\sigma_{ij}^\varepsilon = \sum_{k,l=1}^{3} C_{ijkl} e_{kl}(\underline{u}^\varepsilon) + \sum_{k=1}^{3} \partial_k \phi^\varepsilon Q_{kij}, \quad D_i^\varepsilon = \sum_{k,l=1}^{3} Q_{ikl} e_{kl}(\underline{u}^\varepsilon) - \sum_{k=1}^{3} d_{ik} \partial_k \phi^\varepsilon,$$

for $i, j = 1, 2, 3$. The equilibrium equations are

$$-\mathrm{d}\underline{\mathrm{i}}\mathrm{v}\ \underline{\underline{\sigma}}^\varepsilon = \underline{f}, \quad \mathrm{div}\underline{D}^\varepsilon = 0 \quad \text{in } P,$$

E-mail address: alm@lncc.br

0307-904X/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.apm.2011.09.092

with the boundary conditions

$$\begin{aligned}&\underline{u}^{\varepsilon}=0, \quad \underline{D}^{\varepsilon}\cdot\underline{n}=0 \quad \text{on } \partial P_{\mathrm{L}},\\ &\underline{\underline{\sigma}}^{\varepsilon}\underline{n}=\underline{g}, \quad \phi^{\varepsilon}=\phi_{bc} \quad \text{on } \partial P_{\pm}.\end{aligned}$$

The rigidity tensor and the infinitesimal strain tensor are given by

$$\underline{\underline{\underline{\underline{C}}}}\,\underline{\underline{e}}(\underline{u}^{\varepsilon})=2\mu\underline{\underline{e}}(\underline{u}^{\varepsilon})+\lambda\operatorname{div}\underline{u}^{\varepsilon}\,\underline{\underline{\delta}}, \quad \underline{\underline{e}}(\underline{u}^{\varepsilon})=\frac{1}{2}\left(\underline{\underline{\nabla}}+\underline{\underline{\nabla}}^{t}\right)\underline{u}^{\varepsilon},$$

where $\underline{\underline{\delta}}$ is the identity matrix, and the Lamé coefficients μ, λ are such that $\underline{\underline{\underline{\underline{C}}}}$ is coercive. For simplicity, we follow [6] and assume that the dielectric tensor is of the form

$$\underline{\underline{d}}=\begin{pmatrix} d & 0 & 0\\ 0 & d & 0\\ 0 & 0 & d_{33}\end{pmatrix},$$

where d and d_{33} are positive constants. We also assume that the piezoelectric tensor is such that the only nonzero constants are [6]

$$Q_{333}, \quad Q_{113}=Q_{223}, \quad \text{and } Q_{311}=Q_{322},$$

and that the symmetry relations

$$Q_{ijk}=Q_{ikj} \quad \text{for } i,k,j=1,2,3,$$

hold.

As stated above, our goal is to develop, in a consistent mathematical framework, two-dimensional models for the problem just described. There is an extensive literature dealing with the simpler problem of linearly elastic plates. Regarding the modeling of piezoelectric plates, there are derivations based on geometric and mechanical *a priori* assumptions, see for instance [7,8], or [5] and references therein.

On the mathematical side, some authors generalized the asymptotic arguments that Ciarlet and collaborators used for the linearly elastic plate problem [9]. In particular, Sène [1] showed that as the plate thickness goes to zero, the solution converges in a proper sense to the solution to a biharmonic (20) and a membrane (19) problems. See also Maugin and Attou [6], Weller and Licht [10], Canon and Lenczner [11], and Figueiredo and Leal [3] for further developments using such approach.

The way we proceed is different since it is not "asymptotic" in principle, and we find our models using mixed variational formulations. The approach is based on firm mathematical grounds, and the equations form a sequence of *hierarchical models* that become more accurate as the order of the model grow. See [12], and also [13–17] and references therein, for linearly elastic plates. See [18,5] for a review of the engineering literature that resorts to variational arguments.

Before proceeding, we need to introduce some notation. The 3 × 3 symmetric tensors are denoted in Greek letters with double underbars, as in $\underline{\underline{\sigma}}$, $\underline{\underline{\tau}}$. The symbol $\underline{\underline{\delta}}$ denote the identity tensor. For 2 × 2 symmetric tensors, we use Greek letters with double under-tildes. Similarly, we write vectors in italic letters. If they belong to $\mathbb{R}^3$, they have an under bar and if they belong to $\mathbb{R}^2$, they have an under-tilde. We can then decompose each tensor and vector as in

$$\underline{\underline{\sigma}}=\begin{pmatrix}\underset{\approx}{\sigma} & \underset{\sim}{\sigma}\\ \times & \sigma_{33}\end{pmatrix}, \quad \underline{u}=\begin{pmatrix}\underset{\sim}{u}\\ u_3\end{pmatrix}.$$

We use four under bars (four under tildes) for fourth order tensors acting on 3 × 3 (2 × 2) symmetric tensors. Similar notation holds for third order tensors, and the operators divergence and gradient obey similar notation rules.

Accordingly, if $O\subset\mathbb{R}^d$, $d=1,2,3$, is an open set, then $\underline{\underline{L}}^2(O)$ is the set of 3 × 3 symmetric matrices which components are square integrable functions in O, and $\underline{L}^2(O)$ and $L^2(O)$ are the set of vector and scalar square integrable functions defined in O. Similar definitions hold for $\underline{\underline{H}}^s(O)$, $\underline{H}^s(O)$ and $H^s(O)$, the Sobolev space of order s, for a real number s. We denote the norms of those spaces by $\|\cdot\|_{L^2(O)}$ and $\|\cdot\|_{H^s(O)}$, and the semi-norms by $|\cdot|_{H^s(O)}$.

We denote by c an arbitrary positive constant that might depend on Ω, $\underline{f}$, $\underline{g}$, ϕ_{bc}, and on the material parameters, but does not depend on ε, $\underline{u}$, ϕ, etc.

We now briefly describe the contents of the present paper. In Section 2 we rewrite the piezoelectric problem in a variational form, and define a two-dimensional model. After that, in Section 3, we discuss the asymptotic consistency of the model. Section 4 presents a final discussion, and in the Appendix we perform the computations that led to our model.

2. Variational formulations and hierarchical modeling

Our first step is to rewrite the piezoelectric problem in a variational form. Let

$$\underline{V}(P)=\left\{\underline{v}\in\underline{H}^1(P):\underline{v}=0 \text{ on } \partial P_{\mathrm{L}}\right\}, \quad \Psi_{\phi_{bc}}(P)=\left\{\psi\in H^1(P):\psi=\phi_{bc} \text{ on } \partial P_{\pm}\right\},$$

and we endow these spaces with the $H^1(P)$ norm. We search for $(\underline{u}^\varepsilon, \phi^\varepsilon) \in \underline{V}(P) \times \Psi_{\phi_{bc}}(P)$ such that

$$a\big((\underline{u}^\epsilon, \phi^\epsilon), (\underline{v}, \psi)\big) = l(\underline{v}, \psi) \quad \text{for all } (\underline{v}, \psi) \in \underline{V}(P) \times \Psi_0(P), \tag{2}$$

where

$$\begin{aligned} a((\underline{u}^\varepsilon, \phi^\varepsilon), (\underline{v}, \psi)) &= \int_P \Big[\underline{\underline{\underline{C}}}\ \underline{\underline{e}}(\underline{u}^\varepsilon) + \underline{\nabla}\phi^\varepsilon \underline{\underline{\underline{Q}}}\Big] : \underline{\underline{e}}(\underline{v})\, d\underline{x} - \int_P \Big[\underline{\underline{\underline{Q}}}\ \underline{\underline{e}}(\underline{u}^\varepsilon) - \underline{\underline{d}}\ \underline{\nabla}\phi^\epsilon\Big] \cdot \underline{\nabla}\psi\, d\underline{x}, \\ l(\underline{v}, \psi) &= \int_P \underline{f} \cdot \underline{v}\, d\underline{x} + \int_{\partial P_\pm} \underline{g} \cdot \underline{v}\, d\underline{x}. \end{aligned} \tag{3}$$

Existence and uniqueness of solution follows immediately from Lax–Milgram Theorem since

$$\begin{aligned} a((v, \psi), (v, \psi)) &= \int_P (\underline{\underline{\underline{C}}}\ \underline{\underline{e}}(v) + \underline{\nabla}\psi \underline{\underline{\underline{Q}}}) : \underline{\underline{e}}(v)\, d\underline{x} - \int_P (\underline{\underline{\underline{Q}}}\ \underline{\underline{e}}(v) - \underline{\underline{d}}\ \underline{\nabla}\psi) \cdot \underline{\nabla}\psi\, d\underline{x} = \int_P \underline{\underline{\underline{C}}}\ \underline{\underline{e}}(v) : \underline{\underline{e}}(v)\, d\underline{x} + \int_P \underline{\underline{d}}\ \underline{\nabla}\psi \cdot \underline{\nabla}\psi\, d\underline{x} \\ &\geqslant c\Big(\|v\|^2_{H^1(P)} + \|\psi\|^2_{H^1(P)}\Big), \end{aligned}$$

for all $(\underline{v}, \psi) \in \underline{V}(P) \times \Psi_0(P)$. Above, we used that

$$(\underline{\underline{\underline{Q}}}\ \underline{\underline{\tau}}) \cdot \underline{v} = (\underline{v}\underline{\underline{\underline{Q}}}) : \underline{\underline{\tau}}, \tag{4}$$

for all $\underline{\underline{\tau}} \in \mathbb{R}^{3\times3}_{\text{sym}},\ \underline{v} \in \mathbb{R}^3$.

We now develop a mixed formulation for the same problem. Note that $\underline{\underline{\sigma}}^\epsilon \in \underline{\underline{L}}^2(P),\ \underline{D}^\varepsilon \in \underline{L}^2(P),\ \underline{u}^\varepsilon \in \underline{V}(P),\ \phi^\epsilon \in \Psi_{\phi_{bc}}(P)$ satisfy

$$\int_P \underline{\underline{\underline{A}}}\ \underline{\underline{\sigma}}^\varepsilon : \underline{\underline{\tau}}\, d\underline{x} - \int_P \underline{\underline{e}}(\underline{u}^\varepsilon) : \underline{\underline{\tau}}\, d\underline{x} - \int_P \underline{\nabla}\phi^\varepsilon \underline{\underline{\underline{Q}}} : \underline{\underline{\underline{A}}}\ \underline{\underline{\tau}}\, d\underline{x} = 0 \quad \text{for all } \underline{\underline{\tau}} \in \underline{\underline{L}}^2(P), \tag{5i}$$

$$\int_P \underline{D}^\epsilon \cdot \underline{H}\, d\underline{x} - \int_P \underline{\underline{\underline{Q}}}\ \underline{\underline{e}}(\underline{u}^\varepsilon) \cdot \underline{H}\, d\underline{x} + \int_P \underline{\underline{d}}\ \underline{\nabla}\phi^\epsilon \cdot \underline{H}\, d\underline{x} = 0 \quad \text{for all } \underline{H} \in \underline{L}^2(P), \tag{5ii}$$

$$\int_P \underline{\underline{\sigma}}^\varepsilon : \underline{\underline{e}}(\underline{v})\, d\underline{x} = \int_P \underline{f} \cdot \underline{v}\, d\underline{x} + \int_{\partial P_\pm} \underline{g} \cdot \underline{v}\, d\underline{x} \quad \text{for all } \underline{v} \in \underline{V}(P), \tag{5iii}$$

$$\int_P \underline{D}^\epsilon \cdot \underline{\nabla}\psi\, d\underline{x} = 0 \quad \text{for all } \psi \in \Psi_0(P), \tag{5iv}$$

where $\underline{\underline{\underline{A}}} = \underline{\underline{\underline{C}}}^{-1}$.

If we set

$$\begin{aligned} b\Big((\underline{\underline{\sigma}}^\varepsilon, \underline{D}^\varepsilon), (\underline{\underline{\tau}}, \underline{H})\Big) &= \int_P \underline{\underline{\underline{A}}}\ \underline{\underline{\sigma}}^\varepsilon : \underline{\underline{\tau}}\, d\underline{x} + \int_P \underline{D}^\varepsilon \cdot \underline{H}\, d\underline{x}, \\ b_1\Big((\underline{\underline{\tau}}, \underline{H}), (\underline{u}^\varepsilon, \phi^\varepsilon)\Big) &= -\int_P \underline{\underline{e}}(\underline{u}^\epsilon) : \underline{\underline{\tau}}\, d\underline{x} - \int_P \underline{\nabla}\phi^\varepsilon \underline{\underline{\underline{Q}}} : \underline{\underline{\underline{A}}}\ \underline{\underline{\tau}}\, d\underline{x} - \int_P \underline{\underline{\underline{Q}}}\ \underline{\underline{e}}(\underline{u}^\varepsilon) \cdot \underline{H}\, d\underline{x} + \int_P \underline{\underline{d}}\ \underline{\nabla}\phi^\epsilon \cdot \underline{H}\, d\underline{x}, \\ b_2\Big((\underline{\underline{\sigma}}^\varepsilon, \underline{D}^\varepsilon), (\underline{v}, \psi)\Big) &= \int_P \underline{\underline{\sigma}}^\varepsilon : \underline{\underline{e}}(\underline{v})\, d\underline{x} + \int_P \underline{D}^\varepsilon \cdot \underline{\nabla}\psi\, d\underline{x}, \end{aligned}$$

we have that

$$\begin{aligned} b\Big((\underline{\underline{\sigma}}^\varepsilon, \underline{D}^\varepsilon), (\underline{\underline{\tau}}, \underline{H})\Big) + b_1\Big((\underline{\underline{\tau}}, \underline{H}), (\underline{u}^\varepsilon, \phi^\varepsilon)\Big) &= 0, \\ b_2\Big((\underline{\underline{\sigma}}^\varepsilon, \underline{D}^\varepsilon), (\underline{v}, \psi)\Big) &= l(\underline{v}, \psi), \end{aligned} \tag{6}$$

for all $(\underline{\underline{\tau}}, \underline{H}) \in \underline{\underline{L}}^2(P) \times \underline{L}^2(P)$ and $(\underline{v}, \psi) \in \underline{V}(P) \times \Psi_0(P)$.

To show existence and uniqueness of solution of the above mixed formulation, it is enough to follow [19–21], and show that $b(\cdot,\cdot)$ is coercive (it is!), and that for all $(\underline{v}, \psi) \in \underline{V}(P) \times \Psi_0(P)$,

$$\sup_{(\underline{\underline{\tau}}, \underline{H}) \in \underline{\underline{L}}^2(P) \times \underline{L}^2(P)} \frac{b_\alpha\Big((\underline{\underline{\tau}}, \underline{H}), (\underline{v}, \psi)\Big)}{\|(\underline{\underline{\tau}}, \underline{H})\|_{L^2(P) \times L^2(P)}} \geqslant c\|(\underline{v}, \psi)\|_{\underline{V}(P) \times \Psi_0(P)} \quad \text{for } \alpha = 1, 2.$$

The above inf–sup condition is trivial for $\alpha = 2$ since $\underline{\underline{e}}(\underline{V}(P)) \subset \underline{\underline{L}}^2(P)$ and $\underline{\nabla}(\Psi_0(P)) \subset \underline{L}^2(P)$. For $\alpha = 1$, it is sufficient to notice that

$$\begin{aligned} b_1\Big((-\underline{\underline{\underline{C}}}\ \underline{\underline{e}}(\underline{v}), \underline{\nabla}\psi), (\underline{v}, \psi)\Big) &= \int_P \underline{\underline{\underline{C}}}\ \underline{\underline{e}}(\underline{v}) : \underline{\underline{e}}(\underline{v})\, d\underline{x} + \int_P \underline{\nabla}\psi \underline{\underline{\underline{Q}}} : \underline{\underline{e}}(\underline{v})\, d\underline{x} - \int_P \underline{\underline{\underline{Q}}}\ \underline{\underline{e}}(\underline{v}) \cdot \underline{\nabla}\psi\, d\underline{x} + \int_P \underline{\underline{d}}\ \underline{\nabla}\psi \cdot \underline{\nabla}\psi\, d\underline{x} \\ &\geqslant c\Big(\|\underline{v}\|^2_{H^1(P)} + \|\psi\|^2_{H^1(P)}\Big), \end{aligned} \tag{7}$$

where we again apply (4).

Solving the mixed problem (6) within subspaces of functions that are polynomials in the transverse direction we derive piezoelectric plate models. For instance, let

$$
\begin{aligned}
&\underline{V}(P,p)=\left\{\underline{v}\in\underline{V}(P):\deg_3\underset{\sim}{v}\leqslant p,\ \deg_3 v_3\leqslant p+1\right\},\\
&\Psi_{\phi_{bc}}(P,p)=\left\{\psi\in\Psi_{\phi_{bc}}(P):\deg_3\psi\leqslant p+1\right\},\\
&\underline{\underline{L}}^2(P,p)=\left\{\underline{\underline{\tau}}\in\underline{\underline{L}}^2(P):\deg_3\underset{\approx}{\tau}\leqslant p,\ \deg_3\underset{\sim}{\tau}\leqslant p+1,\ \deg_3\tau_{33}\leqslant p\right\},\\
&\underline{L}^2(P,p)=\left\{\underline{H}\in\underline{L}^2(P):\deg_3\underset{\sim}{H}\leqslant p+1,\ \deg_3 H_3\leqslant p\right\}.
\end{aligned}
\tag{8}
$$

For $\underset{\sim}{v}\in\underset{\sim}{L}^2(P)$ we write $\deg_3\underset{\sim}{v}\leqslant p$ meaning that the components of v are polynomials of degree at most p with coefficients in Ω. Similar interpretation holds for the other tensors. The representation below indicates the degrees of $\underline{v}$, ψ, $\underline{\underline{\tau}}$, $\underline{H}$ in the spaces (8):

$$
\deg\underline{v}=\begin{pmatrix}p\\p+1\end{pmatrix},\quad \deg\psi=(p+1),
$$

$$
\deg\underline{\underline{\tau}}=\begin{pmatrix}p&p+1\\p+1&p\end{pmatrix},\quad \deg\underline{H}=\begin{pmatrix}p+1\\p\end{pmatrix}.
$$

We now search for $\underline{\underline{\sigma}}\in\underline{\underline{L}}^2(P,p),\ \underline{D}\in\underline{L}^2(P,p),\ \underline{u}\in\underline{V}(P,p),\ \phi\in\Psi_{\phi_{bc}}(P,p)$ such that

$$
\begin{aligned}
&b\Big((\underline{\underline{\sigma}},\underline{D}),(\underline{\underline{\tau}},\underline{H})\Big)+b_1\Big((\underline{\underline{\tau}},\underline{H}),(\underline{u},\phi)\Big)=0,\\
&b_2\Big((\underline{\underline{\sigma}},\underline{D}),(\underline{v},\psi)\Big)=l(\underline{v},\psi),
\end{aligned}
\tag{9}
$$

for all $(\underline{\underline{\tau}},\underline{H})\in\underline{\underline{L}}^2(P,p)\times\underline{L}^2(P,p)$ and $(\underline{v},\psi)\in\underline{V}(P,p)\times\Psi_0(P,p)$.

The degrees in (8) are one possibility, the simplest we could find. Other combinations of polynomial degrees yield different models, but not all combinations yield well-posed problems. Moreover, even if the final equations are well-posed, the model might not be "asymptotically consistent", in a sense that we make clear further ahead.

As in the original formulation (6), it follows for the spaces in (8) that $\underline{\underline{e}}(\underline{V}(P,p))\subset\underline{\underline{L}}^2(P,p)$, and $\underline{\nabla}(\Psi_0(P),p)\subset\underline{L}^2(P,p)$. Also, (7) holds for $(\underline{v},\psi)\in\ V(P,p)\times\Psi_0(P,p)$. Thus, the inf–sup conditions hold and the model problem (9) is well-posed for all p.

Also note that since

$$
\underline{\underline{\underline{\underline{C}}}}\,\underline{\underline{e}}(\underline{V}(P,p))-\underline{\nabla}\Psi_{\phi_{bc}}(P,p)\underline{\underline{\underline{Q}}}\subset\underline{\underline{L}}^2(P,p),\quad \underline{\underline{\underline{Q}}}\,\underline{\underline{e}}(\underline{V}(P,p))+\underline{\underline{d}}\,\underline{\nabla}\Psi_{\phi_{bc}}(P,p)\subset\underline{L}^2(P,p),
\tag{10}
$$

the constitutive Eq. (1) are enforced exactly.

Remark. From (10), our model (9) could be defined simply by solving (2) restricted to $\underline{V}(P,p),\ \Psi_{\phi_{bc}}(P,p)$. Since that leads to a well-posed problem, then so is (9).

Before presenting the simplest of such models we define

$$
\begin{aligned}
&\underline{g}^0=\frac{1}{2}\Big[\underline{g}(\underset{\sim}{x},\varepsilon)+\underline{g}(\underset{\sim}{x},-\varepsilon)\Big],\quad \underline{g}^1=\frac{1}{2}\Big[\underline{g}(\underset{\sim}{x},\varepsilon)-\underline{g}(\underset{\sim}{x},-\varepsilon)\Big],\\
&\underline{f}^0(\underset{\sim}{x})=\varepsilon^{-1}\int_{-\varepsilon}^{\varepsilon}\underline{f}(\underline{x})\,dx_3,\quad \underline{f}^1(\underset{\sim}{x})=\varepsilon^{-1}\int_{-\varepsilon}^{\varepsilon}\underline{f}(\underline{x})x_3\,dx_3,\quad f_3^2(\underset{\sim}{x})=\varepsilon^{-1}\int_{-\varepsilon}^{\varepsilon}f_3(\underline{x})L_2(x_3)\,dx_3,
\end{aligned}
$$

where $L_2(z)=(3z^2-\varepsilon^2)/2$. Similar definitions hold for ϕ_{bc}^0 and ϕ_{bc}^1. Let $\underset{\approx}{\underset{\approx}{A}}$ be the two-dimensional version of the compliance tensor with the inverse

$$
\underset{\approx}{\underset{\approx}{A}}^{-1}\underset{\approx}{\tau}=2\mu\left[\underset{\approx}{\tau}+\frac{\lambda}{2\mu+\lambda}\operatorname{tr}(\underset{\approx}{\tau})\,\underset{\approx}{\delta}\right].
\tag{11}
$$

Here, $\operatorname{tr}(\cdot)$ indicates the trace operator.

As in the linearly elastic plate modeling, the solution decouples in bending and stretching components, so we consider each part separately. We show next the resulting equations for $p=1$, but postpone the details to the Appendix. Assume the approximate displacement $\underline{u}$, and electrical potential ϕ are given by

$$
\underline{u}(\underline{x})=\begin{pmatrix}\underset{\sim}{\eta}(\underset{\sim}{x})\\ \rho(\underset{\sim}{x})x_3\end{pmatrix}+\begin{pmatrix}-\underset{\sim}{\theta}(\underset{\sim}{x})x_3\\ \omega(\underset{\sim}{x})+\omega_2(\underset{\sim}{x})L_2(x_3)\end{pmatrix},\quad \phi(\underline{x})=\phi_{bc}^0(\underset{\sim}{x})+\varepsilon^{-1}x_3\phi_{bc}^1(\underset{\sim}{x})+(\varepsilon^2-L_2)\phi_2(\underset{\sim}{x}),
\tag{12}
$$

where $\underset{\sim}{\eta},\ \rho,\ \underset{\sim}{\theta},\ \omega,\ \omega_2,\ \phi_2$ are unknown. Also, the approximate stress tensor $\underline{\underline{\sigma}}$, and electrical displacement $\underline{D}$ are as

$$
\begin{aligned}
\underline{\underline{\sigma}}(\underline{x}) &= \begin{pmatrix} \underset{\approx}{\sigma}^0(\underset{\sim}{x}) & \underset{\sim}{\sigma}^1(\underset{\sim}{x})x_3 \\ \underset{\sim}{\sigma}^1(\underset{\sim}{x})^t x_3 & \sigma^0_{33}(\underset{\sim}{x}) \end{pmatrix} + \begin{pmatrix} \underset{\approx}{\sigma}^1(\underset{\sim}{x})x_3 & \underset{\sim}{\sigma}^0(\underset{\sim}{x}) + \underset{\sim}{\sigma}^2(\underset{\sim}{x})L_2(x_3) \\ \left[\underset{\sim}{\sigma}^0(\underset{\sim}{x}) + \underset{\sim}{\sigma}^2(\underset{\sim}{x})L_2(x_3)\right]^t & \sigma^1_{33}(\underset{\sim}{x})x_3 \end{pmatrix}, \\
\underline{D}(\underline{x}) &= \begin{pmatrix} \underset{\sim}{D}^1(\underset{\sim}{x})x_3 \\ D^0_3(\underset{\sim}{x}) \end{pmatrix} + \begin{pmatrix} \underset{\sim}{D}^0(\underset{\sim}{x}) + \underset{\sim}{D}^2(\underset{\sim}{x})L_2(x_3) \\ D^1_3(\underset{\sim}{x})x_3 \end{pmatrix},
\end{aligned}
\tag{13}
$$

where $\underset{\approx}{\sigma}^0,\ \underset{\sim}{\sigma}^1,\ \sigma^0_{33},\ \underset{\approx}{\sigma}^1,\ \underset{\sim}{\sigma}^0, \underset{\sim}{\sigma}^2,\ \sigma^1_{33},\ \underset{\sim}{D}^1,\ D^0_3,\ \underset{\sim}{D}^0,\ \underset{\sim}{D}^2,\ D^1_3$ need to be determined.

For the stretching part, we find that $\underset{\sim}{\eta}$ and ρ satisfy the equations

$$
\begin{aligned}
&-\underset{\sim}{\operatorname{div}} \underset{\approx}{\underset{\approx}{A}}^{-1} \underset{\approx}{e}(\underset{\sim}{\eta}) - \frac{\lambda^2}{2\mu+\lambda} \underset{\sim}{\nabla} \operatorname{div} \underset{\sim}{\eta} - \lambda \underset{\sim}{\nabla} \rho = \underset{\sim}{l} \quad \text{in } \Omega, \\
&-\frac{\varepsilon^2}{3}\mu\Delta\rho + \lambda \operatorname{div} \underset{\sim}{\eta} + (2\mu+\lambda)\rho = l_3 \quad \text{in } \Omega, \\
&\underset{\sim}{\eta} = 0, \quad \rho = 0 \quad \text{on } \partial\Omega,
\end{aligned}
\tag{14}
$$

where

$$
\underset{\sim}{l} = \varepsilon^{-1}\left(\frac{1}{2}\underset{\sim}{f}^0 + \underset{\sim}{g}^0 + Q_{311} \underset{\sim}{\nabla} \phi^1_{bc}\right), \quad l_3 = \frac{1}{2}f^1_3 + g^1_3 - \varepsilon^{-1}Q_{333}\phi^1_{bc} + \frac{\varepsilon}{3}Q_{113}\Delta\phi^1_{bc}.
$$

After finding $\underset{\sim}{\eta}$ and ρ, the stress tensor and the electrical displacement are computable from

$$
\underset{\approx}{\sigma}^0 = \underset{\approx}{\underset{\approx}{A}}^{-1} \underset{\approx}{e}(\underset{\sim}{\eta}) + \frac{\lambda^2}{2\mu+\lambda} \operatorname{div} \underset{\sim}{\eta}\, \underset{\approx}{\delta} + \lambda\rho\, \underset{\approx}{\delta} + \varepsilon^{-1}Q_{311}\phi^1_{bc}\, \underset{\approx}{\delta}, \tag{15i}
$$

$$
\underset{\sim}{\sigma}^1 = \mu \underset{\sim}{\nabla} \rho + Q_{113}\varepsilon^{-1} \underset{\sim}{\nabla} \phi^1_{bc}, \tag{15ii}
$$

$$
\sigma^0_{33} = \lambda \operatorname{div} \underset{\sim}{\eta} + (2\mu+\lambda)\rho + \varepsilon^{-1}Q_{333}\phi^1_{bc}, \tag{15iii}
$$

$$
\underset{\sim}{D}^1 = Q_{113} \underset{\sim}{\nabla} \rho - \varepsilon^{-1} d \underset{\sim}{\nabla} \phi^1_{bc}, \quad D^0_3 = Q_{311} \operatorname{div} \underset{\sim}{\eta} + Q_{333}\rho - \varepsilon^{-1}d_{33}\phi^1_{bc}. \tag{15iv}
$$

For the bending part, $\underset{\sim}{\theta},\ \omega,\ \omega_2,\ \phi_2$ solve

$$
-\frac{\varepsilon^3}{3} \underset{\sim}{\operatorname{div}} \underset{\approx}{\underset{\approx}{A}}^{-1} \underset{\approx}{e}(\underset{\sim}{\theta}) + \varepsilon\mu(\underset{\sim}{\theta} - \underset{\sim}{\nabla}\omega) + \varepsilon^3 \frac{\lambda}{3} \underset{\sim}{\nabla} \left(-\frac{\lambda}{2\mu+\lambda} \operatorname{div} \underset{\sim}{\theta} + 3\omega_2\right) - \varepsilon^3(Q_{113} + Q_{311}) \underset{\sim}{\nabla} \phi_2 = \underset{\sim}{F}, \tag{16i}
$$

$$
\varepsilon\mu \operatorname{div}(\underset{\sim}{\theta} - \underset{\sim}{\nabla}\omega) - \varepsilon^3 Q_{113}\Delta\phi_2 = F_3, \tag{16ii}
$$

$$
-\frac{\varepsilon^5}{5}\mu\Delta\omega_2 + \varepsilon^3(2\mu+\lambda)\left(-\frac{\lambda}{2\mu+\lambda} \operatorname{div} \underset{\sim}{\theta} + 3\omega_2\right) + \frac{\varepsilon^5}{5}Q_{113}\Delta\phi_2 - 3\varepsilon^3 Q_{333}\phi_2 = F_4, \tag{16iii}
$$

$$
Q_{113} \operatorname{div}(-\underset{\sim}{\theta} + \underset{\sim}{\nabla}\omega) - Q_{311} \operatorname{div} \underset{\sim}{\theta} - \frac{\varepsilon^2}{5}Q_{113}\Delta\omega_2 + 3Q_{333}\omega_2 - \frac{6\varepsilon^2}{5} d\Delta\phi_2 + 3d_{33}\phi_2 = F_5, \tag{16iv}
$$

where

$$
\begin{aligned}
&\underset{\sim}{F} = -\frac{\varepsilon}{2}(\underset{\sim}{f}^1 + 2\underset{\sim}{g}^1) + \varepsilon Q_{113} \underset{\sim}{\nabla} \phi^0_{bc}, \quad F_3 = \frac{\varepsilon}{2}f^0_3 + g^0_3 + \varepsilon Q_{113}\Delta\phi^0_{bc}, \\
&F_4 = \frac{\varepsilon}{2}f^2_3 + \varepsilon^2 g^0_3, \quad F_5 = d\Delta\phi^0_{bc}.
\end{aligned}
$$

with the boundary conditions

$$
\underset{\sim}{\theta} = 0, \quad \omega = \omega_2 = 0, \quad Q_{113}\frac{\partial}{\partial n}\left[\omega - \frac{\varepsilon^2}{5}\omega_2\right] - \frac{6}{5}d\varepsilon^2\frac{\partial\phi_2}{\partial n} = d\frac{\partial\phi^0_{bc}}{\partial n} \quad \text{on } \partial\Omega.
$$

Given $\underset{\sim}{\theta},\ \omega,\ \omega_2$ and ϕ_2, the stress tensor and the electrical displacement are easily calculated as below:

$$\underset{\approx}{\sigma}^1 = -\underset{\approx}{\underset{\approx}{A}}^{-1}\,\underset{\approx}{e}(\underset{\sim}{\theta}) - \frac{\lambda^2}{2\mu+\lambda}\operatorname{div}\underset{\sim}{\theta}\,\underset{\approx}{\delta} + 3\lambda\omega_2\,\underset{\approx}{\delta} - 3Q_{311}\phi_2\,\underset{\approx}{\delta}, \tag{17i}$$

$$\underset{\sim}{\sigma}^0 = \mu(-\underset{\sim}{\theta} + \underset{\sim}{\nabla}\,\omega) + Q_{113}(\underset{\sim}{\nabla}\,\phi^0_{bc} + \varepsilon^2\,\underset{\sim}{\nabla}\,\phi_2), \tag{17ii}$$

$$\underset{\sim}{\sigma}^2 = \mu\,\underset{\sim}{\nabla}\,\omega_2 - Q_{113}\,\underset{\sim}{\nabla}\,\phi_2, \tag{17iii}$$

$$\sigma^1_{33} = -\lambda\operatorname{div}\theta - 3Q_{333}\phi_2 + 3(2\mu+\lambda)\omega_2, \tag{17iv}$$

$$\underset{\sim}{D}^0 = Q_{113}(-\underset{\sim}{\theta} + \underset{\sim}{\nabla}\,\omega) - d\,\underset{\sim}{\nabla}(\phi^0_{bc} + \varepsilon^2\phi_2), \tag{17v}$$

$$\underset{\sim}{D}^2 = Q_{113}\,\underset{\sim}{\nabla}\,\omega_2 + d\,\underset{\sim}{\nabla}\,\phi_2, \tag{17vi}$$

$$D_3 = -Q_{311}\operatorname{div}\underset{\sim}{\theta} + 3Q_{333}\omega_2 + 3d_{33}\phi_2. \tag{17vii}$$

3. Asymptotic consistency

Considering the sequence of plate problem parametrized by the thickness ε, it is possible to show that the three-dimensional solution converges in a proper sense to a solution of two-dimensional problems. This was done by Sène [1] for the piezoelectric problem, as we point out in the Introduction. We present here the limit equations.

After a proper scaling (22), the asymptotic limits of $\underline{u}^\epsilon$ and ϕ^ϵ are $\underline{u}_{KL}$ and ϕ_{KL}, where

$$\begin{gathered}\underline{u}_{KL}(\underline{x}) = \begin{pmatrix} \varepsilon\,\underset{\sim}{\zeta}(\underset{\sim}{x}) - x_3\,\underset{\sim}{\nabla}\,\zeta_3(\underset{\sim}{x}) \\ \zeta_3(\underset{\sim}{x}) \end{pmatrix}, \\ \phi_{KL}(\underline{x}) = \phi^0_{bc} + \varepsilon^{-1}x_3\phi^1_{bc} + \left(Q_{311} - \frac{\lambda}{\lambda+2\mu}Q_{333}\right)\frac{\varepsilon^2 - x_3^2}{2p_{33}}\Delta\zeta_3, \quad p_{33} = \frac{(Q_{333})^2}{\lambda+2\mu} + d_{33}.\end{gathered} \tag{18}$$

The function $\underset{\sim}{\zeta}$ solves

$$-\varepsilon\,\underset{\sim}{\operatorname{div}}\,\underset{\approx}{\underset{\approx}{A}}^{-1}\,\underset{\approx}{e}(\underset{\sim}{\zeta}) = \underset{\sim}{l} - \varepsilon^{-1}\frac{\lambda}{\lambda+2\mu}Q_{333}\,\underset{\sim}{\nabla}\,\phi^1_{bc} \quad \text{in } \Omega, \quad \underset{\sim}{\zeta} = 0 \quad \text{on } \partial\Omega, \tag{19}$$

and ζ_3 solves the biharmonic equation

$$\varepsilon^3 B\Delta^2\zeta_3 = \varepsilon\operatorname{div}\underset{\sim}{f}^1 + \varepsilon f^0_3 + \varepsilon\operatorname{div}\underset{\sim}{g}^1 + g^0_3 \quad \text{in}\Omega, \quad \zeta_3 = \frac{\partial\zeta_3}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{20}$$

where

$$B = \frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)} + \frac{2}{3p_{33}}\left(Q_{311} - \frac{\lambda}{\lambda+2\mu}Q_{333}\right)^2. \tag{21}$$

An important issue in dimensional reduction modeling is the *asymptotic consistency*, i.e., the relative modeling error, say in the $L^2(P)$ norm, should go to zero with ε. That means that the solution of the model should have the same asymptotic behavior as the original three-dimensional solution. Not all assumptions on the subspaces of $\underline{V}(P)$ etc., lead to consistent models. For an instance of this phenomenon see [22].

Theorems 1, 2 below state that the present models are consistent, i.e., as ε goes to zero, they converge to the same limit as the exact three-dimensional equations. A drawback of the arguments used in our proofs is that there are no convergence rates; see [23,9,7] where similar arguments are used.

To investigate the consistency, we make the following scaling assumptions on the loads [24]:

$$\begin{gathered}\underline{f}(\underline{x}) = \left(\varepsilon\,\underset{\sim}{\check{f}}(\underset{\sim}{x}, \varepsilon^{-1}x_3), \varepsilon^2\check{f}_3(\underset{\sim}{x}, \varepsilon^{-1}x_3)\right), \quad \underline{g}(\underline{x}) = \left(\varepsilon^2\,\underset{\sim}{\check{g}}(\underset{\sim}{x}, \varepsilon^{-1}x_3), \varepsilon^3\check{g}_3(\underset{\sim}{x}, \varepsilon^{-1}x_3)\right), \\ \phi_{bc}(\underline{x}) = \varepsilon^2\check{\phi}_{bc}(\underset{\sim}{x}, \varepsilon^{-1}x_3),\end{gathered} \tag{22}$$

where $\underline{\check{f}} : \Omega \times (-1,1) \to \mathbb{R}^3,\ \underline{\check{g}} : \Omega \times \{-1,1\} \to \mathbb{R}^3$, and $\check{\phi}_{bc} : \Omega \times \{-1,1\} \to \mathbb{R}$ are all ε-independent functions. Based on such assumptions, we gather that

$$\begin{gathered}\underset{\sim}{f}^0, \underset{\sim}{l}, l_3 \sim \varepsilon \quad F_5, \underset{\sim}{f}^1, f^0_3, \underset{\sim}{g}^0, \underset{\sim}{g}^1, \phi^0_{bc}, \phi^1_{bc} \sim \varepsilon^2, \quad \underset{\sim}{F}, F_3, f^1_3, g^0_3, g^1_3 \sim \varepsilon^3, \\ f^2_3 \sim \varepsilon^4, \quad F_4 \sim \varepsilon^5,\end{gathered} \tag{23}$$

Above, for a given function $g : \Omega \to \mathbb{R}$, we write $g \sim \varepsilon^i$ if $\varepsilon^{-i}g$ is independent of ε.

Theorem 1. *Assume that the plate is under a nontrivial pure stretching regime, that is,* $\underline{u}_{KL} = \varepsilon(\underset{\sim}{\zeta}, 0) \neq 0$. *Then, under the scaling assumptions* (22), *the following relative errors converge to zero:*

$$\lim_{\varepsilon\to 0} \frac{\| \underset{\sim}{\eta} - \varepsilon \underset{\sim}{\zeta} \|_{H^1(\Omega)}}{\| \varepsilon \underset{\sim}{\zeta} \|_{H^1(\Omega)}} = 0, \quad \lim_{\varepsilon\to 0} \frac{\| \underline{u}_{KL} - \underline{u}_S \|_{L^2(P)}}{\| \underline{u}_{KL} \|_{L^2(P)}} = 0, \tag{24}$$

where $\underline{u}_S = (\underset{\sim}{\eta}, x_3 \rho)$.

Proof. From (19), (23) we gather that ζ is ε-independent. Next, multiplying the first equation in (14) by $(2\mu+\lambda)\,\hat{\underset{\sim}{\eta}}$, the second by $(2\mu+\lambda)\hat{\rho}$, and integrating by parts we obtain the weak formulation

$$B_s\Big((\underset{\sim}{\eta}, \rho); (\hat{\underset{\sim}{\eta}}, \hat{\rho}) \Big) = L_s(\hat{\underset{\sim}{\eta}}, \hat{\rho}) \quad \text{for all } \hat{\underset{\sim}{\eta}} \in \underset{\sim}{H}_0^1(\Omega),\ \hat{\rho} \in H_0^1(\Omega), \tag{25}$$

where

$$\begin{aligned} B_s\Big((\underset{\sim}{\eta}, \rho); (\hat{\underset{\sim}{\eta}}, \hat{\rho}) \Big) &= (2\mu+\lambda) \int_\Omega \underset{\underset{\approx}{\approx}}{A}^{-1}\, \underset{\approx}{e}(\underset{\sim}{\eta}) : \underset{\approx}{e}(\hat{\underset{\sim}{\eta}})\, d\underset{\sim}{x} \\ &\quad + \frac{\varepsilon^2}{3}\mu(2\mu+\lambda) \int_\Omega \underset{\sim}{\nabla}\rho \cdot \underset{\sim}{\nabla}\hat{\rho}\, d\underset{\sim}{x} + \int_\Omega [\lambda \operatorname{div} \underset{\sim}{\eta} + (2\mu+\lambda)\rho][\lambda \operatorname{div} \hat{\underset{\sim}{\eta}} + (2\mu+\lambda)\hat{\rho}]\, d\underset{\sim}{x}, \end{aligned}$$

and $L_s(\hat{\underset{\sim}{\eta}}, \hat{\rho}) = (2\mu+\lambda) \int_\Omega \underset{\sim}{l} \cdot \hat{\underset{\sim}{\eta}} + l_3 \hat{\rho}\, d\underset{\sim}{x}$. Thus

$$\begin{aligned} B_s\Big((\underset{\sim}{\eta}, \rho); (\underset{\sim}{\eta}, \rho) \Big) &= (2\mu+\lambda) \int_\Omega \underset{\underset{\approx}{\approx}}{A}^{-1}\, \underset{\approx}{e}(\underset{\sim}{\eta}) : \underset{\approx}{e}(\underset{\sim}{\eta})\, d\underset{\sim}{x} + \frac{\varepsilon^2}{3}\mu(2\mu+\lambda) \int_\Omega |\underset{\sim}{\nabla}\rho|^2\, d\underset{\sim}{x} + \int_\Omega [\lambda \operatorname{div} \underset{\sim}{\eta} + (2\mu+\lambda)\rho]^2\, d\underset{\sim}{x} \\ &= \int_\Omega (2\mu+\lambda)\, \underset{\sim}{l} \cdot \underset{\sim}{\eta}\, d\underset{\sim}{x} + \int_\Omega l_3 [\lambda \operatorname{div} \underset{\sim}{\eta} + (2\mu+\lambda)\rho]\, d\underset{\sim}{x} - \int_\Omega l_3 \lambda \operatorname{div} \underset{\sim}{\eta}\, d\underset{\sim}{x}, \end{aligned}$$

and the stability estimate

$$\| \underset{\sim}{\eta} \|_{H^1(\Omega)} + \| \lambda \operatorname{div} \underset{\sim}{\eta} + (2\mu+\lambda)\rho \|_{L^2(\Omega)} \leqslant c\Big(\| \underset{\sim}{l} \|_{H^{-1}(\Omega)} + \| l_3 \|_{L^2(\Omega)} \Big)$$

holds. It also follows that

$$\begin{aligned} &\varepsilon^2 \|\rho\|^2_{H^1(\Omega)} \leqslant B_s\Big((\underset{\sim}{\eta}, \rho); (\underset{\sim}{\eta}, \rho) \Big) \leqslant c(\| \underset{\sim}{l} \|_{H^{-1}(\Omega)} + \| l_3 \|_{L^2(\Omega)})^2, \\ &\|(2\mu+\lambda)\rho\|_{L^2(\Omega)} \leqslant \| \lambda \operatorname{div} \underset{\sim}{\eta} + (2\mu+\lambda)\rho \|_{L^2(\Omega)} + \| \lambda \operatorname{div} \underset{\sim}{\eta} \|_{L^2(\Omega)} \leqslant c(\| \underset{\sim}{l} \|_{H^{-1}(\Omega)} + \| l_3 \|_{L^2(\Omega)}), \end{aligned}$$

and thus, from (23)

$$\| \underset{\sim}{\eta} \|_{H^1(\Omega)} \leqslant c\varepsilon, \quad \|\rho\|_{H^1(\Omega)} \leqslant c, \quad \|\rho\|_{L^2(\Omega)} \leqslant c\varepsilon. \tag{26}$$

Then, denoting by $\rightharpoonup$ the weak limit as $\varepsilon \to 0$, there exist $\bar{\eta} \in H_0^1(\Omega)$, $\bar{\rho} \in L^2(\Omega)$, and subsequences of η, ρ (that we still denote by $\underset{\sim}{\eta}$, ρ), such that $\varepsilon^{-1} \underset{\sim}{\eta} \rightharpoonup \underset{\sim}{\bar{\eta}}$ in $H_0^1(\Omega)$, and $\varepsilon^{-1}\rho \rightharpoonup \bar{\rho}$ in $L^2(\Omega)$. So, taking the limit $\varepsilon \to 0$ in (25), we gather that

$$\begin{aligned} &(2\mu+\lambda) \int_\Omega \underset{\underset{\approx}{\approx}}{A}^{-1}\, \underset{\approx}{e}(\underset{\sim}{\bar{\eta}}) : \underset{\approx}{e}(\hat{\underset{\sim}{\eta}})\, d\underset{\sim}{x} + \int_\Omega \Big[\lambda \operatorname{div} \underset{\sim}{\bar{\eta}} + (2\mu+\lambda)\bar{\rho} \Big] \Big[\lambda \operatorname{div} \hat{\underset{\sim}{\eta}} + (2\mu+\lambda)\hat{\rho} \Big]\, d\underset{\sim}{x} \\ &\qquad = (2\mu+\lambda) \int_\Omega \underset{\sim}{\bar{l}} \cdot \hat{\underset{\sim}{\eta}} + \bar{l}_3 \hat{\rho}\, d\underset{\sim}{x} \quad \text{for all } \hat{\underset{\sim}{\eta}} \in \underset{\sim}{H}_0^1(\Omega),\ \hat{\rho} \in H_0^1(\Omega), \end{aligned} \tag{27}$$

where, from (22),

$$\begin{aligned} \underset{\sim}{\bar{l}} &= \frac{1}{2} \int_{-1}^{1} \underset{\sim}{\check{f}}(\underset{\sim}{x}, x_3)\, dx_3 + \frac{1}{2} \Big[\underset{\sim}{\check{g}}(\underset{\sim}{x}, 1) + \underset{\sim}{\check{g}}(\underset{\sim}{x}, -1) \Big] + \frac{1}{2} Q_{311}\, \underset{\sim}{\nabla} \Big[\check{\phi}_{bc}(\underset{\sim}{x}, 1) - \check{\phi}_{bc}(\underset{\sim}{x}, -1) \Big], \\ \bar{l}_3 &= -\frac{1}{2} Q_{333} \Big[\check{\phi}_{bc}(\underset{\sim}{x}, 1) - \check{\phi}_{bc}(\underset{\sim}{x}, -1) \Big] \end{aligned}$$

are independent of ε. From (27), taking $\hat{\underset{\sim}{\eta}} = 0$, note that $\lambda \operatorname{div} \underset{\sim}{\bar{\eta}} + (2\mu+\lambda)\bar{\rho} = \bar{l}_3$, and thus

$$\int_\Omega \underset{\underset{\approx}{\approx}}{A}^{-1}\, \underset{\approx}{e}(\underset{\sim}{\bar{\eta}}) : \underset{\approx}{e}(\hat{\underset{\sim}{\eta}})\, d\underset{\sim}{x} = \int_\Omega \underset{\sim}{\bar{l}} \cdot \hat{\underset{\sim}{\eta}} - \frac{\lambda}{2\mu+\lambda} \bar{l}_3 \operatorname{div} \hat{\underset{\sim}{\eta}}\, d\underset{\sim}{x} \quad \text{for all } \hat{\underset{\sim}{\eta}} \in \underset{\sim}{H}_0^1(\Omega).$$

So, from (19) we conclude that $\underset{\sim}{\bar{\eta}} = \underset{\sim}{\zeta}$, and then $\varepsilon^{-1} \underset{\sim}{\eta} \rightharpoonup \underset{\sim}{\zeta}$.

Now, from standard arguments, since $\underset{\sim}{\zeta},\ \bar{\rho}$ are uniquely defined, it follows that not only a subsequence, but the whole sequence $\underset{\sim}{\eta},\ \rho$ converge to $\underset{\sim}{\zeta},\ \bar{\rho}$. Indeed, if that was not the case, then it would be possible to find a subsequence not converging to $\underset{\sim}{\zeta},\ \bar{\rho}$, and again extract a subsequence from the latter converging to $\underset{\sim}{\zeta},\ \bar{\rho}$, reaching a contradiction.

We next show that the weak convergences are actually strong. First, let δ be a fixed positive number, and $\chi \in H_0^1(\Omega)$ be a smooth cut-off function such that $\chi(\underset{\sim}{x}) = 1$ for $\operatorname{dist}(\underset{\sim}{x}, \partial\Omega) > \delta$, where $\operatorname{dist}(\underset{\sim}{x}, \partial\Omega)$ stands for the distance from $\underset{\sim}{x}$ to the boundary $\partial\Omega$. We also impose that $\lim_{\delta\to 0}\|1-\chi\|_{L^2(\Omega)} = 0$.

Observe that

$$\begin{aligned}(2\mu+\lambda)\int_\Omega \underset{\approx}{\underset{\approx}{A}}^{-1}\,\underset{\approx}{e}(\varepsilon^{-1}\underset{\sim}{\eta}-\underset{\sim}{\zeta}):\underset{\approx}{e}(\varepsilon^{-1}\underset{\sim}{\eta}-\underset{\sim}{\zeta})\,d\underset{\sim}{x}+\frac{\varepsilon^2}{3}\mu(2\mu+\lambda)\int_\Omega|\underset{\sim}{\nabla}(\varepsilon^{-1}\rho-\chi\bar{\rho})|^2\,d\underset{\sim}{x}\\ +\int_\Omega\left[\lambda\operatorname{div}(\varepsilon^{-1}\underset{\sim}{\eta}-\underset{\sim}{\zeta})+(2\mu+\lambda)(\varepsilon^{-1}\rho-\chi\bar{\rho})\right]^2 d\underset{\sim}{x} = B_s\left(\varepsilon^{-1}\underset{\sim}{\eta}-\underset{\sim}{\zeta},\varepsilon^{-1}\rho-\chi\bar{\rho};\varepsilon^{-1}\underset{\sim}{\eta}-\underset{\sim}{\zeta},\varepsilon^{-1}\rho-\chi\bar{\rho}\right)\\ =\varepsilon^{-1}L_s(\underset{\sim}{\eta}-\underset{\sim}{\zeta},\varepsilon^{-1}\rho-\chi\bar{\rho})-B_s\left(\underset{\sim}{\zeta},\chi\bar{\rho};\varepsilon^{-1}\underset{\sim}{\eta}-\underset{\sim}{\zeta},\varepsilon^{-1}\rho-\chi\bar{\rho}\right).\end{aligned}\tag{28}$$

Note now that

$$\varepsilon^{-1}L_s\left(\underset{\sim}{\eta}-\underset{\sim}{\zeta},\varepsilon^{-1}\rho-\chi\bar{\rho}\right)=\varepsilon^{-1}(2\mu+\lambda)\int_\Omega \underset{\sim}{l}\cdot(\varepsilon^{-1}\underset{\sim}{\eta}-\underset{\sim}{\zeta})+l_3(\varepsilon^{-1}\rho-\chi\bar{\rho})\,d\underset{\sim}{x}\to(2\mu+\lambda)\int_\Omega \bar{l}_3(1-\chi)\bar{\rho}\,d\underset{\sim}{x}\tag{29}$$

as $\varepsilon\to 0$. Next,

$$\begin{aligned}B_s\left(\underset{\sim}{\zeta},\chi\bar{\rho};\varepsilon^{-1}\underset{\sim}{\eta}-\underset{\sim}{\zeta},\varepsilon^{-1}\rho-\chi\bar{\rho}\right)&=(2\mu+\lambda)\int_\Omega \underset{\approx}{\underset{\approx}{A}}^{-1}\,\underset{\approx}{e}(\underset{\sim}{\zeta}):\underset{\approx}{e}(\varepsilon^{-1}\underset{\sim}{\eta}-\underset{\sim}{\zeta})\,d\underset{\sim}{x}+\frac{\varepsilon^2}{3}\mu(2\mu+\lambda)\int_\Omega\underset{\sim}{\nabla}(\chi\bar{\rho})\cdot\underset{\sim}{\nabla}(\varepsilon^{-1}\rho-\chi\bar{\rho})\,d\underset{\sim}{x}\\ &\quad+\int_\Omega\left[\lambda\operatorname{div}\underset{\sim}{\zeta}+(2\mu+\lambda)\chi\bar{\rho}\right]\left[\lambda\operatorname{div}(\varepsilon^{-1}\underset{\sim}{\eta}-\underset{\sim}{\zeta})+(2\mu+\lambda)(\varepsilon^{-1}\rho-\chi\bar{\rho})\right]d\underset{\sim}{x}\\ &\quad\to(2\mu+\lambda)\int_\Omega\left[\lambda\operatorname{div}\underset{\sim}{\zeta}+(2\mu+\lambda)\chi\bar{\rho}\right](1-\chi)\bar{\rho}\,d\underset{\sim}{x}\end{aligned}\tag{30}$$

as $\varepsilon\to 0$. Finally,

$$\|\varepsilon^{-1}\underset{\sim}{\eta}-\underset{\sim}{\zeta}\|_{H^1(\Omega)}+\|\varepsilon^{-1}\rho-\bar{\rho}\|_{L^2(\Omega)}\leqslant\|\varepsilon^{-1}\underset{\sim}{\eta}-\underset{\sim}{\zeta}\|_{H^1(\Omega)}+\|\varepsilon^{-1}\rho-\chi\bar{\rho}\|_{L^2(\Omega)}+\|\chi\bar{\rho}-\bar{\rho}\|_{L^2(\Omega)}.$$

From (28)–(30),

$$\lim_{\varepsilon\to 0}\left(\|\varepsilon^{-1}\underset{\sim}{\eta}-\underset{\sim}{\zeta}\|^2_{H^1(\Omega)}+\|\varepsilon^{-1}\rho-\bar{\rho}\|^2_{L^2(\Omega)}\right)\leqslant c(2\mu+\lambda)\int_\Omega\left[\lambda\operatorname{div}\underset{\sim}{\zeta}+(2\mu+\lambda)\chi\bar{\rho}+\bar{l}_3\right](1-\chi)\bar{\rho}\,d\underset{\sim}{x}+c\|(\chi-1)\bar{\rho}\|^2_{L^2(\Omega)}.$$

Note that the left hand side of the above expression does not depend on δ, and taking the limit as $\delta\to 0$, we have that

$$\lim_{\varepsilon\to 0}\left(\|\varepsilon^{-1}\underset{\sim}{\eta}-\underset{\sim}{\zeta}\|_{H^1(\Omega)}+\|\varepsilon^{-1}\rho-\bar{\rho}\|_{L^2(\Omega)}\right)=0,$$

and the strong convergences $\varepsilon^{-1}\underset{\sim}{\eta}\to\underset{\sim}{\zeta}$ in $H_0^1(\Omega)$ and $\rho\to\bar{\rho}$ in $L^2(\Omega)$ hold.

Finally, to check the relative convergence rates, it is enough to use that $\underset{\sim}{\zeta}$ does not depend on ε, and that

$$\|\underline{u}_{KL}\|^2_{L^2(P)}=\int_\Omega\int_{-\varepsilon}^{\varepsilon}|\varepsilon\underset{\sim}{\zeta}|^2\,dx_3\,d\underset{\sim}{x}\geqslant c\varepsilon^3.$$

Also,

$$\begin{aligned}\|\underline{u}_{KL}-\underline{u}_S\|^2_{L^2(P)}&=\int_\Omega\int_{-\varepsilon}^{\varepsilon}|\varepsilon\underset{\sim}{\zeta}-\underset{\sim}{\eta}|^2+x_3^2\rho^2\,dx_3\,d\underset{\sim}{x}=\int_\Omega 2\varepsilon|\varepsilon\underset{\sim}{\zeta}-\underset{\sim}{\eta}|^2+\frac{3}{2}\varepsilon^3\rho^2\,d\underset{\sim}{x}=2\varepsilon\|\varepsilon\underset{\sim}{\zeta}-\underset{\sim}{\eta}\|^2_{L^2(\Omega)}+\frac{3}{2}\varepsilon^3\|\rho\|^2_{L^2(\Omega)}\\ &\leqslant 2\varepsilon^3\|\underset{\sim}{\zeta}-\varepsilon^{-1}\underset{\sim}{\eta}\|^2_{L^2(\Omega)}+c\varepsilon^5\end{aligned}$$

where we used (26). Since $\|\underset{\sim}{\zeta}-\varepsilon^{-1}\underset{\sim}{\eta}\|_{L^2(\Omega)}\to 0$, the result follows. □

Remark. Although not stated in Theorem 1, the consistency of the electric potential is obvious since, in the pure stretching regime, the model approximation for ϕ^ϵ and its asymptotic limit ϕ_{KL} coincide, cf. (12), (18).

Theorem 2. *Assume that the plate is under a nontrivial pure bending regime, that is,* $\underline{u}_{KL}=(-x_3\underset{\sim}{\nabla}\zeta_3,\zeta_3)\neq 0$. *Let* $\phi_{KL}(\underline{x})=\phi^0_{bc}+c_1(\varepsilon^2-x_3^2)\Delta\zeta_3$, *where we define the constant* $c_1=[Q_{311}-\lambda Q_{333}/(\lambda+2\mu)]/(2p_{33})$. *Then, under the scaling assumptions (22), the relative error converges to zero:*

$$\lim_{\varepsilon\to 0}\frac{\|\omega-\zeta_3\|_{H^1(\Omega)}}{\|\zeta_3\|_{H^1(\Omega)}}=0,\quad \lim_{\varepsilon\to 0}\frac{\|\underset{\sim}{\theta}-\underset{\sim}{\nabla}\zeta_3\|_{H^1(\Omega)}}{\|\underset{\sim}{\nabla}\zeta_3\|_{H^1(\Omega)}}=0, \tag{31}$$

$$\lim_{\varepsilon\to 0}\frac{\|\underline{u}_{KL}-\underline{u}_B\|_{L^2(P)}}{\|\underline{u}_{KL}\|_{L^2(P)}}=0,\quad \lim_{\varepsilon\to 0}\frac{\|\phi_B-\phi_{KL}\|_{L^2(P)}}{\|\phi_{KL}\|_{L^2(P)}}=0, \tag{32}$$

where $\underline{u}_B(\underline{x})=\Big(-\underset{\sim}{\theta}(\underset{\sim}{x})x_3,\omega(\underset{\sim}{x})+\omega_2(\underset{\sim}{x})L_2(x_3)\Big)$*, and* $\phi_B(\underline{x})=\phi^0_{bc}(\underset{\sim}{x})+\phi_2(\underset{\sim}{x})(\varepsilon^2-L_2(x_3))$*.*

Proof. The weak formulation of the bending problem (16) is given by

$$B_b\Big((\underset{\sim}{\theta},\omega,\omega_2,\phi_2);(\hat{\underset{\sim}{\theta}},\hat{\omega},\hat{\omega}_2,\phi_2)\Big)=L_b(\hat{\underset{\sim}{\theta}},\hat{\omega},\hat{\omega}_2,\phi_2), \tag{33}$$

where the bilinear form

$$\begin{aligned}B_b\Big((\underset{\sim}{\theta},\omega,\omega_2,\phi_2);(\hat{\underset{\sim}{\theta}},\hat{\omega},\hat{\omega}_2,\phi_2)\Big)=\int_\Omega\Bigg\{&\frac{1}{3}\underset{\approx}{\underset{\approx}{A}}^{-1}\underset{\approx}{e}(\underset{\sim}{\theta}):\underset{\approx}{e}(\hat{\underset{\sim}{\theta}})+\varepsilon^{-2}\mu(\underset{\sim}{\theta}-\underset{\sim}{\nabla}\omega)\cdot\Big(\hat{\underset{\sim}{\theta}}-\underset{\sim}{\nabla}\hat{\omega}\Big)\\&+\frac{1}{3(2\mu+\lambda)}(-\lambda\operatorname{div}\underset{\sim}{\theta}+3(2\mu+\lambda)\omega_2\Big)\Big(-\lambda\operatorname{div}\hat{\underset{\sim}{\theta}}+3(2\mu+\lambda)\hat{\omega}_2\Big)\\&-(Q_{113}+Q_{311})\underset{\sim}{\nabla}\phi_2\cdot\hat{\underset{\sim}{\theta}}+Q_{113}\underset{\sim}{\nabla}\phi_2\cdot\underset{\sim}{\nabla}\hat{\omega}+\varepsilon^2\frac{\mu}{5}\underset{\sim}{\nabla}\omega_2\cdot\underset{\sim}{\nabla}\hat{\omega}_2\\&-\frac{\varepsilon^2}{5}Q_{113}\underset{\sim}{\nabla}\phi_2\cdot\underset{\sim}{\nabla}\hat{\omega}_2-3Q_{333}\phi_2\hat{\omega}_2-Q_{113}(-\underset{\sim}{\theta}+\underset{\sim}{\nabla}\omega)\cdot\underset{\sim}{\nabla}\hat{\phi}_2-Q_{311}\operatorname{div}\underset{\sim}{\theta}\,\hat{\phi}_2\\&+\frac{\varepsilon^2}{5}Q_{113}\underset{\sim}{\nabla}\omega_2\cdot\underset{\sim}{\nabla}\hat{\phi}_2+3Q_{333}\omega_2\hat{\phi}_2+\frac{6\varepsilon^2}{5}d\underset{\sim}{\nabla}\phi_2\cdot\underset{\sim}{\nabla}\hat{\phi}_2+3d_{33}\phi_2\hat{\phi}_2\Bigg\}d\underset{\sim}{x},\end{aligned}$$

and the linear form

$$L_b(\hat{\underset{\sim}{\theta}},\hat{\omega},\hat{\omega}_2,\hat{\phi}_2)=\int_\Omega\Big\{\varepsilon^{-3}\underset{\sim}{F}\cdot\hat{\underset{\sim}{\theta}}+\varepsilon^{-3}F_3\hat{\omega}+\varepsilon^{-3}F_4\hat{\omega}_2+F_5\hat{\phi}_2\Big\}d\underset{\sim}{x}+\int_{\partial\Omega}d\frac{\partial\phi^0_{bc}}{\partial n}\hat{\phi}_2\,d\underset{\sim}{x}.$$

Note that

$$\begin{aligned}B_b\Big((\underset{\sim}{\theta},\omega,\omega_2,\phi_2);(\underset{\sim}{\theta},\omega,\omega_2,\phi_2)\Big)=\int_\Omega\Bigg\{&\frac{1}{3}\underset{\approx}{\underset{\approx}{A}}^{-1}|\underset{\approx}{e}(\underset{\sim}{\theta})|^2+\varepsilon^{-2}\mu|\underset{\sim}{\theta}-\underset{\sim}{\nabla}\omega|^2+\frac{1}{3(2\mu+\lambda)}\Big(-\lambda\operatorname{div}\underset{\sim}{\theta}+3(2\mu+\lambda)\omega_2\Big)^2\\&+\varepsilon^2\frac{\mu}{5}|\underset{\sim}{\nabla}\omega_2|^2+\frac{6\varepsilon^2}{5}d|\underset{\sim}{\nabla}\phi_2|^2+3d_{33}\phi_2^2\Bigg\}d\underset{\sim}{x}.\end{aligned} \tag{34}$$

Thus,

$$\|\underset{\sim}{\theta}\|_{H^1(\Omega)}+\|-\lambda\operatorname{div}\underset{\sim}{\theta}+3(2\mu+\lambda)\omega_2\|_{L^2(\Omega)}+\|\varepsilon\omega_2\|_{H^1(\Omega)}+\|\varepsilon\underset{\sim}{\nabla}\phi_2\|_{L^2(\Omega)}+\|\phi_2\|_{L^2(\Omega)}\leqslant c.$$

Also,

$$\begin{aligned}&\|\varepsilon^{-1}(\underset{\sim}{\theta}-\underset{\sim}{\nabla}\omega)\|_{L^2(\Omega)}\leqslant L_b(\underset{\sim}{\theta},\omega,\omega_2,\phi_2)\leqslant c,\\&\|\underset{\sim}{\nabla}\omega\|_{L^2(\Omega)}\leqslant\|\varepsilon^{-1}(\underset{\sim}{\theta}-\underset{\sim}{\nabla}\omega)\|_{L^2(\Omega)}+\|\underset{\sim}{\theta}\|_{H^1(\Omega)}\leqslant c,\\&\|\omega_2\|_{L^2(\Omega)}\leqslant c\Big[\|-\lambda\operatorname{div}\underset{\sim}{\theta}+3(2\mu+\lambda)\omega_2\|_{L^2(\Omega)}+\|\underset{\sim}{\theta}\|_{H^1(\Omega)}\Big]\leqslant c.\end{aligned}$$

Then there exist $\bar{\underset{\sim}{\theta}}$, $\bar{\omega}$, $\bar{\underset{\sim}{\chi}}$, and subsequences of $\underset{\sim}{\theta}$, ω, $\varepsilon^{-1}(\underset{\sim}{\theta}-\underset{\sim}{\nabla}\omega)$ such that the following weak convergences hold:

$$\begin{aligned}&\underset{\sim}{\theta}\rightharpoonup\bar{\theta}\text{ in }H^1_0(\Omega),\quad\omega\rightharpoonup\bar{\omega}\text{ in }H^1_0(\Omega),\quad\varepsilon^{-1}(\underset{\sim}{\theta}-\underset{\sim}{\nabla}\omega)\rightharpoonup\bar{\underset{\sim}{\chi}}\text{ in }L^2(\Omega),\\&\omega_2\rightharpoonup\bar{\omega}_2\text{ in }L^2(\Omega),\quad\phi_2\rightharpoonup\bar{\phi}_2\text{ in }L^2(\Omega),\end{aligned}$$

and then $\bar{\underset{\sim}{\theta}}=\underset{\sim}{\nabla}\bar{\omega}$, and $\bar{\omega}\in H^2_0(\Omega)$. Note also that since $\varepsilon\underset{\sim}{\nabla}\omega_2$ and $\varepsilon\underset{\sim}{\nabla}\phi_2$ are bounded in $L^2(\Omega)$, then $\varepsilon^2\underset{\sim}{\nabla}\omega_2$ and $\varepsilon^2\underset{\sim}{\nabla}\phi_2$ both converge to zero.

It follows from (33) that

$$\int_\Omega\Big\{\Big(-\lambda\operatorname{div}\underset{\sim}{\theta}+3(2\mu+\lambda)\omega_2\Big)\hat{\omega}_2+\varepsilon^2\frac{\mu}{5}\underset{\sim}{\nabla}\omega_2\cdot\underset{\sim}{\nabla}\hat{\omega}_2-\frac{\varepsilon^2}{5}Q_{113}\underset{\sim}{\nabla}\phi_2\cdot\underset{\sim}{\nabla}\hat{\omega}_2-3Q_{333}\phi_2\hat{\omega}_2\Big\}d\underset{\sim}{x}=\varepsilon^{-3}\int_\Omega F_4\hat{\omega}_2\,d\underset{\sim}{x}$$

for all $\hat{\omega}_2\in H^1_0$. Thus, taking $\varepsilon\to 0$, we gather that

$$-\lambda\operatorname{div}\bar{\underset{\sim}{\theta}}+3(2\mu+\lambda)\bar{\omega}_2-3Q_{333}\bar{\phi}_2=0. \tag{35}$$

Similarly, from (33),

$$\int_\Omega \left\{ -Q_{113}(-\underset{\sim}{\theta}+\underset{\sim}{\nabla}\,\omega)\cdot\underset{\sim}{\nabla}\,\hat{\phi}_2 - Q_{311}\operatorname{div}\underset{\sim}{\theta}\,\hat{\phi}_2 + \frac{\varepsilon^2}{5}Q_{113}\,\underset{\sim}{\nabla}\,\omega_2\cdot\underset{\sim}{\nabla}\,\hat{\phi}_2 + 3Q_{333}\omega_2\hat{\phi}_2 + \frac{6\varepsilon^2}{5}d\,\underset{\sim}{\nabla}\,\phi_2\cdot\underset{\sim}{\nabla}\,\hat{\phi}_2 + 3d_{33}\phi_2\hat{\phi}_2 \right\} d\underset{\sim}{x}$$
$$= \int_\Omega F_5\hat{\phi}_2\,d\underset{\sim}{x} + \int_{\partial\Omega} d\frac{\partial\phi^0_{bc}}{\partial n}\hat{\phi}_2\,d\underset{\sim}{x},$$

for all $\hat{\phi}_2 \in H^1_0$. Thus, taking $\varepsilon \to 0$,

$$-Q_{311}\operatorname{div}\underset{\sim}{\bar{\theta}} + 3Q_{333}\bar{\omega}_2 + 3d_{33}\bar{\phi}_2 = 0. \tag{36}$$

Then, from (35), (36),

$$\begin{aligned} \bar{\phi}_2 &= \frac{1}{3p_{33}}\left(Q_{311} - Q_{333}\frac{\lambda}{2\mu+\lambda}\right)\operatorname{div}\bar{\theta}, \\ \bar{\omega}_2 &= \frac{1}{3(2\mu+\lambda)}\left[\lambda + \frac{Q_{333}}{p_{33}}\left(Q_{311} - Q_{333}\frac{\lambda}{2\mu+\lambda}\right)\right]\operatorname{div}\bar{\theta}, \end{aligned} \tag{37}$$

where p_{33} is defined in (18).

Now, from (33),

$$\int_\Omega \left\{ \frac{1}{3}\underset{\approx}{\underset{\approx}{A}}^{-1}\,\underset{\approx}{e}(\underset{\sim}{\theta}) : \underset{\approx}{e}(\underset{\sim}{\hat{\theta}}) + \varepsilon^{-2}\mu(\underset{\sim}{\theta} - \underset{\sim}{\nabla}\,\omega)\cdot(\underset{\sim}{\hat{\theta}} - \underset{\sim}{\nabla}\,\hat{\omega}) - \frac{\lambda}{3(2\mu+\lambda)}\left(-\lambda\operatorname{div}\underset{\sim}{\theta} + 3(2\mu+\lambda)\omega_2\right)\operatorname{div}\underset{\sim}{\hat{\theta}} \right.$$
$$\left. -\,(Q_{113}+Q_{311})\,\underset{\sim}{\nabla}\,\phi_2\cdot\underset{\sim}{\hat{\theta}} + Q_{113}\,\underset{\sim}{\nabla}\,\phi_2\cdot\underset{\sim}{\nabla}\,\hat{\omega} \right\} d\underset{\sim}{x} = \int_\Omega \varepsilon^{-3}\left\{\underset{\sim}{F}\,\underset{\sim}{\hat{\theta}} + F_3\hat{\omega}\right\} d\underset{\sim}{x},$$

for all $\underset{\sim}{\hat{\theta}} \in \underset{\sim}{H}^1_0(\Omega)$, $\hat{\omega} \in H^2_0(\Omega)$. Taking $\underset{\sim}{\hat{\theta}} = \underset{\sim}{\nabla}\,\hat{\omega}$, where $\hat{\omega} \in H^2_0(\Omega)$, we gather that

$$\int_\Omega \left\{ \frac{1}{3}\underset{\approx}{\underset{\approx}{A}}^{-1}\,\underset{\approx}{e}(\underset{\sim}{\theta}) : \underset{\approx}{e}(\underset{\sim}{\nabla}\,\hat{\omega}) - \frac{\lambda}{3(2\mu+\lambda)}\left(-\lambda\operatorname{div}\underset{\sim}{\theta} + 3(2\mu+\lambda)\omega_2\right)\Delta\hat{\omega} + Q_{311}\phi_2\cdot\Delta\hat{\omega} \right\} d\underset{\sim}{x} = \int_\Omega \varepsilon^{-3}\left\{-\operatorname{div}\underset{\sim}{F} + F_3\right\}\hat{\omega}\,d\underset{\sim}{x},$$

for all $\hat{\omega} \in H^2_0(\Omega)$. Making $\varepsilon \to 0$, we gather that

$$\int_\Omega \left\{ \frac{1}{3}\underset{\approx}{\underset{\approx}{A}}^{-1}\,\underset{\approx}{e}(\underset{\sim}{\nabla}\,\bar{\omega}) : \underset{\approx}{e}(\underset{\sim}{\nabla}\,\hat{\omega}) - \frac{\lambda}{3(2\mu+\lambda)}(-\lambda\Delta\bar{\omega} + 3(2\mu+\lambda)\bar{\omega}_2)\Delta\hat{\omega} + Q_{311}\bar{\phi}_2\Delta\hat{\omega} \right\} d\underset{\sim}{x} = \int_\Omega \overline{F}\hat{\omega}\,d\underset{\sim}{x},$$

where

$$\overline{F} = \frac{1}{2}\left\{ \int_{-1}^{1}\left[\check{f}_3(\underline{x}) + \operatorname{div}\underset{\sim}{\check{f}}(\underline{x})x_3\right]dx_3 + \operatorname{div}\left[\underset{\sim}{\check{g}}(\underset{\sim}{x},1) - \underset{\sim}{\check{g}}(\underset{\sim}{x},-1)\right] + \left[\check{g}_3(\underset{\sim}{x},\varepsilon) + \check{g}_3(\underset{\sim}{x},-\varepsilon)\right] \right\}.$$

Using (37),

$$\int_\Omega \left\{ \frac{1}{3}\underset{\approx}{\underset{\approx}{A}}^{-1}\,\underset{\approx}{e}(\underset{\sim}{\nabla}\,\bar{\omega}) : \underset{\approx}{e}(\underset{\sim}{\nabla}\,\hat{\omega}) + \frac{1}{3p_{33}}\left(Q_{311} - Q_{333}\frac{\lambda}{2\mu+\lambda}\right)^2\Delta\bar{\omega}\Delta\hat{\omega} \right\} d\underset{\sim}{x} = \int_\Omega \overline{F}\hat{\omega}\,d\underset{\sim}{x},$$

The above equation is simply the weak form of

$$\frac{1}{3}\operatorname{div}\underset{\sim}{\operatorname{div}}\underset{\approx}{\underset{\approx}{A}}^{-1}\,\underset{\approx}{e}(\underset{\sim}{\nabla}\,\bar{\omega}) + \frac{1}{3p_{33}}\left(Q_{311} - Q_{333}\frac{\lambda}{2\mu+\lambda}\right)^2\Delta^2\bar{\omega} = \overline{F}, \tag{38}$$

and it follows from the identity

$$\operatorname{div}\underset{\sim}{\operatorname{div}}\underset{\approx}{\underset{\approx}{A}}^{-1}\,\underset{\approx}{e}(\underset{\sim}{\nabla}\,\bar{\omega}) = \frac{4\mu(\mu+\lambda)}{2\mu+\lambda}\Delta^2\bar{\omega},$$

that $\bar{\omega} = \zeta_3$. Again, as in the stretching case, since $\bar{\omega}$, $\bar{\phi}$ are uniquely defined, the whole sequences ω, ϕ converge weakly to $\bar{\omega}$, $\bar{\phi}$.

We next prove that the strong convergence. As in the proof of Theorem 1, for $\delta > 0$ let $\chi \in H^1_0(\Omega)$ be a smooth cut-off function where $\chi(\underset{\sim}{x}) = 1$ for $\operatorname{dist}(\underset{\sim}{x}, \partial\Omega) > \delta$. Observe that

$$B\Big((\underset{\sim}{\theta} - \underset{\sim}{\bar{\theta}}, \omega - \bar{\omega}, \omega_2 - \chi\bar{\omega}_2, \phi_2 - \bar{\phi}_2); (\underset{\sim}{\theta} - \underset{\sim}{\bar{\theta}}, \omega - \bar{\omega}, \omega_2 - \chi\bar{\omega}_2, \phi_2 - \bar{\phi}_2)\Big)$$
$$= L(\underset{\sim}{\theta} - \underset{\sim}{\bar{\theta}}, \omega - \bar{\omega}, \omega_2 - \chi\bar{\omega}_2, \phi_2 - \bar{\phi}_2) - B\Big((\underset{\sim}{\bar{\theta}}, \bar{\omega}, \chi\bar{\omega}_2, \bar{\phi}_2); (\underset{\sim}{\theta} - \underset{\sim}{\bar{\theta}}, \omega - \bar{\omega}, \omega_2 - \bar{\omega}_2, \phi_2 - \bar{\phi}_2)\Big).$$

Now we note that

$$\begin{aligned}&B\Big((\underset{\sim}{\bar\theta},\bar\omega,\chi\bar\omega_2,\bar\phi_2);(\underset{\sim}{\theta}-\underset{\sim}{\bar\theta},\omega-\bar\omega,\omega_2-\chi\bar\omega_2,\phi_2-\bar\phi_2)\Big)=\int_\Omega\Bigg\{\frac{1}{3}\underset{\approx}{\underset{\approx}{A}}^{-1}\underset{\approx}{e}(\underset{\sim}{\bar\theta}):\underset{\approx}{e}(\underset{\sim}{\theta}-\underset{\sim}{\bar\theta})+\frac{1}{3(2\mu+\lambda)}\Big(-\lambda\mathrm{div}\underset{\sim}{\bar\theta}+3(2\mu+\lambda)\chi\bar\omega_2\Big)\\
&\Big(-\lambda\mathrm{div}(\underset{\sim}{\theta}-\underset{\sim}{\bar\theta})+3(2\mu+\lambda)(\omega_2-\chi\bar\omega_2)\Big)\\
&\quad+(Q_{113}+Q_{311})\bar\phi_2\mathrm{div}(\underset{\sim}{\theta}-\underset{\sim}{\bar\theta})+Q_{113}\underset{\sim}{\nabla}\bar\phi_2\cdot\underset{\sim}{\nabla}(\omega-\bar\omega)+\varepsilon^2\frac{\mu}{5}\underset{\sim}{\nabla}(\chi\bar\omega_2)\cdot\Big(\underset{\sim}{\nabla}\omega_2-\underset{\sim}{\nabla}(\chi\bar\omega_2)\Big)\\
&\quad-\frac{\varepsilon^2}{5}Q_{113}\underset{\sim}{\nabla}\bar\phi_2\cdot\Big(\underset{\sim}{\nabla}\omega_2-\underset{\sim}{\nabla}(\chi\bar\omega_2)\Big)-3Q_{333}\bar\phi_2(\omega_2-\chi\bar\omega_2)-Q_{113}\mathrm{div}\underset{\sim}{\bar\theta}(\phi_2-\bar\phi_2)-Q_{113}\underset{\sim}{\nabla}\bar\omega\cdot(\underset{\sim}{\nabla}\phi_2-\underset{\sim}{\nabla}\bar\phi_2)\\
&\quad-Q_{311}\mathrm{div}\underset{\sim}{\bar\theta}(\phi_2-\bar\phi_2)+\frac{\varepsilon^2}{5}Q_{113}\underset{\sim}{\nabla}(\chi\bar\omega_2)\cdot(\underset{\sim}{\nabla}\phi_2-\underset{\sim}{\nabla}\bar\phi_2)+3Q_{333}\chi\bar\omega_2(\phi_2-\bar\phi_2)+\frac{6\varepsilon^2}{5}d\underset{\sim}{\nabla}\bar\phi_2\cdot(\underset{\sim}{\nabla}\phi_2-\underset{\sim}{\nabla}\bar\phi_2)\\
&\quad+3d_{33}\bar\phi_2(\phi_2-\bar\phi_2)\Bigg\}d\underset{\sim}{x}\to\int_\Omega\Big\{\Big(-\lambda\mathrm{div}\underset{\sim}{\bar\theta}+3(2\mu+\lambda)\chi\bar\omega_2\Big)(1-\chi)\bar\omega_2-3Q_{333}\bar\phi_2(1-\chi)\bar\omega_2\Big\}d\underset{\sim}{x}\end{aligned}\tag{39}$$

as $\varepsilon\to 0$. To check the above convergence, it is necessary to use the weak convergences shown before, and also the fact that $\varepsilon\underset{\sim}{\nabla}\omega_2$ and $\varepsilon\underset{\sim}{\nabla}\phi_2$ are uniformly bounded, and thus $\varepsilon^2\underset{\sim}{\nabla}\omega_2$ and $\varepsilon^2\underset{\sim}{\nabla}\phi_2$ converge to zero. It is also necessary to see that, integrating by parts,

$$\int_\Omega\underset{\sim}{\nabla}\bar\omega\cdot(\underset{\sim}{\nabla}\phi_2-\underset{\sim}{\nabla}\bar\phi_2)\,d\underset{\sim}{x}=-\int_\Omega\Delta\bar\omega(\phi_2-\bar\phi_2)\,d\underset{\sim}{x}\to 0,$$

where we also used that $\bar\omega\in H_0^2(\Omega)$.

Now, from (34), Korn's inequality and (39),

$$\lim_{\varepsilon\to 0}\Big(\|\underset{\sim}{\theta}-\underset{\sim}{\bar\theta}\|_{H^1(\Omega)}^2+\|\omega-\bar\omega\|_{H^1(\Omega)}^2\Big)\leqslant\int_\Omega\Big\{\Big(-\lambda\mathrm{div}\underset{\sim}{\bar\theta}+3(2\mu+\lambda)\chi\bar\omega_2\Big)(1-\chi)\bar\omega_2-3Q_{333}\bar\phi_2(1-\chi)\bar\omega_2\Big\}d\underset{\sim}{x}.$$

Taking the limit as $\delta\to 0$ in the above inequality, we obtain that the convergences $\underset{\sim}{\theta}\to\underset{\sim}{\bar\theta}$, and $\omega\to\bar\omega$ are strong. Since $\underset{\sim}{\bar\theta}=\underset{\sim}{\nabla}\bar\omega$ and $\bar\omega=\zeta_3$, we obtain (31).

We now proceed to prove (32). Using that ζ_3 does not depend on ε, then

$$\|\underline{u}_{KL}\|_{L^2(P)}^2=\int_\Omega\int_{-\varepsilon}^{\varepsilon}|x_3\underset{\sim}{\nabla}\zeta_3|^2+|\zeta_3|^2\,dx_3\,d\underset{\sim}{x}=\frac{2}{3}\varepsilon^3\|\underset{\sim}{\nabla}\zeta_3\|_{L^2(\Omega)}^2+2\varepsilon\|\zeta_3\|_{L^2(\Omega)}^2\geqslant c\varepsilon.$$

Also,

$$\begin{aligned}\|\underline{u}_{KL}-\underline{u}_B\|_{L^2(P)}^2&=\int_\Omega\int_{-\varepsilon}^{\varepsilon}|-x_3(\underset{\sim}{\nabla}\zeta_3-\underset{\sim}{\phi})|^2+(\zeta_3-\omega-\omega_2L_2(x_3))^2\,dx_3\,d\underset{\sim}{x}\\&=\frac{2}{3}\varepsilon^3\|\underset{\sim}{\nabla}\zeta_3-\underset{\sim}{\phi}\|_{L^2(\Omega)}^2+2\varepsilon\|\zeta_3-\omega\|_{L^2(\Omega)}^2+\frac{2}{5}\varepsilon^5\|\omega_2\|_{L^2(\Omega)}^2\leqslant 2\varepsilon\|\zeta_3-\omega\|_{L^2(\Omega)}^2+c\varepsilon^3.\end{aligned}$$

Since $\|\zeta_3-\omega\|_{L^2(P)}\to 0$, the first limit in (31) follows.

Finally, to estimate $\|\phi_B-\phi_{KL}\|_{L^2(P)}/\|\phi_{KL}\|_{L^2(P)}$, note first that $\|\phi_{KL}\|_{L^2(P)}\geqslant c\varepsilon^{5/2}$. Then,

$$\phi_B(\underline{x})-\phi_{KL}(\underline{x})=\left[\frac{3}{2}\phi_2(\underset{\sim}{x})-c_1\Delta\zeta_3(\underset{\sim}{x})\right](\varepsilon^2-x_3^2),$$

and

$$\|\phi_B-\phi_{KL}\|_{L^2(P)}^2=\int_{-\varepsilon}^{\varepsilon}(\varepsilon^2-x_3^2)^2\,dx_3\int_\Omega\left[\frac{3}{2}\phi_2(\underset{\sim}{x})-c_1\Delta\zeta_3(\underset{\sim}{x})\right]^2d\underset{\sim}{x}\leqslant c\varepsilon^5\left\|\frac{3}{2}\phi_2-c_1\Delta\zeta_3\right\|_{L^2(\Omega)}^2.$$

Finally, since $(3/2)\bar\phi_2=c_1\Delta\zeta_3$, we have

$$\frac{\|\phi_B-\phi_{KL}\|_{L^2(P)}}{\|\phi_{KL}\|_{L^2(P)}}\leqslant c\left\|\frac{3}{2}\phi_2-c_1\Delta\zeta_3\right\|_{L^2(\Omega)}=c\|\phi_2-\bar\phi_2\|_{L^2(\Omega)},$$

and the the second limit in (31) follows since $\lim_{\varepsilon\to 0}\phi_2=\bar\phi_2$ in $L^2(\Omega)$. □

Although the above theorem show that (16) yields a "correct model", in the asymptotic sense, its formulation is not as simple as one would hope for due to the many unknowns involved. It is desirable to obtain a further reduced model, i.e., a system of equations only in terms of ω and $\underset{\sim}{\theta}$, as in Reissner–Mindlin models, and only afterwards compute the other quantities of interest. And indeed, it is somewhat straightforward [25,23] to show that

$$\begin{aligned} &-\frac{\varepsilon^3}{3}\,\underset{\sim}{\operatorname{div}}\,\underset{\approx}{\underset{\approx}{A}}^{-1}\,\underset{\approx}{e}(\underset{\sim}{\theta}) + \varepsilon\mu(\underset{\sim}{\theta} - \underset{\sim}{\nabla}\,\omega) - c_2\varepsilon^3\,\underset{\sim}{\nabla}\,\operatorname{div}\theta = \underset{\sim}{F}, \\ &2\varepsilon\mu\operatorname{div}(\underset{\sim}{\theta} - \underset{\sim}{\nabla}\,\omega) = F_3, \end{aligned} \tag{40}$$

where

$$c_2 = \frac{1}{3p_{33}}\left(Q_{311} - Q_{333}\,\frac{\lambda}{2\mu+\lambda}\right)^2,$$

yields a consistent model. One could also define

$$\begin{aligned} \omega_2 &= \frac{1}{3(2\mu+\lambda)}\left[\lambda + \frac{Q_{333}}{p_{33}}\left(Q_{311} - Q_{333}\,\frac{\lambda}{2\mu+\lambda}\right)\right]\operatorname{div}\theta, \\ \phi_2 &= \frac{1}{3p_{33}}\left(Q_{311} - Q_{333}\,\frac{\lambda}{2\mu+\lambda}\right)\operatorname{div}\theta. \end{aligned}$$

Of course such procedure is *ad hoc*, and thus not fully satisfactory. Nonetheless, it yields a consistent model that is as simple as the usual Reissner–Mindlin models for elasticity, what is good news.

4. Discussion

The holy grail of dimensional reduction is to obtain the simplest possible model that is "good enough" for most application, and computationally feasible. Our criteria for "good enough" is asymptotic consistency, and here the references [1,2] play a leading role. It is thus wise to ask if it is possible to derive a simpler model that is asymptotically consistent through variational arguments.

A positive answer to the above question would be good news. Indeed, Alessandrini et al. [12] obtains a simpler linearly elastic plate model, denoted $\mathrm{HR}_1(1)$. For sure, such work considers no piezoelectricity, but the stretching equations involves only a two-dimensional vector unknown, instead of the coupled system (14). And for the bending part, the $\mathrm{HR}_1(1)$ model of [12] requires solving for three scalar unknowns instead of the four unknowns required in (16) (disregarding the electrical potential contribution). Unfortunately, for the present piezoelectric problem, such simpler assumptions on the load does not lead to consistent models.

Using the notation of [12], our model is closer to the more complicated $\mathrm{HR}_3(1)$, which is the simplest consistent minimum energy model. In the case of bending of linearly elastic plates, this is the $(1,1,2)$ model of Babuska and Li [15,26]. See also [17] for a complete description of such models.

To be fair with the present derivation, it led to the intriguing system (40), which we recall, is consistent. Is there a variational way to derive such system without *ad hoc* considerations? How good is this model, i.e., what is the convergence rate of its solutions to the exact, three-dimensional solutions?

An alternative to derive models is to use variants of Hellinger–Reisner principle, as in [12,27], but it is not so clear how to do so for piezoelectric materials. As far as we know, except for [22], there is no general result that predicts whether a hierarchical model will be consistent or not. So, it seems that developing simple consistent models is a matter of trying to guess the right spaces and formulation, and check, a posteriori, if the resulting equations are consistent.

Another point worth discussing concerns the boundary layers present in both the solutions of the hierarchical and the exact three-dimensional problem. As can be seen in [25,28] for the elasticity case, both solutions have a nontrivial structure, and its not simple to compare them. For the Poisson case, this was investigated in [13,16,17]. It is interesting to see that, in the asymptotic limit, the boundary layers "disappear", casting doubts on how well asymptotic limit models can perform—see [29] for further aspects of this discussion.

Another consequence of the presence of boundary layers is on what norms convergence holds. For instance, even assuming the functions in the statement of Theorems 1, 2 are smooth, if higher norms were considered, the boundary layer part of the solution would prevail, spoiling any hope of convergence. Such boundary layer influence can be eliminated by considering interior estimates. This was done in [30] for elastic plates, and in [13,16,17] for the Poisson problem.

Finally, we understand that our choices for rigidity, dielectric and piezoelectric tensors are not as general as would be desirable to model "real life" materials. But we hope that even in this simpler setting, our modeling efforts shed some light and help in the quest of developing provably good plate models.

Acknowledgements

The first author gratefully acknowledges the partial support of CNPq, Grant No. 308670/2007–8, and also FAPERJ, Grant No. E-26/102.255/2009.

Appendix A

In this Appendix we provide the main steps to derive (14), (15), (16), and (17) from (9), for $p = 1$.
From (11) we gather that

$$\underset{\approx}{\underset{\approx}{A}}\,\underset{\approx}{\tau} = \frac{1}{2\mu}\left[\underset{\approx}{\tau} - \frac{\lambda}{2\mu+3\lambda}\operatorname{tr}(\underset{\approx}{\tau})\,\underset{\approx}{\delta}\right], \quad \underline{\underline{\underline{A}}}\,\underline{\underline{\tau}} = \begin{pmatrix} \underset{\approx}{\underset{\approx}{A}}\,\underset{\approx}{\tau} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\tau_{33}\,\underset{\approx}{\delta} & \frac{1}{2\mu}\,\underset{\sim}{\tau} \\ \frac{1}{2\mu}\,\underset{\sim}{\tau}^t & \frac{\mu+\lambda}{\mu(2\mu+3\lambda)}\tau_{33} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\operatorname{tr}(\underset{\approx}{\tau}) \end{pmatrix}.$$

A.1. A stretching model

Assume that (12), (13) holds, and let

$$\underline{v} = \begin{pmatrix} \underset{\sim}{v} \\ v_3 x_3 \end{pmatrix}, \quad \underline{\underline{\tau}} = \begin{pmatrix} \underset{\approx}{\tau} & \underset{\sim}{\tau}^1 x_3 \\ (\underset{\sim}{\tau}^1)^t x_3 & \tau_{33} \end{pmatrix}, \quad \underline{H} = \begin{pmatrix} \underset{\sim}{H} x_3 \\ H_3 \end{pmatrix},$$

where $\underline{v} \in \underline{V}(P,1)$, $\underline{\underline{\tau}} \in \underline{\underline{L}}^2(P,1)$ and $\underline{H} \in \underline{L}^2(P,1)$. From (12), have

$$\underline{\nabla\phi}\,\underline{\underline{Q}} = \begin{pmatrix} Q_{311}\varepsilon^{-1}\phi^1_{bc}\,\underset{\approx}{\delta} & Q_{113}\varepsilon^{-1}x_3\,\underset{\sim}{\nabla}\,\phi^1_{bc} \\ Q_{113}\varepsilon^{-1}x_3\,\underset{\sim}{\nabla}^t\phi^1_{bc} & Q_{333}\varepsilon^{-1}\phi^1_{bc} \end{pmatrix}.$$

Using the first constitutive Eq. (5i) we integrate with respect to x_3 and find

$$\begin{aligned} &\underset{\approx}{\underset{\approx}{A}}\,\underset{\approx}{\sigma}^0 - \frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma^0_{33}\,\underset{\approx}{\delta} - \underset{\approx}{e}(\underset{\sim}{\eta}) - \varepsilon^{-1}Q_{311}\phi^1_{bc}\,\underset{\approx}{\underset{\approx}{\underset{\approx}{A}}}\,\underset{\approx}{\delta} + \varepsilon^{-1}Q_{333}\phi^1_{bc}\frac{\lambda}{2\mu(2\mu+3\lambda)}\,\underset{\approx}{\delta} = 0, \frac{2\varepsilon^3}{3}\frac{1}{\mu}\underset{\sim}{\sigma}^1 - \frac{2\varepsilon^3}{3}\,\underset{\sim}{\nabla}\,\rho - \frac{2Q_{113}}{3\mu}\varepsilon^2\,\underset{\sim}{\nabla}\,\phi^1_{bc} \\ &= 0, -\varepsilon\frac{\lambda}{\mu(2\mu+3\lambda)}\operatorname{tr}(\underset{\approx}{\sigma}^0) + 2\varepsilon\frac{\mu+\lambda}{\mu(2\mu+3\lambda)}\sigma^0_{33} - 2\varepsilon\rho + \frac{2\lambda Q_{311}}{\mu(2\mu+3\lambda)}\phi^1_{bc} - 2Q_{333}\phi^1_{bc}\frac{\mu+\lambda}{\mu(2\mu+3\lambda)} = 0. \end{aligned}$$

Thus, (15i), (15ii), (15iii) follows. Similarly, (5ii) yields (15iv).
Analogously to the constitutive equation, integrating the equilibrium equation in x_3, we find the equilibrium condition

$$\begin{aligned} &\int_\Omega 2\varepsilon\underset{\approx}{\sigma}^0 : \underset{\approx}{e}(\underset{\sim}{v}) + \frac{2\varepsilon^3}{3}\underset{\sim}{\sigma}^1\cdot\underset{\sim}{\nabla}\,v_3 + 2\varepsilon\sigma^0_{33}v_3\,d\underset{\sim}{x} = \int_\Omega(\underset{\sim}{f}^0 + 2\underset{\sim}{g}^0)\cdot\underset{\sim}{v} + (\varepsilon f^1_3 + 2\varepsilon g^1_3)v^1_3\,d\underset{\sim}{x}, \\ &\text{for all } \underset{\sim}{v} \in \underset{\sim}{H}^1_0(\Omega) \text{ and all } v_3 \in H^1_0(\Omega). \end{aligned}$$

Hence, from (15i), (15ii), (15iii), we obtain (14).

A.2. A bending model

We assume again (12), (13), and that

$$\underline{v} = \begin{pmatrix} x_3\,\underset{\sim}{v} \\ v^0_3 + v^2_3 L_2(x_3) \end{pmatrix}, \quad \underline{\underline{\tau}} = \begin{pmatrix} \underset{\approx}{\tau}x_3 & \underset{\sim}{\tau}^0 + \underset{\sim}{\tau}^2 L_2(x_3) \\ \left[\underset{\sim}{\tau}^0 + \underset{\sim}{\tau}^2 L_2(x_3)\right]^t & \tau_{33}x_3 \end{pmatrix},$$

$$\underline{H} = \begin{pmatrix} \underset{\sim}{H}^0 + \underset{\sim}{H}^2 L_2(x_3) \\ H_3 x_3 \end{pmatrix}, \quad \psi = (\varepsilon^2 - L_2)\psi^2,$$

where $\underline{v} \in \underline{V}(P,1)$, $\underline{\underline{\tau}} \in \underline{\underline{L}}^2(P,1)$, $\underline{H} \in \underline{L}^2(P,1)$, and $\psi \in \Psi_0(P,1)$. From (12),

$$\underline{\nabla\phi}\,\underline{\underline{Q}} = \begin{pmatrix} -3x_3Q_{311}\phi_2\,\underset{\approx}{\delta} & \left[\underset{\sim}{\nabla}\,\phi^0_{bc} + (\varepsilon^2 - L_2)\,\underset{\sim}{\nabla}\,\phi_2\right]Q_{113} \\ \left[\underset{\sim}{\nabla}\,\phi^0_{bc} + (\varepsilon^2 - L_2)\,\underset{\sim}{\nabla}\,\phi_2\right]^t Q_{113} & -3x_3Q_{333}\phi_2 \end{pmatrix},$$

$$\underline{\underline{\underline{Q}}}\;\underline{\underline{e}}(\underline{u}) = \begin{pmatrix} Q_{113}(-\theta + \underset{\sim}{\nabla}\,\omega + \underset{\sim}{\nabla}\,\omega_2 L_2) \\ -Q_{311}x_3\mathrm{div}\theta + Q_{333}\omega_2 L_2' \end{pmatrix}.$$

Integrating (5i) in the transverse direction we gather

$$\begin{aligned}
&\frac{2\varepsilon^3}{3}\underset{\underset{\approx}{\approx}}{A}\,\underset{\approx}{\sigma}^1 - \frac{\varepsilon^3\lambda}{3\mu(2\mu+3\lambda)}\sigma_{33}^1\,\underset{\approx}{\delta} + \frac{2\varepsilon^3}{3}\,\underset{\approx}{e}(\theta) + 2\varepsilon^3 Q_{311}\phi_2\,\underset{\underset{\approx}{\approx}}{A}\,\underset{\approx}{\delta} - \frac{\varepsilon^3\lambda Q_{333}}{\mu(2\mu+3\lambda)}\phi_2\,\underset{\approx}{\delta} = 0,\\
&\frac{2\varepsilon}{\mu}\underset{\sim}{\sigma}^0 - 2\varepsilon(-\theta + \underset{\sim}{\nabla}\,\omega) - \frac{2\varepsilon Q_{113}}{\mu}(\underset{\sim}{\nabla}\,\phi_{bc}^0 + \varepsilon^2\,\underset{\sim}{\nabla}\,\phi_2) = 0,\\
&\frac{2\varepsilon^5}{5\mu}\underset{\sim}{\sigma}^2 - \frac{2\varepsilon^5}{5}\,\underset{\sim}{\nabla}\,\omega_2 + \frac{2\varepsilon^5}{5\mu}Q_{113}\,\underset{\sim}{\nabla}\,\phi_2 = 0,\\
&-\frac{\varepsilon^3\lambda}{3\mu(2\mu+3\lambda)}\mathrm{tr}(\underset{\approx}{\sigma}^1) + \frac{2\varepsilon^3(\mu+\lambda)}{3\mu(2\mu+3\lambda)}\sigma_{33}^1 - 2\varepsilon^3\omega_2 - \frac{2\varepsilon^3\lambda Q_{311}}{\mu(2\mu+3\lambda)}\phi_2 + \frac{2\varepsilon^3(\mu+\lambda)Q_{333}}{\mu(2\mu+3\lambda)}\phi_2 = 0.
\end{aligned}$$

and then 17i, 17ii, 17iii, 17iv, holds. In the same fashion, 17v, 17vi, 17vii follow from (5ii).

To find now the equilibrium conditions we integrate (5iii) and then

$$\begin{aligned}
\int_\Omega \frac{2\varepsilon^3}{3}\underset{\approx}{\sigma}^1 : \underset{\approx}{e}(\underset{\sim}{v}) + 2\varepsilon\underset{\sim}{\sigma}^0\cdot(\underset{\sim}{v} + \underset{\sim}{\nabla}\,v_3^0) + \frac{2\varepsilon^5}{5}\underset{\sim}{\sigma}^2\cdot\underset{\sim}{\nabla}\,v_3^2 + 2\varepsilon^3\sigma_{33}^1 v_3^2\,d\underset{\sim}{x}\\
= \int_\Omega \varepsilon\underset{\sim}{f}^1\cdot\underset{\sim}{v} + \varepsilon f_3^0 v_3^0 + \varepsilon f_3^2 v_3^2 + 2\varepsilon\underset{\sim}{g}^1\cdot\underset{\sim}{v} + 2g_3^0 v_3^0 + 2\varepsilon^2 g_3^0 v_3^2\,d\underset{\sim}{x}.
\end{aligned}$$

Eqs. 16i, 16ii, 16iii, follow then from 17i, 17ii, 17iii, 17iv.

Finally from (5iv),

$$\int_\Omega 2\varepsilon^3\underset{\sim}{D}^0\cdot\underset{\sim}{\nabla}\,\psi^2 - \frac{2\varepsilon^5}{5}\underset{\sim}{D}^2\cdot\underset{\sim}{\nabla}\,\psi^2 - 2\varepsilon^3 D_3\psi^2\,d\underset{\sim}{x} = 0,$$

i.e.,

$$-\mathrm{div}\underset{\sim}{D}^0 + \frac{\varepsilon^2}{5}\mathrm{div}\underset{\sim}{D}^2 - D_3 = 0 \quad \text{in } \Omega, \quad (\underset{\sim}{D}^0 - \frac{\varepsilon^2}{5}\underset{\sim}{D}^2)\cdot\underset{\sim}{n} = 0 \quad \text{on } \partial\Omega,$$

and then (16i) results from 17v, 17vi, 17vii.

References

[1] A. Sène, Modelling piezoeletric static thin plates, Asymptotic Anal. 25 (2001) 1–20.
[2] A. Raoult, Abdou Sène, Modelling of piezoeletric plates including magnetic effects, Asymptotic Anal. 34 (2003) 1–40.
[3] I.M.N. Figueiredo, C.M.F. Leal, A piezoelectric anisotropic plate model, Asymptot. Anal. 44 (3–4) (2005) 327346.
[4] S.V. Gopinathan, V.V. Varadan, V.K. Varadan, A review and critique of theories for piezoelectric laminates, Smart Mater. Struct. 9 (2000) 24–48.
[5] J. Wang, J.S. Yang, Higher-order theories of piezoelectric plates and applications, Appl. Mech. Rev. 53 (4) (2000) 87–99.
[6] G.A. Maugin, D. Attou, An asymptotic theory of thin piezoelectric plates, Quart. J. Mech. Appl. Math. 43 (3) (1990) 347–362.
[7] F. Auricchio, P. Bisegna, C. Lovadina, Finite element approximation of piezoelectric plates, Int. J. Numer. Methods Eng. 50 (6) (2001) 1469–1499.
[8] S. Roccella, P. Gaudenzi, On the formulation of a piezoelectric plate model, J. Intell. Mater. Syst. Struct. 16 (2005) 285–290.
[9] P.G. Ciarlet, Mathematical Elasticity, Volume II: Theory of Plates, Studies in Mathematics and its Applications, Vol. 27, North-Holland, 1997.
[10] T. Weller, C. Licht, Analyse Asymptotique de plaques minces linéairement piézoélectriques, C.R. Acad. Paris Ser. I 335 (2002) 309–314.
[11] E. Canon, M. Lenczner, Models of elastic plates with piezoelectric inclusions. I. Models without homogenization, Math. Comput. Model. 26 (5) (1997) 79–106.
[12] S.M. Alessandrini, D.N. Arnold, R.S. Falk, A.L. Madureira, Derivation and Justification of Plate Models by Variational Methods, Centre de Recherches Mathematiques, CRM Proceedings and Lecture Notes, 1999.
[13] D.N. Arnold, A.L. Madureira, Asymptotic estimates of hierarchical modeling, Math. Models Methods Appl. Sci. 13 (9) (2003) 1325–1350.
[14] I. Babuška, L. Li, Hierarchical modelling of plates, Comput. Struct. 40 (1991) 419–430.
[15] I. Babuška, L. Li, The problem of plate modeling: Theoretical and computational results, Comput. Methods Appl. Mech. Eng. 100 (1992) 249–273.
[16] A.L. Madureira, Hierarchical modeling based on mixed principles: asymptotic error estimates, Math. Models Methods Appl. Sci. 15 (7) (2005) 985–1008.
[17] A.L. Madureira, Asymptotics and Hierarchical Modeling of Thin Domains, Ph.D. Dissertation, The Pennsylvania State University, University Park, Pa., 1999.
[18] P. Bisegna, G. Caruso, Evaluation of high-order theories of piezoelectric plates in bending and in stretching, Int. J. Solids Struct. 38 (2001) 8805–8830.
[19] R. Nicolaides, Existence, uniqueness and approximation for generalized saddle point problems, SIAM J. Numer. Anal. 19 (2) (1982) 349–357.
[20] C. Bernardi, C. Canuto, Y. Maday, Generalized inf–sup conditions for Chebyshev spectral approximation of the Stokes problem, SIAM J. Numer. Anal. 25 (6) (1988) 1237–1271.
[21] F. Brezzi, M. Fortin, Mixed and hybrid finite element methods, Springer Ser. Comput. Math. 15 (1991).
[22] J.-C. Paumier, A. Raoult, Asymptotic consistency of the polynomial approximation in the linearized plate theory. Application to the Reissner–Mindlin model, Élasticité, viscoélasticité et contrôle optimal (Lyon, 1995), ESAIM Proc. 2, PP. 203–213 (electronic), 1997.
[23] D. Chenais, J.-C. Paumier, On the locking phenomenon for a class of elliptic problems, Numer. Math. 67 (4) (1994) 427–440.
[24] B. Miara, Justification of the asymptotic analysis of elastic plates. I. The linear case, Asymptotic Anal. 9 (1) (1994) 4760. 73K10 (73C02).
[25] D.N. Arnold, R.S. Falk, Asymptotic analysis of the boundary layer for the Reissner-Mindlin plate model, SIAM J. Math. Anal. 27 (2) (1996) 486–514. MR1377485 (97i:73064).
[26] C. Schwab, A posteriori modeling error estimation for hierarchic plate models, Numer. Math. 74 (2) (1996) 221–259.
[27] F. Auricchio, C. Lovadina, A.L. Madureira, An asymptotically optimal model for isotropic heterogeneous linearly elastic plates, M2AN Math. Model. Numer. Anal. 38 (5) (2004) 877–897.

[28] M. Dauge, I. Gruais, Asymptotics of arbitrary order for a thin elastic clamped plate. I. Optimal error estimates, Asymptotic Anal. 13 (2) (1996) 167197.
[29] D.N. Arnold, A.L. Madureira, S. Zhang, On the range of applicability of the Reissner–Mindlin and Kirchhoff–Love plate bending models, J. Elastic. 67 (3) (2002/2003) 171185.
[30] C. Chen, Asymptotic convergence for the Kirchhoff plate model, Ph.D. Dissertation, The Pennsylvania State University, University Park, Pa., 1995.