LINEAR STABILITY OF THE WATER WAVE PROBLEM

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March 1, 2002

Abstract. The water wave problem is a free-surface problem which involves finding the profile of a inviscid and incompressible fluid. We describe the problem using Lagrangian coordinates, and the wave profile is governed by a nonlinear system of pseudo-differential equations. In this note, we apply the ideas of Beale, Hou and Lowengrub (J.T. Beale, T.Y. Hou and J. Lowengrub, *Growth Rates for the Linearized Motion of Fluids away from Equilibrium*, Communications on Pure and Applied Mathematics **XLVI** (1993), pages 1269–1301) to the periodic wave case. We show that linear perturbations around an exact solution can grow, but at a bounded rate. This analysis is essential for designing stable numerical methods for the water wave problem.

1. The Model. The aim of this work is to model the water wave phenomena, a free surface problem. See [Hou], [LaL], [Lam], [MaP] and [Sto] for further details. Let $\Omega_t \subset \mathbb{R}^2$ be the open region occupied by the fluid and Γ_t be the free surface given by $\Gamma_t = \{(x(\alpha,t), y(\alpha,t))\ ;\ \alpha \in \mathbb{R}\}$. Define also the free surface subset given by $\overline{\Gamma} = \{(x(\alpha,t), y(\alpha,t)) \in \Gamma_t\ ;\ \alpha \in [-\pi,\pi)\}$, where $(-\pi,\pi)$ is the parameter domain. Let $\mathbf{u} = (u_1, u_2)$ be the fluid velocity, p its pressure and g the gravity acceleration acting on the fluid.

We consider the flow as being inviscid and incompressible, described by the Euler equation:

$$\left.\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\boldsymbol{\nabla}\mathbf{u})\,\mathbf{u} + \boldsymbol{\nabla} p &= -g\mathbf{e}_2 \\ \nabla.\mathbf{u} &= 0 \end{aligned}\right. \qquad \text{in } \Omega_t,$$

where $\mathbf{e}_2 = (0,1)^T$. If the given initial flow is irrotational, then, by Kelvin's Theorem (see [Lan]), the velocity is irrotational for all times. So the flow is potential, i.e., $\exists\, \phi : \Omega_t \longrightarrow \mathbb{R}$ such that $\mathbf{u} = \boldsymbol{\nabla}\phi$. The incompressibility of $\mathbf{u}$ implies that ϕ is harmonic.

The boundary conditions are given by the following:

- The fluid is assumed to be periodic.
- We will consider the periodic wave in infinite depth. The normal velocity is zero at the "bottom".

I would like to thank Douglas N. Arnold, for presenting this problem to me, and for several inputs, specially in the formulation setup.

- At the free surface, the boundary conditions are the kinematical and the dynamical (Bernoulli's equation):

$$\frac{\partial}{\partial t}(x,y)^T = \nabla\phi\left((x,y),t\right)$$
$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + p + gy = 0.$$

We can now write down the full water wave problem. Given the initial wave profile $\left(x^0, y^0\right)$ with $x^0(\alpha+2\pi) = x^0(\alpha) + 2\pi$ and $y^0(\alpha+2\pi) = y^0(\alpha)$, and the potential $\phi^0(x,y)$ satisfying $\phi^0(x+2\pi,y) = \phi^0(x,y)$ as initial conditions, we have:

$$\begin{aligned}
&\Delta\phi = 0 \quad \text{in } \Omega_t,\\
&\frac{\partial\phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\left((x,y),t\right)\right|^2 + p + gy = 0 \quad \text{on } \Gamma_t,\\
&\frac{\partial(x,y)}{\partial t} = \nabla\phi\left((x,y),t\right) \quad \text{on } \Gamma_t,\\
&\frac{\partial\phi}{\partial n} = 0 \quad \text{at the bottom},\\
&\phi\left((x,y),0\right) = \phi^0(x,y) \quad \text{for } (x,y)\in\Omega_0 \text{ and}\\
&\left(x(\alpha,0), y(\alpha,0)\right) = \left(x^0(\alpha), y^0(\alpha)\right), \quad \alpha\in[-\pi,\pi).
\end{aligned} \tag{1.1}$$

In order to find the wave profile velocity, we use a boundary integral method, computing the velocity using only data from the wave profile, therefore reducing the dimension of the problem by one. Also, the wave evolution equations are written down using complex variables, what makes the equations more compact and easier to deal with.

To simplify the notation, the dependence of the variables on time will not be made explicit in the formulas below.

Definitions. Let $z = x + iy$ and assume that the wave profile is also parametrized by $z(\alpha)$. Let $\phi(z) = \phi(x,y) + i\boldsymbol{\psi}(x,y)$ be the complex potential, where $\boldsymbol{\psi}$ is the stream function (see [Bat], page 107). We use the operators $\Re$ and $\Im$ to denote the real and imaginary parts of complex numbers.

Let $G(x_0, y_0; x, y)$ be a fundamental solution for the Laplace's equation, satisfying the periodic conditions:

$$G\left((x_0,y_0),(x,y)+(2\pi,0)\right) = G\left((x_0,y_0)+(2\pi,0),(x,y)\right) = G(x_0,y_0;x,y),$$

and vanishing its normal derivative at the bottom. It is straightforward to see that

$$G(x_0,y_0;x,y) = \frac{1}{2\pi}\log\left|e^{-i(x_0+iy_0)} - e^{-i(x+iy)}\right|$$

satisfies these conditions.

It is just a matter of computation to see the following result (and we defer its proof to the Appendix):

Lemma 1.1. *If $z = x + iy$ and $z_0 = x_0 + iy_0$, then*

$$\frac{1}{2\pi}\frac{-ie^{-iz}}{e^{-iz_0} - e^{-iz}} = -\frac{\partial G}{\partial x}(x_0, y_0; x, y) + i\frac{\partial G}{\partial y}(x_0, y_0; x, y).$$

Assume that the potential is given by a double layer potential with density μ:

$$\phi(x_0, y_0) = \int_{\overline{\Gamma}} \frac{\partial G}{\partial n_{(x,y)}}(x_0, y_0; x, y)\ \mu(x, y)\ d\overline{\Gamma}_{(x,y)}\ \text{, where } (x_0, y_0) \in \Omega_t. \tag{1.2}$$

Then, Lemma (1.1) implies that for $z_0 = x_0 + iy_0$:

$$\phi(x_0, y_0) = \frac{1}{2\pi}\Re\int_{-\pi}^{\pi} \frac{-ie^{-iz(\alpha)}}{e^{-iz_0} - e^{-iz(\alpha)}}\left(y_\alpha(\alpha) - ix_\alpha(\alpha)\right)\mu\left(x(\alpha), y(\alpha)\right)d\alpha \tag{1.3}$$

and then

$$\phi(z_0) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{z_\alpha(\alpha)\, e^{-iz(\alpha)}}{e^{-iz_0} - e^{-iz(\alpha)}}\mu\left(x(\alpha), y(\alpha)\right)d\alpha.$$

Above, and henceforward, y_α denotes the derivative of y with respect to α. The same holds for x_α, etc.

The conjugate of the complex wave profile velocity $\mathbf{w}(x + iy) = u(x, y) - iv(x, y)$ is given by (see [Bat], [BMO]):

$$\mathbf{w}(z_0) = \frac{d\phi(z_0)}{dz_0} = -e^{-iz(\alpha_0)}\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{e^{-iz_0} - e^{-iz(\alpha)}}\mu_\alpha\left(x(\alpha), y(\alpha)\right)d\alpha, \tag{1.4}$$

where an integration by parts was performed.

Taking the limit $(x_0, y_0) \to \overline{\Gamma}$ in (1.3) and (1.4), we have the jump:

$$\phi(x_0, y_0) = -\frac{1}{2\pi}\Re\int_{-\pi}^{\pi} \frac{z_\alpha(\alpha)\, e^{-iz(\alpha)}}{e^{-iz(\alpha_0)} - e^{-iz(\alpha)}}\mu\left(x(\alpha), y(\alpha)\right)d\alpha + \frac{1}{2}\mu(x_0, y_0) \tag{1.5}$$

and

$$\mathbf{w}(z(\alpha_0)) = -e^{-iz(\alpha_0)}\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{e^{-iz_0} - e^{-iz(\alpha)}}\mu_\alpha\left(x(\alpha), y(\alpha)\right)d\alpha + \frac{\mu_\alpha(x_0, y_0)}{2z_\alpha(\alpha_0)} \tag{1.6}$$

where $z(\alpha_0) = x_0 + iy_0$.

Definitions. With some abuse of notation, let $\phi(\alpha, t) = \phi\left(\left(x(\alpha, t), y(\alpha, t)\right), t\right)$ be the values of ϕ on the wave profile. Similarly, we define

$$w(\alpha, t) = \mathbf{w}\left(z(\alpha, t), t\right) \quad \text{and} \quad \mu(\alpha, t) = \mu\left(\left(x(\alpha, t), y(\alpha, t)\right), t\right).$$

Define also the operator:

$$\mathcal{K}(z, \rho)(\alpha_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{-iz_\alpha(\alpha)\, e^{-iz(\alpha)}}{e^{-iz(\alpha_0)} - e^{-iz(\alpha)}}\rho(\alpha)\ d\alpha. \tag{1.7}$$

Using equations (1.5), (1.6), and the definitions above, it is easy to see that the free boundary evolves by:

$$\begin{aligned}
&\frac{\partial \overline{z}}{\partial t} = e^{-iz}\mathcal{K}\left(z, \frac{\mu_\alpha}{iz_\alpha e^{-iz}}\right) + \frac{\mu_\alpha}{2z_\alpha} = w, \\
&\frac{\partial \phi}{\partial t} = \frac{|w|^2}{2} - gy, \\
&\phi = \Im\, \mathcal{K}\left(z, \mu\right) + \frac{\mu}{2}.
\end{aligned} \tag{1.8}$$

Remark 1.2. Baker, Meiron and Orszag [BMO], differentiated the integral equation for μ with respect to time, used Bernoulli's equation and found another integral equation of the second kind describing the evolution in time for μ.

Remark 1.3. There is a slightly different way to write down this problem. If we substitute $\gamma = \mu_\alpha$ and use the integral equation given by:

$$\phi_\alpha = \Im\left(iz_\alpha e^{-iz}\mathcal{K}\left(z, \frac{\gamma}{iz_\alpha e^{-iz}}\right)\right) + \frac{\gamma}{2},$$

we arrive at a formulation used by other authors, including [Hou].

2. Analysis of linear perturbations. To perform this analysis, we follow closely the steps of [BHL]. First of all, denote by A_0 any bounded operator in $\mathbb{H}^0(-\pi,\pi)$ and by $A_{-\infty}$ any smoothing operator, i.e., $A_{-\infty} : \mathbb{H}^0(-\pi,\pi) \to \mathbb{H}^s(-\pi,\pi)$ is bounded. Finally, $\|\,.\,\|_s$ gives the usual $\mathbb{H}^s(-\pi,\pi)$ norm.

Along this paper, we assume that the following conditions hold with $m \geq 3$:

$$\begin{aligned}
&z(\alpha_0, t) \neq z(\alpha, t) \quad \text{for } \alpha_0 \neq \alpha, \\
&z(\alpha_0, t) - \alpha \in C\left(0, T; \mathbb{H}^m(\mathbb{R})\right), \\
&\phi,\, w \in C\left(0, T; \mathbb{H}^m(\mathbb{R})\right), \text{ and} \\
&z_\alpha(\alpha, t) \neq 0
\end{aligned}$$

Definition. Let $\mathcal{H}$ be the "Hilbert-like" operator

$$\mathcal{H}\rho = \frac{1}{\pi}\int_{-\pi}^{\pi} \rho_\alpha(\alpha) \log\left|2\sin\frac{\alpha_0 - \alpha}{2}\right| d\alpha. \tag{2.1}$$

The action of $\mathcal{H}$ can be computed using Fourier coefficients:

$$\widehat{\mathcal{H}\rho}(n) = -i\, sgn(n)\, \hat{\rho}(n).$$

Lemma 2.1. *The operator $\mathcal{K}$ defined in (1.7) can be decomposed in a sum of the operator $\mathcal{H}$ defined in (2.1), and a smoothing operator, i.e.:*

$$\mathcal{K}(z,\rho)(\alpha_0) = \frac{\mathcal{H}\rho}{2} + \mathcal{K}_1(z,\rho),$$

where the operator $\mathcal{K}_1(z,\cdot):\mathbb{H}^s\to\mathbb{H}^{m-2}$ is bounded for $0\le s\le m-2$.

Proof.

$$
\begin{aligned}
\mathcal{K}(z,\rho)(\alpha_0) &= -\frac{1}{2\pi}\int_{-\pi}^{\pi}\rho(\alpha)\frac{d}{d\alpha}\log\left(e^{-iz(\alpha_0)}-e^{-iz(\alpha)}\right)d\alpha\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\rho_\alpha(\alpha)\log\left(e^{-iz(\alpha_0)}-e^{-iz(\alpha)}\right)d\alpha\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\rho_\alpha(\alpha)\log\left|2\sin\frac{\alpha_0-\alpha}{2}\right|d\alpha+\mathcal{K}_1(z,\rho)\\
&= \frac{\mathcal{H}}{2}\rho+\mathcal{K}_1(z,\rho).
\end{aligned}
$$

The boundedness of $\mathcal{K}_1$ follows from arguments as in [BHL], and then the result holds. □

Lemma 2.2.

The commutator $[\mathcal{H},\rho]f=\mathcal{H}(\rho f)-\rho\mathcal{H}(f)$ is a smoothing operator. If $\rho\in\mathbb{H}^s(\mathbb{R})$, $s\ge1$, then $[\mathcal{H},\rho]:\mathbb{H}^0(\mathbb{R})\to\mathbb{H}^{s-1}(\mathbb{R})$ is bounded. If $s\ge2$, then the commutator is bounded from $\mathbb{H}^{-1}(\mathbb{R})\to\mathbb{H}^{s-2}(\mathbb{R})$

Proof. See [BHL]. □

Given a general operator $A(z,\phi,\mu)$, we find its formal variation $A(z,\phi,\mu)^{\cdot}$ by substituting z, μ and ϕ by $z+\epsilon\dot z$, $\mu+\epsilon\dot\mu$ and $\phi+\epsilon\dot\phi$ in the operator A and keep the $O(\epsilon)$ terms. The resulting operator depends on z, ϕ and μ, and is linear with respect to $\dot z$, $\dot\phi$, and $\dot\mu$.

Lemma 2.3. *The variation of the operator $\mathcal{K}$ defined in (1.7) is given by:*

$$
\mathcal{K}(z,\rho)^{\cdot}(\alpha_0)=\mathcal{K}(z,\dot\rho)(\alpha)-\frac{i}{2\pi}\int_{-\pi}^{\pi}\rho_\alpha(\alpha)\frac{\dot z(\alpha_0)e^{-iz(\alpha_0)}-\dot z(\alpha)e^{-iz(\alpha)}}{e^{-iz(\alpha_0)}-e^{-iz(\alpha)}}d\alpha.
$$

Proof. The linear variation is given by:

$$
\begin{aligned}
\mathcal{K}(z,\rho)^{\cdot}(\alpha_0)=\mathcal{K}(z,\dot\rho)(\alpha_0)-i\frac{1}{2\pi}\int_{-\pi}^{\pi}\rho(\alpha)&\left[\frac{\dot z_\alpha(\alpha)e^{-iz(\alpha)}-iz_\alpha(\alpha)\dot z(\alpha)e^{-iz(\alpha)}}{e^{-iz(\alpha_0)}-e^{-iz(\alpha)}}\right.\\
&\left.-\frac{z_\alpha(\alpha)e^{-iz(\alpha)}}{\left(e^{-iz(\alpha_0)}-e^{-iz(\alpha)}\right)^2}\left(-i\dot z(\alpha_0)e^{-iz(\alpha_0)}+i\dot z(\alpha)e^{-iz(\alpha)}\right)\right]d\alpha,
\end{aligned}
$$

and an integration by parts yields the desired result. □

Using Lemmas (2.1) and (2.3), we conclude the following result.

Corollary 2.4.

$$
\mathcal{K}(z,\rho)^{\cdot}=\frac{\mathcal{H}}{2}(\dot\rho)(\alpha)+A_0(\dot z).
$$

Let $\rho=\mu_\alpha/z_\alpha$. Then the velocity linear variation can be computed as follows. From (1.8-i), Corollary (2.4) and Lemma (2.2), we have

$$
\begin{aligned}
\dot w &= -ie^{-iz}\mathcal{K}\left(z,\frac{\rho}{e^{-iz}}\right)^{\cdot}+\frac{\dot\rho}{2}+A_0(\dot z)\\
&= -ie^{-iz}\frac{\mathcal{H}}{2}\left(\frac{\dot\rho}{e^{-iz}}\right)+\frac{\dot\rho}{2}+A_0(\dot z)\\
&= \frac{1}{2}(-i\mathcal{H}+I)(\dot\rho)+A_0(\dot z)+A_{-\infty}(\dot\rho).
\end{aligned}
$$

But $\dot{\rho} = (\dot{\mu}_\alpha / z_\alpha) - (\mu_\alpha / z_\alpha^2)\dot{z}_\alpha$ and then, using Lemma (2.2) again it follows that

$$\dot{w} = \frac{1}{2z_\alpha}\left(-i\mathcal{H} + I\right)\left(\dot{\mu}_\alpha - \frac{\mu_\alpha}{z_\alpha}\dot{z}_\alpha\right) + A_0\left(\dot{z}\right) + A_{-\infty}\left(\dot{\mu}\right). \tag{2.2}$$

To replace $\dot{\mu}$, we need the variation of the integral equation (1.8-iii):

$$\begin{aligned}\dot{\phi} &= \Im\,\mathcal{K}\left(z, \dot{\mu}\right) - \Im\frac{i}{2\pi}\int_{-\pi}^{\pi}\mu_\alpha\left(\alpha\right)\frac{\dot{z}\left(\alpha_0\right)e^{-iz(\alpha_0)} - \dot{z}\left(\alpha\right)e^{-iz(\alpha)}}{e^{-iz(\alpha_0)} - e^{-iz(\alpha)}}\,d\alpha + \frac{\dot{\mu}}{2}\\ &= \frac{\dot{\mu}}{2} + \Im\,\mathcal{K}\left(z, \dot{\mu}\right) + \Im\,\left(\dot{z}e^{-iz}w_0\right) - \Im\,\mathcal{H}\left(\frac{\mu_\alpha}{2z_\alpha}\dot{z}\right) + A_{-\infty}\left(\dot{z}\right),\end{aligned} \tag{2.3}$$

where we define

$$w_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{-i\mu_\alpha\left(\alpha\right)}{e^{-iz(\alpha_0)} - e^{-iz(\alpha)}}\,d\alpha = \mathcal{K}\left(z, \frac{\mu_\alpha}{z_\alpha e^{-iz}}\right)\left(\alpha_0\right).$$

We have that $\Im\,\mathcal{K}$ is a smoothing operator as its kernel is given by the normal derivative of the fundamental solution G. Then, taking the derivative with respect to α in (2.3), we get:

$$\dot{\mu}_\alpha = 2\dot{\phi}_\alpha - 2\Im\,\left(\dot{z}_\alpha e^{-iz}w_0\right) + \Im\,\mathcal{H}\left(\frac{\mu_\alpha}{z_\alpha}\dot{z}_\alpha\right) + A_0\left(\dot{z}\right) + A_{-\infty}\left(\dot{\mu}\right). \tag{2.4}$$

The integral equation (1.8-iii) can be solved for $\dot{\mu}$ with $|\dot{\mu}| \le C\left(|\dot{z}| + \left|\dot{\phi}\right|\right)$ (see [BMO]), and then, substituting (2.4) in (2.2), the following holds:

$$\begin{aligned}\dot{w} =& \frac{1}{2z_\alpha}\left(i\mathcal{H} + I\right)\left(2\dot{\phi}_\alpha - 2\Im\,\left(\dot{z}_\alpha e^{-iz}w_0\right) + \Im\,\mathcal{H}\left(\frac{\mu_\alpha}{z_\alpha}\dot{z}_\alpha\right) - \frac{\mu_\alpha}{z_\alpha}\dot{z}_\alpha\right)\\ &+ A_0\left(\dot{z}\right) + A_{-\infty}\left(\dot{\phi}\right).\end{aligned}$$

If we define

$$A\left(\dot{z}, \dot{\phi}\right) = A_0\left(\dot{z}\right) + A_{-\infty}\left(\dot{\phi}\right), \tag{2.5}$$

then, using the operator identity $\left(-i\mathcal{H} + I\right)\Im\,\mathcal{H} = i\left(I - i\mathcal{H}\right)\Im$, after a straightforward computation we get

$$\begin{aligned}\dot{w} =& \frac{1}{z_\alpha}\left(-i\mathcal{H} + I\right)\left(\dot{\phi}_\alpha - \Im\,\left(\dot{z}_\alpha e^{-iz}w_0\right) + i\Im\,\left(\frac{\mu_\alpha}{2z_\alpha}\dot{z}_\alpha\right) - \frac{\mu_\alpha}{2z_\alpha}\dot{z}_\alpha\right) + A\left(\dot{z}, \dot{\phi}\right)\\ =& \frac{1}{z_\alpha}\left(-i\mathcal{H} + I\right)\left(\dot{\phi}_\alpha - \Im\,\left(\dot{z}_\alpha e^{-iz}\mathcal{K}\left(z, \frac{\mu_\alpha}{z_\alpha e^{-iz}}\right)\right) - \Re\,\left(\frac{\mu_\alpha}{2z_\alpha}\dot{z}_\alpha\right)\right) + A\left(\dot{z}, \dot{\phi}\right)\\ =& \frac{1}{z_\alpha}\left(-i\mathcal{H} + I\right)\left(\dot{\phi}_\alpha + \Re\,\left(\dot{z}_\alpha\left(ie^{-iz}\mathcal{K}\left(z, \frac{\mu_\alpha}{z_\alpha e^{-iz}}\right) - \frac{\mu_\alpha}{2z_\alpha}\right)\right)\right) + A\left(\dot{z}, \dot{\phi}\right)\\ =& \frac{1}{z_\alpha}\left(-i\mathcal{H} + I\right)\left(\dot{\phi}_\alpha - \Re\,\left(\dot{z}_\alpha w\right)\right) + A\left(\dot{z}, \dot{\phi}\right).\end{aligned}$$

Defining

$$\dot{F} = \dot{\phi} - \Re\,\left(\dot{z}w\right) = \dot{\phi} - u\dot{x} - v\dot{y}, \tag{2.6}$$

we finally have

$$\dot{w} = \frac{1}{z_\alpha}\left(-i\mathcal{H} + I\right)F_\alpha + A\left(\dot{z}, \dot{\phi}\right). \tag{2.7}$$

In the remaining of this note, we will follow the ideas of [BHL] to prove the bounded growth of the linear variations.

Definitions. Let $\sigma = \left(x_\alpha^2 + y_\alpha^2\right)^{-1/2}$. Let the normal and tangential components of the wave profile by defined by

$$\dot{w}^T = \sigma\left(x_\alpha \dot{u} + y_\alpha \dot{v}\right), \quad \dot{w}^N = \sigma\left(-y_\alpha \dot{u} + x_\alpha \dot{v}\right), \tag{2.8}$$

$$\dot{z}^T = \sigma\left(x_\alpha \dot{x} + y_\alpha \dot{y}\right), \quad \dot{z}^N = \sigma\left(-y_\alpha \dot{x} + x_\alpha \dot{y}\right). \tag{2.9}$$

From (2.7)–(2.9), we have that

$$\dot{w}^T - iw^N = \sigma\left(-i\mathcal{H} + I\right) F_\alpha + A\left(\dot{z}, \dot{\phi}\right) \tag{2.10}$$

Differentiating (2.9) in time, and using (2.10) we get

$$\begin{aligned} \dot{z}_t^T &= \dot{w}^T + \left(\sigma x_\alpha\right)_t \dot{x} + \left(\sigma y_\alpha\right)_t \dot{y} = \sigma \dot{F}_\alpha + A(\dot{\phi}, \dot{z}) \quad \text{and similarly,} \\ \dot{z}_t^N &= \sigma \mathcal{H} \dot{F}_\alpha + A(\dot{\phi}, \dot{z}). \end{aligned}$$

Noting that the evolution equations for z^T and z^N are similar, we define

$$\dot{\delta} = \dot{z}^T + \mathcal{H}\dot{z}^N, \tag{2.11}$$

and then we have

$$\dot{\delta}_t = A(\dot{\phi}, \dot{z}).$$

Let us now write down the evolution equation for $\dot{F}$ in terms of the new variables $\dot{\delta}$ and $\dot{z}^N$. From the definition of $\dot{F}$ in (2.6), $\dot{F}_t = \dot{\phi}_t - u\dot{u} - v\dot{v} - u_t\dot{x} - v_t\dot{y}$. The variation of the Bernoulli equation (1.8-ii) is given by $\dot{\phi}_t = u\dot{u} + v\dot{v} - g\dot{y}$, and then

$$\dot{F}_t = -g\dot{y} - u_t\dot{x} - v_t\dot{y}.$$

Note that the Lagrangian velocity (u, v) satisfies the Euler equations, i.e.,

$$u_t + p_x = 0 \quad \text{and} \quad v_t + p_y + g = 0,$$

and so $\dot{F}_t = \boldsymbol{\nabla} p \cdot (\dot{x}, \dot{y})$. Assuming that $p = 0$ on the interface, we gather that $\boldsymbol{\nabla} p$ is normal to the wave profile, and then

$$\dot{F}_t = -c(\alpha, t)\dot{z}^N,$$

where $c(\alpha, t) = -\boldsymbol{\nabla} p \cdot N = (u_t, v_t + g) \cdot N$. We will assume that:

$$c(\alpha, t) \geq c_0 > 0 \tag{2.12}$$

The linearized system of evolution equations is then given by:

$$\begin{aligned} \dot{z}_t^N &= \sigma\mathcal{H}\dot{F}_\alpha + A(\dot{\phi}, \dot{z}), \\ \dot{\delta}_t &= A(\dot{\phi}, \dot{z}), \\ \dot{F}_t &= -c(\alpha, t)\dot{z}^N. \end{aligned} \tag{2.13}$$

Let $\Lambda = \mathcal{H}D_\alpha$. As Λ has Fourier symbol $|k|$, then

$$\left((\Lambda+1)\,\dot{f},\dot{f}\right) \le \left\|\dot{f}\right\|_{\frac{1}{2}} \le C\left((\Lambda+1)\,\dot{f},\dot{f}\right), \tag{2.14}$$

where $(\cdot,\cdot)$ denotes the L^2 inner product.

Defining

$$y_0(t) = \left\|(c(\alpha,t)/\sigma)^{\frac{1}{2}}\dot{z}^N(t)\right\|_0^2 + \left\|\dot{\delta}(t)\right\|_0^2 + \left((\Lambda+1)\,\dot{F}(t),\dot{F}(t)\right),$$

we have by simple differentiation in time, and by equations (2.13) that:

$$\frac{1}{2}\frac{dy_0}{dt} = \left(\Lambda\dot{F}, c(\alpha,t)\dot{z}^N\right) - \left(c(\alpha,t)\dot{z}^N,(\Lambda+1)\,\dot{F}\right) + \left(f_1,\dot{z}^N\right) + \left(f_2,\dot{\delta}\right),$$

where $\|f_j\|_0 \le C\left(\|\dot{z}^N\|_0 + \left\|\dot{\delta}\right\|_0 + \left\|\dot{F}\right\|_0\right)$, for $j=1,2$. The terms in Λ cancels out and we find that, for some constant M_0 we have:

$$\begin{aligned}\frac{1}{2}\frac{dy_0}{dt} &= -\left(c(\alpha,t)\dot{z}^N,\dot{F}\right) + \left(f_1,\dot{z}^N\right) + \left(f_2,\dot{\delta}\right) \le C\left(\|\dot{z}^N\|_0 + \left\|\dot{\delta}\right\|_0 + \left\|\dot{F}\right\|_0\right)^2 \\ &\le M_0\; y_0\,(t)\,,\end{aligned}$$

and then the following estimate holds:

$$y_0(t) \le e^{M_0 t} y_0(0). \tag{2.15}$$

Using condition (2.12) and estimates (2.14), (2.15), we can bound the growth of the variation:

$$\|\dot{z}^N(t)\|_0^2 + \left\|\dot{\delta}(t)\right\|_0^2 + \left\|\dot{F}(t)\right\|_{\frac{1}{2}}^2 \le \tilde{B}_0(t)\left(\|\dot{z}^N(0)\|_0^2 + \left\|\dot{\delta}(0)\right\|_0^2 + \left\|\dot{F}(0)\right\|_{\frac{1}{2}}^2\right). \tag{2.16}$$

In the original variables, we have

$$\|\dot{z}(t)\|_0^2 + \left\|\dot{\phi}(t)\right\|_0^2 \le B_0(t)\left(\|\dot{z}(0)\|_{\frac{1}{2}}^2 + \left\|\dot{\phi}(0)\right\|_{\frac{1}{2}}^2\right). \tag{2.17}$$

The above estimate (2.17) follows from (2.16) and from the definitions of the new variables (2.6), (2.9) and (2.11). The extra $\|\cdot\|_{\frac{1}{2}}$ on the right hand side of (2.17) is due to the estimate $\left\|\dot{F}\right\|_{\frac{1}{2}} \le \|\dot{z}\|_{\frac{1}{2}} + \left\|\dot{\phi}\right\|_{\frac{1}{2}}$ that follows from the definition of $\dot{F}$ in (2.6).

We can obtain estimates in higher norms in a similar way (see [BHL]), and then prove the

Theorem 2.5. *Assuming that the flow satisfies*

$$\begin{aligned}&z(\alpha_0,t) \ne z(\alpha,t) \quad \textit{for } \alpha_0 \ne \alpha,\\ &z(\alpha_0,t) - \alpha \in C\left(0,T;\mathbb{H}^m(\mathbb{R})\right),\\ &\phi,\, w \in C\left(0,T;\mathbb{H}^m(\mathbb{R})\right), \ \textit{and}\\ &z_\alpha(\alpha,t) \ne 0\end{aligned}$$

with $m \ge 3$. Also assume $(u_t, v_t + g)\cdot N \ge c_0 > 0$.

Then, for $0 \le s \le m-3$, the solution of the linearized evolution equations satisfy:

$$\|\dot{z}(t)\|_s^2 + \left\|\dot{\phi}(t)\right\|_s^2 \le B_s(t)\left(\|\dot{z}(0)\|_{s+\frac{1}{2}}^2 + \left\|\dot{\phi}(0)\right\|_{s+\frac{1}{2}}^2\right).$$

Appendix

Lemma 1.1. *If $z = x + iy$ and $z_0 = x_0 + iy_0$, then*

$$\frac{1}{2\pi}\frac{-ie^{-iz}}{e^{-iz_0} - e^{-iz}} = -\frac{\partial G}{\partial x}(x_0, y_0; x, y) + i\frac{\partial G}{\partial y}(x_0, y_0; x, y).$$

Proof. This follows from a straightforward computation. The partial derivatives of G are given by

$$\frac{\partial G}{\partial x} = \frac{1}{2\pi}\frac{\left(e^{y_0}\cos x_0 - e^{y}\cos x\right)e^{y}\sin x - \left(e^{y_0}\sin x_0 - e^{y}\sin x\right)e^{y}\cos x}{\left[\left(e^{y_0}\cos x_0 - e^{y}\cos x\right)^2 + \left(e^{y_0}\sin x_0 - e^{y}\sin x\right)^2\right]},$$

$$\frac{\partial G}{\partial y} = \frac{1}{2\pi}\frac{-\left(e^{y_0}\cos x_0 - e^{y}\cos x\right)e^{y}\cos x - \left(e^{y_0}\sin x_0 - e^{y}\sin x\right)e^{y}\sin x}{\left[\left(e^{y_0}\cos x_0 - e^{y}\cos x\right)^2 + \left(e^{y_0}\sin x_0 - e^{y}\sin x\right)^2\right]}.$$

On the other hand,

$$\begin{aligned}
\frac{-ie^{-iz}}{e^{-iz_0} - e^{-iz}} &= -\frac{ie^{y}\left(\cos x - i\sin x\right)}{\left|e^{-iz(\alpha_0)} - e^{-iz(\alpha)}\right|^2} \\
&\qquad \left[\left(e^{y_0}\cos x_0 - e^{y}\cos x\right) + i\left(e^{y_0}\sin x_0 - e^{y}\sin x\right)\right] \\
&= -\frac{ie^{y}}{\left|e^{-iz(\alpha_0)} - e^{-iz(\alpha)}\right|^2}\left[\left(e^{y}\cos x - e^{y_0}\cos x_0\right)\cos x + \left(e^{y}\sin x - e^{y_0}\sin x_0\right)\sin x\right. \\
&\qquad \left. +i\left[-\left(e^{y}\cos x - e^{y_0}\cos x_0\right)\sin x + \left(e^{y}\sin x - e^{y_0}\sin x_0\right)\cos x\right]\right] \\
&= 2\pi i\left(\frac{\partial G}{\partial y} + i\frac{\partial G}{\partial x}\right),
\end{aligned}$$

and the result follows. □

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