AN IMPROVED BIHARMONIC MODEL—INCORPORATING HIGHER ORDER RESPONSES OF THE PLATE BENDING PHENOMENA

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ABSTRACT. We modify the usual biharmonic model, still frequently used in the engineering community to model linearly elastic plates. In its traditional form, the biharmonic model diverges in general, since it does not incorporate shear effects. Our changes make it convergent for all loads, under a usual assumption on how the loads depend on the thickness. The idea is to add "higher order" terms that appear in the asymptotic expansion of the exact solution. The changes can be readily incorporated to engineering codes, without degrading the computational performance.

INTRODUCTION

The use of dimension reduction is quite common in the modeling of thin, three-dimensional elastic structures. This reduction consists in posing a two-dimensional problem, which solution is somehow extended to the three-dimensional domain. The extended solution is expected then to be close to the original solution.

One of the best known model for linearly elastic plate bending is the biharmonic model, also known as Kirchhoff or Kirchhoff–Love. It first appeared around 1850, but a proof of convergence, although under certain load restrictions, came only in 1959, with the work of Morgenstern [10]. Twenty years latter, Ciarlet and Destuynder [2] showed that in general, and in a proper sense, the biharmonic model is the asymptotic limit of the three-dimensional linearly elastic plate equations. See also [1, 7].

Although successful, the biharmonic model suffers several criticisms from the engineering community. It is regarded as suitable for "very thin" plates, but not so performing for "moderately thin" plates [1, p. 86]. In the latter case, models of Reissner–Mindlin type are

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usually preferred. In fact, the biharmonic model does not consider shear effects, and gives no information concerning the boundary layer.

The absence of shear is often mentioned as the main drawback of the biharmonic model. In most cases, shear introduces a negligible effect in the bending of a plate. It is indeed much easier to bend a plate by applying a transverse traction than by applying a constant horizontal shear inducing traction. Nevertheless, in some situations, shear is important and even dominant.

In this paper, we aim to incorporate this and other effects in the biharmonic model, by adding extra terms that make the model always convergent, in a sense that we make clear further below. Which terms to add is a nontrivial question, but the answers come from the asymptotic expansion for the exact solution. Remarkably, in most practical situations (the exceptions being when the body forces present a non-polynomial behavior in the transverse direction!), the changes can be easily incorporated in existing computer codes. As in the Kirchhoff–Love model, there is only one biharmonic equation to solve, thus there is no loss in the computational performance.

Consider a three-dimensional, homogeneous, isotropic, linearly elastic plate $P^\varepsilon = \Omega \times (-\varepsilon, \varepsilon)$, where $\Omega \subset \mathbb{R}^2$ is a smoothly bounded domain. A typical point in P^ε is given by $\underline{x} = (\underset{\sim}{x}, x_3)$, where $\underset{\sim}{x} \in \Omega$ and $x_3 \in (-\varepsilon, \varepsilon)$. The plate lateral boundary is given by $\partial P^\varepsilon_{\mathrm{L}} = \partial\Omega \times (-\varepsilon, \varepsilon)$ and its top and bottom by $\partial P^\varepsilon_+ = \Omega \times \{\varepsilon\}$ and $\partial P^\varepsilon_- = \Omega \times \{-\varepsilon\}$.

The plate is subjected to possibly nonzero body force density $\underline{f}^\varepsilon : P^\varepsilon \to \mathbb{R}^3$, and surface force densities $\underline{g}^{+,\varepsilon} : \partial P^\varepsilon_+ \to \mathbb{R}^3$ and $\underline{g}^{-,\varepsilon} : \partial P^\varepsilon_- \to \mathbb{R}^3$, and is clamped along its lateral boundary. The displacement $\underline{u}^\varepsilon : P^\varepsilon \to \mathbb{R}^3$ and stress $\underline{\underline{\sigma}}^\varepsilon : P^\varepsilon \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ satisfy then the following mathematical problem:

$$
\begin{gathered}
\underline{\underline{\sigma}}^\varepsilon = \underline{\underline{\underline{\underline{C}}}}\,\underline{\underline{e}}(\underline{u}^\varepsilon), \quad -\underline{\operatorname{div}}\,\underline{\underline{\sigma}}^\varepsilon = \underline{f}^\varepsilon \text{ in } P^\varepsilon, \\
\underline{u}^\varepsilon = 0 \text{ on } \partial P^\varepsilon_{\mathrm{L}}, \quad \underline{\underline{\sigma}}^\varepsilon \underline{n} = \begin{cases} \underline{g}^{+,\varepsilon} \text{ on } \partial P^\varepsilon_+, \\ \underline{g}^{-,\varepsilon} \text{ on } \partial P^\varepsilon_-. \end{cases}
\end{gathered} \tag{1}
$$

Here, the rigidity tensor and the infinitesimal strain tensor are given by

$$
\underline{\underline{\underline{\underline{C}}}}\,\underline{\underline{e}}(\underline{u}^\varepsilon) = 2\mu\,\underline{\underline{e}}(\underline{u}^\varepsilon) + \lambda \operatorname{div} \underline{u}^\varepsilon\ \underline{\underline{\delta}}, \qquad \underline{\underline{e}}(\underline{u}^\varepsilon) = \frac{1}{2}\big(\underline{\underline{\nabla}} + \underline{\underline{\nabla}}^T\big)\underline{u}^\varepsilon,
$$

where μ, λ are the Lamé coefficients, and $\underline{\underline{\delta}}$ the identity matrix.

It is sensible to pause now, and explain the notation employed in this paper. We use underbars to indicate tensors in three variables, a first-order tensor (or 3-vector) is written with one underbar, a second-order tensor (or 3×3 matrix) with two underbars, etc. We use undertildes in the same way for tensors in two variables, and so a 3-vector can be decomposed into a 2-vector giving the in-plane components and a scalar for the transverse component. Thus,

$$\underline{v} = \begin{pmatrix} \underset{\sim}{v} \\ v_3 \end{pmatrix}.$$

In this paper we consider the case of *pure bending*, i.e., we assume

$$\underset{\sim}{f}^\varepsilon \text{ odd in } x_3, \quad f_3^\varepsilon \text{ even in } x_3,$$
$$\underset{\sim}{g}^\varepsilon(\underset{\sim}{x}) := \underset{\sim}{g}^{+,\varepsilon}(\underset{\sim}{x}, \varepsilon) = -\underset{\sim}{g}^{-,\varepsilon}(\underset{\sim}{x}, -\varepsilon), \quad g_3^\varepsilon(\underset{\sim}{x}) := g_3^{+,\varepsilon}(\underset{\sim}{x}, \varepsilon) = g_3^{-,\varepsilon}(\underset{\sim}{x}, -\varepsilon).$$

In this case, it is easy to check that $\underset{\sim}{u}^\varepsilon$ is odd, and u_3^ε is even in x_3.

An approximation to $\underline{u}^\varepsilon$ is

$$\underline{u}^B(\underline{x}) = \begin{pmatrix} -x_3 \underset{\sim}{\nabla} \zeta^0(\underset{\sim}{x}) \\ \zeta^0(\underset{\sim}{x}) + \lambda(x_3^2/2 - \varepsilon^2/6)\, \Delta\, \zeta^0(\underset{\sim}{x})/(2\mu + \lambda) \end{pmatrix}, \tag{2}$$

where ζ^0 solves the biharmonic equation

$$\begin{aligned} D\varepsilon^3\, \Delta^2\, \zeta^0 &= l_K \qquad \text{in } \Omega, \\ \zeta^0 = \frac{\partial \zeta^0}{\partial n} &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{3}$$

and $D = 8\mu(\mu + \lambda)/[3(2\mu + \lambda)]$. Also,

$$l_K(\underset{\sim}{x}) = \int_{-\varepsilon}^{\varepsilon} f_3^\varepsilon(\underset{\sim}{x}, x_3)\, dx_3 + 2g_3^\varepsilon(\underset{\sim}{x}) + \int_{-\varepsilon}^{\varepsilon} x_3 \operatorname{div} \underset{\sim}{f}^\varepsilon(\underset{\sim}{x}, x_3)\, dx_3 + 2\varepsilon \operatorname{div} \underset{\sim}{g}^\varepsilon(\underset{\sim}{x}).$$

Morgenstern [10] studied the convergence of the approximation to $\underline{u}^\varepsilon$ defined by (2), in the energy norm. Such norm is defined by

$$\|\underline{v}\|_{E(P^\varepsilon)} = \left(\int_{P^\varepsilon} |\, \underline{\underline{e}}(\underline{v})|^2\, d\underline{x} \right)^{1/2}.$$

He ingeniously used the Prager–Synge Theorem to prove convergence for a class of problems where the traction vanishes identically and the body forces act in the transverse direction only, and are constant in each vertical fiber. Although not explicitly mentioned in his paper, a bound of $O(\varepsilon^{1/2})$ for the relative error follows.

It is easy though to see that the biharmonic approximation given by (2) cannot converge to the exact solution in all situations. Consider for instance that there is no body force, that is, $\underline{f}^\varepsilon = 0$, and that the traction is horizontal and constant, that is, $\underset{\sim}{g}^\varepsilon$ is nonzero, constant, and $g_3^\varepsilon = 0$. In this case, it follows from (3) that $\zeta^0 = 0$, and

$$\frac{\|\underline{u}^\varepsilon - \underline{u}^B\|}{\|\underline{u}^\varepsilon\|} = 1.$$

in any norm.

The remedy for this particular situation is surprisingly simple. The displacement

$$\underline{u}^B + \frac{x_3}{\mu}\begin{pmatrix} \underset{\sim}{g}^\varepsilon \\ 0 \end{pmatrix}$$

converges to the exact solution in relative energy norm as $O(\varepsilon^{1/2})$ when the plate is subject to an arbitrary, pure traction loads (no body force). See Theorem 2 further ahead for a precise statement.

In the example above, it would be easy to modify an existing computer code to add the extra term, since it involves no further computation. The extra term is the one with "highest energy" in the asymptotic expansion for the exact solution $\underline{u}^\varepsilon$, for that particular loading. The choice of terms to add depends on a careful analysis of the asymptotic expansions under different situations. A similar analysis of these "higher order responses" was performed in [4], where the authors considered the expansion in a scaled, ε-independent domain.

In a more general setting, if $\underset{\sim}{f}^\varepsilon(\underline{x}) = x_3\check{\underset{\sim}{f}}^\varepsilon(\underset{\sim}{x})$, and f_3^ε is independent of x_3, then we define the approximation

$$\begin{pmatrix} -x_3 \underset{\sim}{\nabla}\,\bar\zeta(\underset{\sim}{x}) \\ \bar\zeta(\underset{\sim}{x}) + \lambda(x_3^2/2 - \varepsilon^2/6)\,\Delta\,\bar\zeta(\underset{\sim}{x})/(2\mu+\lambda) \end{pmatrix} + \frac{1}{\mu}\begin{pmatrix} x_3\underset{\sim}{g}^\varepsilon(\underset{\sim}{x}) + (\varepsilon^2 x_3 - x_3^3/3)\check{\underset{\sim}{f}}^\varepsilon(\underset{\sim}{x})/2 \\ (-x_3^2/2 + \varepsilon^2/6)f_3^\varepsilon(\underset{\sim}{x})/(2\mu+\lambda) \end{pmatrix}, \tag{4}$$

where $\bar\zeta$ solves the biharmonic equation

$$\begin{aligned} D\varepsilon^3\,\Delta^2\,\bar\zeta &= l_K + l_K^{extra} \qquad \text{in } \Omega, \\ \bar\zeta = \frac{\partial\bar\zeta}{\partial n} &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{5}$$

and

$$l_K^{extra}(\underset{\sim}{x}) = \frac{2\lambda}{2\mu+\lambda}\,\Delta\left[\int_{-\varepsilon}^{\varepsilon} x_3^2 f_3^\varepsilon(\underset{\sim}{x}, x_3)\,dx_3 + \varepsilon^2 g_3^\varepsilon(\underset{\sim}{x})\right].$$

Our main result states that under some scaling assumptions for $\underline{f}^\varepsilon$ and $\underline{g}^\varepsilon$, see (6), the approximation defined above converges to the exact solution, in relative energy norm. Again the convergence rate is $O(\varepsilon^{1/2})$.

The outline of the paper is as follows. In Section 1 we recall the main aspects of the asymptotic expansion for the solution of the elasticity problem (1), and then, in Section 2, we use the information gathered from the asymptotic expansion to construct our improved model, in the energy norm. In Section 3 we briefly consider other two norms of interest, the $L^2(P^\varepsilon)$ and $H^1(P^\varepsilon)$ norms. We finally conclude with some remarks and a numerical test in Section 4.

1. The Asymptotic expansion

As we mention in the introduction, to find out the suitable modifications to the biharmonic model, it is essential to look at the asymptotic expansion for the exact solution $\underline{u}^\varepsilon$. The complete description of such expansion is beyond the scope of this paper, we mention only the underlying ideas, and also how some terms are defined. The interested reader can find the details at [5] for clamped plates, and [6] for other boundary conditions. A fundamental step to obtain the asymptotic expansion is to redefine and consider the original problem in a domain that is independent of the small parameter ε. Hence we define the new coordinate $\hat{x}_3 = \varepsilon^{-1} x_3$. We also introduce the functions

$$\begin{aligned} \underset{\sim}{f}(\underset{\sim}{x}, \hat{x}_3) &= \varepsilon^{-1} \underset{\sim}{f}^\varepsilon(\underset{\sim}{x}, x_3), \quad f_3(\underset{\sim}{x}, \hat{x}_3) = \varepsilon^{-2} f_3^\varepsilon(\underset{\sim}{x}, x_3), \\ \underset{\sim}{g}(\underset{\sim}{x}) &= \varepsilon^{-2} \underset{\sim}{g}^\varepsilon(\underset{\sim}{x}), \quad g_3(\underset{\sim}{x}) = \varepsilon^{-3} g_3^\varepsilon(\underset{\sim}{x}). \end{aligned} \tag{6}$$

We assume that $\underline{f}$ and $\underline{g}$ are independent of ε. The exact scaling of $\underline{f}^\varepsilon$ and $\underline{g}^\varepsilon$ is immaterial, since the equations governing the problem are linear, and we shall consider errors in relative norms.

The asymptotic expansion for $\underline{u}^\varepsilon$ is then

$$\begin{aligned} \underline{u}^\varepsilon(\underline{x}) \sim \begin{pmatrix} 0 \\ \zeta^0(\underset{\sim}{x}) \end{pmatrix} &+ \varepsilon \underline{u}^1(\underset{\sim}{x}, \varepsilon^{-1} x_3) + \varepsilon^2 [\underline{u}^2(\underset{\sim}{x}, \varepsilon^{-1} x_3) + \underline{w}^2(\underset{\sim}{x}, \varepsilon^{-1}\rho, \varepsilon^{-1} x_3)] \\ &+ \varepsilon^3 [\underline{u}^3(\underset{\sim}{x}, \varepsilon^{-1} x_3) + \underline{w}^3(\underset{\sim}{x}, \varepsilon^{-1}\rho, \varepsilon^{-1} x_3)] + \cdots . \end{aligned} \tag{7}$$

We proceed to explain the terms present in the expansion. The $\underline{u}^k$ are ε-independent functions defined in the domain $\Omega \times (-1, 1)$, and can be further decomposed as

$$\underline{u}^k(\underset{\sim}{x}, \hat{x}_3) = \mathring{\underline{u}}^k(\underset{\sim}{x}, \hat{x}_3) + \begin{pmatrix} -\hat{x}_3 \underset{\sim}{\nabla} \zeta^{k-1}(\underset{\sim}{x}) \\ \zeta^k(\underset{\sim}{x}) \end{pmatrix},$$

where ζ^k are defined in the domain Ω, and

$$\int_{-1}^{1} \mathring{\underline{u}}^k(\underset{\sim}{x}, \hat{x}_3)\, d\hat{x}_3 = 0 \qquad \text{for all } \underset{\sim}{x} \in \Omega. \tag{8}$$

There are also the boundary correctors $\underline{w}^k$. These correctors represent the boundary layers present in the exact solution, and decay exponentially fast to zero with $\varepsilon^{-1}\rho$, where $\rho = \operatorname{dist}(\underset{\sim}{x}, \partial\Omega)$ is the distance of $\underset{\sim}{x}$ to the boundary $\partial\Omega$.

The computation of each term in the series (7) depends on previously computed terms. The equations determining $\mathring{u}_3^k$ for instance are as follows.

$$
\begin{aligned}
&\frac{2\mu+\lambda}{\mu}\frac{\partial^2}{\partial \hat{x}_3^2}\mathring{u}_3^k(\underset{\sim}{x},\hat{x}_3) = -\,\Delta_{2D}\, u_3^{k-2}(\underset{\sim}{x},\hat{x}_3) - \frac{\mu+\lambda}{\mu}\frac{\partial}{\partial \hat{x}_3}\operatorname{div}\underset{\sim}{u}^{k-1}(\underset{\sim}{x},\hat{x}_3) - \frac{1}{\mu}\delta_{4,k} f_3(\underset{\sim}{x},\hat{x}_3),\\
&\frac{\partial}{\partial \hat{x}_3}\mathring{u}_3^k(\underset{\sim}{x},\hat{x}_3) = -\frac{\lambda}{2\mu+\lambda}\operatorname{div}\underset{\sim}{u}^{k-1}(\underset{\sim}{x},\hat{x}_3) + \hat{x}_3\frac{1}{2\mu+\lambda}\delta_{4,k} g_3(\underset{\sim}{x}) \qquad \text{for } \hat{x}_3\in\{-1,1\}.
\end{aligned} \tag{9}
$$

Above, and throughout this section, we use the operator $\Delta_{2D} = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ to avoid confusion with the three-dimensional laplacian.

To compute $\mathring{\underset{\sim}{u}}^k$, it is necessary to solve the following equations.

$$
\begin{aligned}
&\frac{\partial^2}{\partial \hat{x}_3^2}\mathring{\underset{\sim}{u}}^k(\underset{\sim}{x},\hat{x}_3) = -\,\Delta_{2D}\,\underset{\sim}{u}^{k-2}(\underset{\sim}{x},\hat{x}_3) - \frac{\mu+\lambda}{\mu}\underset{\sim}{\nabla}(\operatorname{div}\underset{\sim}{u}^{k-2} + \frac{\partial}{\partial \hat{x}_3}\mathring{u}_3^{k-1})(\underset{\sim}{x},\hat{x}_3) - \frac{1}{\mu}\delta_{3,k}\underset{\sim}{f}(\underset{\sim}{x},\hat{x}_3),\\
&\qquad\frac{\partial}{\partial \hat{x}_3}\mathring{\underset{\sim}{u}}^k(\underset{\sim}{x},\hat{x}_3) = -\,\underset{\sim}{\nabla}\,\mathring{u}_3^{k-1}(\underset{\sim}{x},\hat{x}_3) + \frac{1}{\mu}\delta_{3,k}\underset{\sim}{g}(\underset{\sim}{x}) \qquad \text{for } \hat{x}_3\in\{-1,1\}.
\end{aligned} \tag{10}
$$

It is a long but somewhat straightforward computation to check that the compatibility conditions of equations (9) and (10) hold if and only if

$$
\begin{aligned}
D\,\Delta_{2D}^2\,\zeta^k = &\int_{-1}^1 \Bigg[\frac{\lambda^*}{4}\hat{x}_3^2\,\Delta_{2D}(\Delta_{2D}\,\mathring{u}_3^k + \frac{\partial}{\partial \hat{x}_3}\operatorname{div}\mathring{\underset{\sim}{u}}^{k+1}) + \frac{3}{2}D\hat{x}_3\,\Delta_{2D}\operatorname{div}\mathring{\underset{\sim}{u}}^{k+1}\\
&+ \frac{\lambda}{2(2\mu+\lambda)}\hat{x}_3^2\delta_{k,2}\,\Delta_{2D}\, f_3 + \delta_{k,0}(\hat{x}_3\operatorname{div}\underset{\sim}{f} + f_3)\Bigg]\,d\hat{x}_3 + \frac{\lambda}{2\mu+\lambda}\delta_{k,2}\,\Delta_{2D}\, g_3 + 2\delta_{k,0}(\operatorname{div}\underset{\sim}{g} + g_3),
\end{aligned} \tag{11}
$$

where $\lambda^* = 2\mu\lambda/(2\mu+\lambda)$.

It is clear from (9) and (10) that, in general, $\mathring{\underline{u}}^k$ do not vanish on the lateral boundary. We introduce then the boundary correctors $\underline{w}^k$, which satisfy $\underline{w}^k = \underline{u}^k$ on the lateral boundary. Also, $\underline{w}^k$ decay to zero as long as the traces of ζ^k and $\partial\zeta^{k-1}/\partial n$ on $\partial\Omega$ are defined properly. There is actually a unique, nontrivial way to define such traces, and hence not only $\underline{w}^k$ are well-defined, but also the ζ^k can be determined from the biharmonic equation (11).

We do not attempt here to rigorously define the boundary correctors, since the many technicalities involved would overshadow our main goal. The following properties are nonetheless important.

$$
\text{Given } \left.\mathring{\underline{u}}^k\right|_{\partial\Omega\times(-1,1)},\ \text{then } \left.\zeta^k\right|_{\partial\Omega},\ \left.\frac{\partial\zeta^{k-1}}{\partial n}\right|_{\partial\Omega},\ \text{and } \underline{w}^k \text{ are well-determined.} \tag{12}
$$

$$
\text{If } \left.\mathring{\underline{u}}^k\right|_{\partial\Omega\times(-1,1)} \equiv 0,\ \text{then } \left.\zeta^k\right|_{\partial\Omega},\ \left.\frac{\partial\zeta^{k-1}}{\partial n}\right|_{\partial\Omega},\ \text{and } \underline{w}^k \text{ vanish identically.} \tag{13}
$$

Remark. In practice, an accurate computation of the boundary correctors is exceedingly expensive. If there is interest in the boundary layer properties of the solution, it is just better to use a direct 3D approach, or, preferably, hierarchical models, which can capture with arbitrary precision the boundary layer part of the solution. See [8, 11] and references therein.

All the terms in the asymptotic expansion are now uniquely determined. Indeed, from equations (8), (9) and (10), we see that $\mathring{\underline{u}}^0 = 0$. Hence, from (8) and (13), $\underline{w}^0 = 0$ and $\zeta^0|_{\partial\Omega} = 0$. Next, from (8), (9), (10), $\mathring{\underline{u}}^1 = 0$, and then $\underline{w}^1 = 0$ and $\zeta^1|_{\partial\Omega} = \partial\zeta^0/\partial n|_{\partial\Omega} = 0$. We can now compute ζ^0 using (11), or equivalently, (3). Proceeding one step further, (8) and (9) imply that $\mathring{u}_3^2(\underset{\sim}{x}, \hat{x}_3) = \lambda\,\Delta\,\zeta^0(\underset{\sim}{x})(\hat{x}_3^2/2 - 1/6)/(2\mu+\lambda)$, and (8), (10) imply that $\underset{\sim}{\mathring{u}}^2 = 0$. Since $\mathring{\underline{u}}^2$ does not necessarily vanish on ∂P_L, a boundary corrector is necessary. So, based on property (12), we determine $\underline{w}^2$, $\zeta^2|_{\partial\Omega}$, and $\partial\zeta^1/\partial n|_{\partial\Omega}$, and these are nontrivial functions in general. Using (11) again, we find ζ^1. Next we determine $\mathring{u}_3^3$ as a quadratic polynomial in $\hat{x}_3$ that depends on ζ^0. If the in plane volume load varies linearly with the transverse variable, i.e., $\underset{\sim}{f}(\underset{\sim}{x}, \hat{x}_3) = \hat{x}_3 \underset{\sim}{\check{f}}(\underset{\sim}{x})$, then

(14)
$$\underset{\sim}{\mathring{u}}^3(\underset{\sim}{x}, \hat{x}_3) = \frac{1}{3(2\mu+\lambda)}\left[\frac{4\mu+3\lambda}{2}\hat{x}_3^3 - (6\mu+5\lambda)\hat{x}_3\right]\underset{\sim}{\nabla}\,\Delta\,\zeta^0(\underset{\sim}{x}) + \frac{1}{\mu}\hat{x}_3\underset{\sim}{g}(\underset{\sim}{x}) - \frac{1}{2\mu}\Big(\frac{1}{3}\hat{x}_3^3 - \hat{x}_3\Big)\underset{\sim}{\check{f}}(\underset{\sim}{x}).$$

This procedure goes on indefinitely, and the general trend is that, given $\mathring{\underline{u}}^0, \cdots, \mathring{\underline{u}}^{k-1}$, $\zeta^0, \cdots, \zeta^{k-2}$, and $\underline{w}^0, \cdots, \underline{w}^{k-1}$, we use (8), (9), (10) to determine $\mathring{\underline{u}}^k$, and then find $\underline{w}^k$, $\zeta^k|_{\partial\Omega}$, and $\partial\zeta^{k-1}/\partial n|_{\partial\Omega}$. Finally, from (11) we compute ζ^{k-1}.

We rewrite the asymptotic expansion (7) in a more convenient form:

$$\begin{aligned}(15)\quad \underline{u}^\varepsilon(\underline{x}) \sim \underline{u}^0_{KL}(\underline{x}) + \varepsilon\underline{u}^1_{KL}(\underline{x}) + \varepsilon^2[\underline{u}^2_{KL}(\underline{x}) + \mathring{\underline{u}}^2(\underset{\sim}{x}, \varepsilon^{-1}x_3) + \underline{w}^2(\underset{\sim}{x}, \varepsilon^{-1}\rho, \varepsilon^{-1}x_3)]\\ + \varepsilon^3[\underline{u}^2_{KL}(\underline{x}) + \mathring{\underline{u}}^3(\underset{\sim}{x}, \varepsilon^{-1}x_3) + \underline{w}^3(\underset{\sim}{x}, \varepsilon^{-1}\rho, \varepsilon^{-1}x_3)] + \cdots,\end{aligned}$$

where

$$\underline{u}^k_{KL}(\underline{x}) = \begin{pmatrix} -x_3\,\underset{\sim}{\nabla}\,\zeta^k(\underset{\sim}{x}) \\ \zeta^k(\underset{\sim}{x}) \end{pmatrix}.$$

Although the expansion (15) is formal, the next theorem shows how to interpret it.

Theorem 1. *For each positive integer N, there exists a constant c, that also depends on Ω, and on $\underline{f}$ and $\underline{g}$, but is independent of ε, such that*

$$(16)\quad \left\|\underline{u}^\varepsilon - \underline{u}^0_{KL}\right\|_{E(P^\varepsilon)} \le c\varepsilon^{3/2}, \qquad \left\|\underline{u}^\varepsilon - \sum_{k=0}^N \varepsilon^k\big(\underline{u}^k_{KL} + \mathring{\underline{u}}^k\big) + \sum_{k=2}^N \varepsilon^k \underline{w}^k\right\|_{E(P^\varepsilon)} \le c\varepsilon^{N+1/2}.$$

The above result basically says that the energy norm of the difference between the exact solution and a truncated asymptotic expansion is of the same order in ε as the term of highest energy norm in the remainder series. Indeed, for each nonnegative k there exists a constant c such that

$$\|\underline{u}^k_{KL}\|_{E(P^\varepsilon)} \le c\varepsilon^{3/2}, \quad \|\mathring{\underline{u}}^k\|_{E(P^\varepsilon)} \le c\varepsilon^{-1/2}, \quad \|\underline{w}^k\|_{E(P^\varepsilon)} \le c, \tag{17}$$

and the right hand sides of the inequalities in (16) are of the same order as $\varepsilon^2\|\mathring{\underline{u}}^2\|_{E(P^\varepsilon)}$ and $\varepsilon^{N+1}\|\mathring{\underline{u}}^{N+1}\|_{E(P^\varepsilon)}$.

2. An improved Biharmonic model

We next identify the leading terms of the asymptotic expansions for the solution of (1) under several load situations. By "leading" we mean the term with highest energy norm.

Case I. We assume first that the loads are such that the right hand side of (3) is nonzero. From (16), (17), the estimate $\|\underline{u}^\varepsilon - \underline{u}^B\|_{E(P^\varepsilon)} \le c\varepsilon^2$ holds, where $\underline{u}^B$ is defined by (2). Under the scaling assumption (6), the function ζ^0 is independent of ε. Hence, $\|\underline{u}^\varepsilon\|_{E(P^\varepsilon)} \ge c\varepsilon^{3/2}$ and

$$\frac{\|\underline{u}^\varepsilon - \underline{u}^B\|_{E(P^\varepsilon)}}{\|\underline{u}^\varepsilon\|_{E(P^\varepsilon)}} \le c\varepsilon^{1/2}. \tag{18}$$

Remark. In the case of a "periodic plate" there are no boundary layers, and therefore the rates of convergence improve. Indeed, in this case,

$$\underline{u}^\varepsilon(\underline{x}) \sim \underline{u}^B(\underline{x}) + \varepsilon^2 \begin{pmatrix} -x_3 \underset{\sim}{\nabla}\, \zeta^2(\underset{\sim}{x}) \\ \zeta^2(\underset{\sim}{x}) \end{pmatrix} + \varepsilon^3 \begin{pmatrix} \mathring{\underset{\sim}{u}}^3(\underset{\sim}{x}, \varepsilon^{-1}x_3) \\ 0 \end{pmatrix} + \cdots,$$

and then the displacement $\underset{\sim}{u}^B$ converges as $O(\varepsilon)$ in the relative energy norm. Post-processing the displacement even further and including the contributions of $\mathring{\underset{\sim}{u}}^3$, the convergence rate is $O(\varepsilon^2)$ [12, 3].

Case II. Assume now that the load is such that

$$\int_{-1}^{1} [\hat{x}_3 \operatorname{div} \underset{\sim}{f}(\underset{\sim}{x}, \hat{x}_3) + f_3(\underset{\sim}{x}, \hat{x}_3)]\, d\hat{x}_3 + 2[\operatorname{div} \underset{\sim}{g}(\underset{\sim}{x}) + g_3(\underset{\sim}{x})] = 0, \tag{19}$$

or equivalently, that $\zeta^0 = 0$. We also assume that $\underset{\sim}{f}$ and $\underset{\sim}{g}$ are not both identically zero. Such situation includes the pure shear case, when $\underset{\sim}{f}$, f_3 and g_3 vanish, but $\underset{\sim}{g}$ is a nonzero constant. The asymptotic expansion is as follows in this case:

$$\underline{u}^\varepsilon(\underline{x}) \sim \underline{u}_{II}(\underline{x}) + \varepsilon^2 \begin{pmatrix} -x_3 \underset{\sim}{\nabla}\, \zeta^2(\underset{\sim}{x}) \\ \zeta^2(\underset{\sim}{x}) \end{pmatrix} + \varepsilon^3 \begin{pmatrix} -x_3 \underset{\sim}{\nabla}\, \zeta^3(\underset{\sim}{x}) \\ \zeta^3(\underset{\sim}{x}) \end{pmatrix} + \varepsilon^3 \underline{w}^3(\underset{\sim}{x}, \varepsilon^{-1}\rho, \varepsilon^{-1}x_3) + \cdots,$$

and the high energy term

$$\underline{u}_{II}(\underset{\sim}{x}, x_3) = \varepsilon^3 \begin{pmatrix} \mathring{\underset{\sim}{u}}^3(\underset{\sim}{x}, \varepsilon^{-1}x_3) \\ 0 \end{pmatrix}, \tag{20}$$

where

$$\begin{gathered} \frac{\partial^2}{\partial \hat{x}_3^2} \mathring{\underset{\sim}{u}}^3(\underset{\sim}{x}, \hat{x}_3) = -\frac{1}{\mu} \underset{\sim}{f}(\underset{\sim}{x}, \hat{x}_3), \\ \frac{\partial}{\partial \hat{x}_3} \mathring{\underset{\sim}{u}}^3(\underset{\sim}{x}, \hat{x}_3) = \frac{1}{\mu} \underset{\sim}{g}(\underset{\sim}{x}) \qquad \text{for } \hat{x}_3 \in \{-1, 1\}, \qquad \int_{-1}^{1} \mathring{\underset{\sim}{u}}^3(\underset{\sim}{x}, \hat{x}_3)\, d\hat{x}_3 = 0 \qquad \text{for } \underset{\sim}{x} \in \Omega. \end{gathered} \tag{21}$$

If for instance $\underset{\sim}{f}(\underset{\sim}{x}, \hat{x}_3) = \check{\underset{\sim}{f}}(\underset{\sim}{x})\hat{x}_3$, then, as a special case of (14),

$$\underset{\sim}{u}_{II}(\underset{\sim}{x}, x_3) = \frac{1}{\mu} \varepsilon^2 x_3 \underset{\sim}{g}(\underset{\sim}{x}) + \frac{1}{2\mu}(\varepsilon^2 x_3 - \frac{1}{3} x_3^3) \check{\underset{\sim}{f}}(\underset{\sim}{x}).$$

It follows from (16), (17) that

$$\frac{\|\underline{u}^\varepsilon - \underline{u}_{II}\|_{E(P^\varepsilon)}}{\|\underline{u}^\varepsilon\|_{E(P^\varepsilon)}} \le c\varepsilon^{1/2}. \tag{22}$$

Note that, in general, the displacement $\underline{u}_{II}$ does not vanish on the lateral boundary, but its fiber average does.

Case III. In addition to (19), assume that both $\underset{\sim}{f}$ and $\underset{\sim}{g}$ are identically zero. Hence, not only $\zeta^0 = 0$, but also $\mathring{\underset{\sim}{u}}^3 = 0$. It follows from (19) that

$$g_3(\underset{\sim}{x}) = -\frac{1}{2} \int_{-1}^{1} f_3(\underset{\sim}{x}, \hat{x}_3)\, d\hat{x}_3. \tag{23}$$

Note that (23) includes the case of a vertical traction sustaining the weight of the plate.

The asymptotic expansion for $\underline{u}^\varepsilon$ has the form

$$\underline{u}^\varepsilon(\underline{x}) \sim \underline{u}_{III}(\underline{x}) + \varepsilon^4 \underline{w}^4(\underset{\sim}{x}, \varepsilon^{-1}\rho, \varepsilon^{-1}x_3) + \cdots,$$

where

$$\underline{u}_{III}(\underline{x}) = \varepsilon^2 \begin{pmatrix} -x_3 \underset{\sim}{\nabla} \zeta^2(\underset{\sim}{x}) \\ \zeta^2(\underset{\sim}{x}) \end{pmatrix} + \varepsilon^4 \begin{pmatrix} 0 \\ \mathring{u}_3^4(\underset{\sim}{x}, \varepsilon^{-1}x_3) \end{pmatrix}. \tag{24}$$

The function ζ^2 satisfies

$$\begin{gathered} D\, \Delta^2 \zeta^2(\underset{\sim}{x}) = \frac{\lambda}{2\mu + \lambda} \Delta \left[\int_{-1}^{1} \frac{\hat{x}_3^2}{2} f_3(\underset{\sim}{x}, \hat{x}_3)\, d\hat{x}_3 + g_3(\underset{\sim}{x}) \right] \qquad \text{in } \Omega, \\ \zeta^2 = \frac{\partial \zeta^2}{\partial n} = 0 \qquad \text{on } \partial\Omega, \end{gathered}$$

and $\mathring{u}_3^4$ is determined by

$$\begin{aligned}
&\frac{\partial^2}{\partial\hat{x}_3^2}\mathring{u}_3^4(\underset{\sim}{x},\hat{x}_3) = \frac{\lambda}{2\mu+\lambda}\,\Delta\,\zeta^2(\underset{\sim}{x}) - \frac{1}{2\mu+\lambda}f_3(\underset{\sim}{x},\hat{x}_3),\\
&\frac{\partial}{\partial\hat{x}_3}\mathring{u}_3^4(\underset{\sim}{x},\hat{x}_3) = \frac{\lambda}{2\mu+\lambda}\hat{x}_3\,\Delta\,\zeta^2(\underset{\sim}{x}) + \frac{1}{2\mu+\lambda}\hat{x}_3 g_3(\underset{\sim}{x}) \qquad \text{for } \hat{x}_3\in\{-1,1\},\\
&\int_{-1}^{1}\mathring{\underline{u}}^4(\underset{\sim}{x},\hat{x}_3)\,d\hat{x}_3 = 0 \qquad \text{for all } \underset{\sim}{x}\in\Omega.
\end{aligned} \tag{25}$$

Again, there is a $\varepsilon^{1/2}$ convergence if the high energy terms are selected as an approximation to $\underline{u}^\varepsilon$:

$$\frac{\|\underline{u}^\varepsilon - \underline{u}_{III}\|_{E(P^\varepsilon)}}{\|\underline{u}^\varepsilon\|_{E(P^\varepsilon)}} \le c\varepsilon^{1/2}. \tag{26}$$

There is no need to consider further cases, since $\underline{u}^B$, $\underline{u}_{II}$ and $\underline{u}_{III}$ are all identically zero only if $\underline{f}$, $\underline{g}$ vanish identically. Based on the asymptotic arguments presented in Cases I, II, and III, we propose an approximation for $\underline{u}^\varepsilon$ that converges for all possible loading combinations. Define the modified biharmonic approximation

$$\underline{u}^{MB} = \underline{u}^B + \underline{u}_{II} + \underline{u}_{III}, \tag{27}$$

where $\underline{u}^B$ is defined by (2), $\underline{u}_{II}$ is as in (20), and $\underline{u}_{III}$ is as in (24). The following theorem holds, and its proof follows from the estimates (18), (22) and (26).

Theorem 2. *Let $\underline{u}^\varepsilon$ be the solution of problem* (1)*, and let $\underline{u}^{MB}$ be defined by* (27)*. Assume that the functions $\underline{f}$ and $\underline{g}$ defined by* (6) *are independent of ε. Then there exists a constant c such that*

$$\frac{\|\underline{u}^\varepsilon - \underline{u}^{MB}\|_{E(P^\varepsilon)}}{\|\underline{u}^\varepsilon\|_{E(P^\varepsilon)}} \le c\varepsilon^{1/2}.$$

Such constant depends on the domain Ω, and on $\underline{f}$ and $\underline{g}$, but is independent of ε.

Remark. In the above theorem, to conclude that the constant is independent of ε, we rely on the premise that $\underline{f}$ and $\underline{g}$ are independent of ε. This is commonly assumed in the literature to justify dimension reduction models. As Ciarlet [1, p. xxiii] points out, "...the magnitude of the components of the applied loads and of the Lamé constants must behave as appropriate powers of the thickness...." This sort of restriction was thoroughly discussed by Miara [9], where she, in her own words, gives "a complete justification of these scalings and assumptions on the data in the linearized case."

3. Convergence in other norms

Of course other norms are also of interest, and the *modus operandi* to construct convergent approximations is the same. We discuss here the $L^2(P^\varepsilon)$ and $H^1(P^\varepsilon)$ norms. In general, to obtain convergence in these norms, it is required to resolve the boundary layer part, and this is not possible by simply solving equations in Ω. Nevertheless, if $\underline{f} = 0$, it is still viable to design convergent models. We present here the final results without going into the details and calculations involved.

For the $L^2(P^\varepsilon)$ and $H^1(P^\varepsilon)$ norms, we have that

$$\|\underline{u}^k_{KL}\|_{L^2(P^\varepsilon)} + \|\mathring{\underline{u}}^k\|_{L^2(P^\varepsilon)} \le c\varepsilon^{1/2}, \quad \|\underline{w}^k\|_{L^2(P^\varepsilon)} \le c\varepsilon,$$
$$\|\underline{u}^k_{KL}\|_{H^1(P^\varepsilon)} \le c\varepsilon^{1/2}, \quad \|\mathring{\underline{u}}^k\|_{H^1(P^\varepsilon)} \le c\varepsilon^{-1/2}, \quad \|\underline{w}^k\|_{H^1(P^\varepsilon)} \le c.$$

Led by the above estimates, we define the approximation

$$\underline{u}^M = \begin{pmatrix} -x_3 \underset{\sim}{\nabla}\, \zeta^0(\underset{\sim}{x}) \\ \zeta^0(\underset{\sim}{x}) \end{pmatrix} + \varepsilon^2 \begin{pmatrix} -x_3 \underset{\sim}{\nabla}\, \zeta^2(\underset{\sim}{x}) \\ \zeta^2(\underset{\sim}{x}) \end{pmatrix} + \frac{1}{\mu}\varepsilon^2 x_3 \begin{pmatrix} \underset{\sim}{g}(\underset{\sim}{x}) \\ 0 \end{pmatrix},$$

where ζ^2 solves (11) with $k = 2$, $\mathring{u}^2_3 = 0$, $\mathring{\underset{\sim}{u}}^3 = (\hat{x}_3/\mu)\underset{\sim}{g}$, and the boundary conditions $\zeta^2 = 0$ and $\partial\zeta^2/\partial n = (\underset{\sim}{g} \cdot \underset{\sim}{n})/\mu$ on $\partial\Omega$.

The convergence then is as follows:

$$\frac{\|\underline{u}^\varepsilon - \underline{u}^M\|_{L^2(P^\varepsilon)}}{\|\underline{u}^\varepsilon\|_{L^2(P^\varepsilon)}} + \frac{\|\underline{u}^\varepsilon - \underline{u}^M\|_{H^1(P^\varepsilon)}}{\|\underline{u}^\varepsilon\|_{H^1(P^\varepsilon)}} \le c\varepsilon^{1/2}.$$

Remark. Although the convergence of $O(\varepsilon^{1/2})$ in the L^2 norm looks pessimistic, it can indeed occur. If however $\zeta^0 \ne 0$, the relative error improves to $O(\varepsilon)$.

4. Conclusions

To illustrate our arguments, we consider a thin (two-dimensional) beam with Lamé coefficients $\mu = 41.67$ and $\lambda = 27.78$, and of thickness 1/10. The beam is subjected to the traction load $(-40, -0.1)$ at the top, and $(40, -0.1)$ at the bottom. There is no volume load. We show in Figure 1 a displaced strip using the elasticity equation, and the approximations given by the usual biharmonic model and the model defined here. It is noticeable that the solution given by the biharmonic is quite far from the exact solution, while the modified model solution almost completely coincides with the exact one.

The traditional biharmonic model gives deceiving results in some situations and a model of Reissner–Mindlin type is to be preferred in general. But if there is some reason to use the biharmonic model, then some extra terms can be added to the model solution to render it convergent in more general situations. In most practical situations (cases I and II), there is

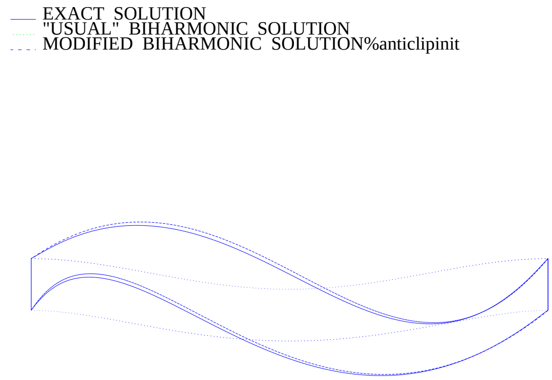

FIGURE 1. Strip displacement

only need to include terms given directly from the load data, and no further computation is needed to make the model convergent in the energy norm.

The technique to find out which terms to add is based on asymptotic expansions, and is flexible enough to give satisfactory answers regardless of the norm which convergence is sought.

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