Adaptive FETI-DP Discretizations

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1 Summary

Major progress has been made recently to make FETI-DP and BDDC preconditioners robust with respect to any variation of coefficients inside and/or across the subdomains. A reason for this success is the adaptive selection of primal constraints technique based on local generalized eigenvalue problems. Here we introduce a mathematical framework to transfer this technique to the field of discretizations. We design discretizations where the number of degrees of freedom is the number of primal constraints on the coarse triangulation and associated basis functions are built on the fine mesh and with a priori energy error estimates independent of the contrast of the coefficients.

2 Hybrid Primal Formulation

Consider the problem of finding the weak solution $u : \Omega \to \mathbb{R}$ of

$$\begin{aligned} -\operatorname{div} \rho \, \boldsymbol{\nabla} \, u = \rho g = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ for $d = 2$ or 3 is an open bounded connected domain with polyhedral boundary $\partial\Omega$, the coefficient ρ satisfies $0 < \rho_{\min} \leq \rho(x) \leq \rho_{\max}$ and g is a given forcing data. Define the ρ-weighted $L^2(\Omega)$-norm by $\|g\|_{L^2_\rho(\Omega)} = \|\rho^{1/2} g\|_{L^2(\Omega)}$ and the energy norm by $\|v\|_{H^1_\rho(\Omega)} = \|\rho^{1/2} \, \boldsymbol{\nabla} \, v\|_{L^2(\Omega)}$. We obtain the following stability result:

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$$\|u\|_{H^1_\rho(\Omega)} \le C_P \|g\|_{L^2_\rho(\Omega)},$$

where C_P is the weighted Poincaré constant of $\|v\|_{L^2_\rho(\Omega)} \le C_P |v|_{H^1_\rho(\Omega)}$ for all $v \in H^1_\rho(\Omega)$ vanishing on $\partial\Omega$.

We start by recasting the continuous problem in a weak formulation that depends on a polyhedral and regular mesh $\mathcal{T}_H$, which can be based on different geometries. Without loss of generality, we adopt above and in the remainder of the text, the terminology of three-dimensional domains, denoting for instance the boundaries of the elements by faces. For a given element $\tau \in \mathcal{T}_H$ let $\partial\tau$ denote its boundary and $\boldsymbol{n}^\tau$ the unit size normal vector that points outward τ. We denote by $\boldsymbol{n}$ the outward normal vector on $\partial\Omega$. Consider now the following spaces:

$$\begin{aligned} H^1(\mathcal{T}_H) &= \{v \in L^2(\Omega) : \, v|_\tau \in H^1(\tau), \, \tau \in \mathcal{T}_H\}, \\ \Lambda(\mathcal{T}_H) &= \Big\{ \prod_{\tau \in \mathcal{T}_H} \boldsymbol{\tau} \cdot \boldsymbol{n}^\tau|_{\partial\tau} : \, \boldsymbol{\tau} \in H(\mathrm{div};\Omega) \Big\} \subsetneq \prod_{\tau \in \mathcal{T}_H} H^{-1/2}(\partial\tau). \end{aligned} \tag{2}$$

For w, $v \in H^1(\mathcal{T}_H)$ and $\mu \in \Lambda(\mathcal{T}_H)$ define

$$(w,v)_{\mathcal{T}_H} = \sum_{\tau \in \mathcal{T}_H} \int_\tau wv \, d\boldsymbol{x} \qquad (\mu, v)_{\partial\mathcal{T}_H} = \sum_{\tau \in \mathcal{T}_H} (\mu, v)_{\partial\tau}, \tag{3}$$

where $(\cdot,\cdot)_{\partial\tau}$ is the dual product involving $H^{-1/2}(\partial\tau)$ and $H^{1/2}(\partial\tau)$. Then

$$(\mu, v)_{\partial\tau} = \int_\tau \mathrm{div}\, \boldsymbol{\sigma} v \, d\boldsymbol{x} + \int_\tau \boldsymbol{\sigma} \cdot \boldsymbol{\nabla}\, v \, d\boldsymbol{x}$$

for all $\boldsymbol{\sigma} \in H(\mathrm{div};\tau)$ such that $\boldsymbol{\sigma} \cdot \boldsymbol{n}^\tau = \mu$. We also define the norms

$$\begin{aligned} \|\boldsymbol{\sigma}\|^2_{H_\rho(\mathrm{div};\Omega)} &= \|\rho^{-1/2}\boldsymbol{\sigma}\|^2_{0,\Omega} + \|\rho^{-1/2}\,\mathrm{div}\,\boldsymbol{\sigma}\|^2_{0,\Omega}, \\ \|\mu\|_{H_\rho^{-1/2}(\mathcal{T}_H)} &= \inf_{\substack{\boldsymbol{\sigma} \in H(\mathrm{div};\Omega) \\ \boldsymbol{\sigma} \cdot \boldsymbol{n}^\tau = \mu \text{ on } \partial\tau,\, \tau \in \mathcal{T}_H}} \|\boldsymbol{\sigma}\|_{H_\rho(\mathrm{div};\Omega)}, \\ |v|^2_{H^1_\rho(\mathcal{T}_H)} &= \sum_{\tau \in \mathcal{T}_H} \|\rho^{1/2}\,\boldsymbol{\nabla}\, v\|^2_{0,\tau}. \end{aligned} \tag{4}$$

We use analogous definitions on subsets of $\mathcal{T}_H$, in particular when the subset consists of a single element τ (and in this case we write τ instead of $\{\tau\}$). We note that since $0 < \rho_{\min} \le \rho(x) \le \rho_{\max}$, the space $H_\rho(\mathrm{div};\Omega)$ and $H^1_\rho(\mathcal{T}_H)$ are equal to the spaces $H(\mathrm{div};\Omega)$ and $H^1(\mathcal{T}_H)$, respectively.

In the primal hybrid formulation [3], $u \in H^1(\mathcal{T}_H)$ and $\lambda \in \Lambda(\mathcal{T}_H)$ are such that

$$\begin{aligned} &(\rho\,\boldsymbol{\nabla}\, u, \boldsymbol{\nabla}\, v)_{\mathcal{T}_H} - (\lambda, v)_{\partial\mathcal{T}_H} = (\rho g, v)_{\mathcal{T}_H} && \text{for all } v \in H^1(\mathcal{T}_H), \\ &(\mu, u)_{\partial\mathcal{T}_H} = 0 && \text{for all } \mu \in \Lambda(\mathcal{T}_H). \end{aligned} \tag{5}$$

Following Theorem 1 of [3], it is possible to show that the solution (u,λ) of (5) is such that $u \in H^1(\Omega)$ and vanishing on $\partial\Omega$ satisfies (1) in the weak sense and $\lambda = \rho \boldsymbol{\nabla}\, u \cdot \boldsymbol{n}^\tau$ for all elements τ.

In the spirit of [3, 4, 5] we consider the decomposition

$$H^1(\mathcal{T}_H) = \mathbb{P}^0(\mathcal{T}_H) \oplus \widetilde{H}^1(\mathcal{T}_H),$$

where $\mathbb{P}^0(\mathcal{T}_H)$ is the space of piecewise constants, and $\widetilde{H}^1(\mathcal{T}_H)$ is its $L^2_\rho(\tau)$ orthogonal complement, i.e., the space of functions with zero ρ-weighted average within each element $\tau \in \mathcal{T}_H$

$$\begin{aligned} \mathbb{P}^0(\mathcal{T}_H) &= \{v \in H^1(\mathcal{T}_H) :\, v|_\tau \text{ is constant}, \, \tau \in \mathcal{T}_H\}, \\ \widetilde{H}^1(\mathcal{T}_H) &= \{\tilde{v} \in H^1(\mathcal{T}_H) :\, \int_\tau \rho \tilde{v}\, d\boldsymbol{x} = 0, \, \tau \in \mathcal{T}_H\}. \end{aligned} \tag{6}$$

We then write $u = u^0 + \tilde{u}$, where $u^0 \in \mathbb{P}^0(\mathcal{T}_H)$ and $\tilde{u} \in \widetilde{H}^1(\mathcal{T}_H)$, and find from (5) that

$$\begin{aligned} (\rho \boldsymbol{\nabla}\, \tilde{u}, \boldsymbol{\nabla}\, \tilde{v})_{\mathcal{T}_H} - (\lambda, \tilde{v})_{\partial\mathcal{T}_H} &= (\rho g, \tilde{v})_{\mathcal{T}_H} && \text{for all } \tilde{v} \in \widetilde{H}^1(\mathcal{T}_H), \\ (\lambda, v^0)_{\partial\mathcal{T}_H} &= -(\rho g, v^0)_{\mathcal{T}_H} && \text{for all } v^0 \in \mathbb{P}^0(\mathcal{T}_H), \\ (\mu, u^0 + \tilde{u})_{\partial\mathcal{T}_H} &= 0 && \text{for all } \mu \in \Lambda(\mathcal{T}_H). \end{aligned} \tag{7}$$

Let $T : \Lambda(\mathcal{T}_H) \to \widetilde{H}^1(\mathcal{T}_H)$ and $\tilde{T} : L^2(\Omega) \to \widetilde{H}^1(\mathcal{T}_H)$ be such that, given $\tau \in \mathcal{T}_H$, $\mu \in \Lambda(\mathcal{T}_H)$ and $g \in L^2_\rho(\Omega)$, for all $\tilde{v} \in \widetilde{H}^1(\mathcal{T}_H)$ we have

$$\int_\tau \rho\, \boldsymbol{\nabla}(T\mu) \cdot \boldsymbol{\nabla}\, \tilde{v}\, d\boldsymbol{x} = (\mu, \tilde{v})_{\partial\tau}, \qquad \int_\tau \rho\, \boldsymbol{\nabla}(\tilde{T}g) \cdot \boldsymbol{\nabla}\, \tilde{v}\, d\boldsymbol{x} = (\rho g, \tilde{v})_\tau. \tag{8}$$

Note from the first equation of (7) that $\tilde{u} = T\lambda + \tilde{T}g$, and substituting in the other two equations of (7), we have that $u^0 \in \mathbb{P}^0(\mathcal{T}_H)$ and $\lambda \in \Lambda(\mathcal{T}_H)$ solve

$$\begin{aligned} (\mu, \gamma T\lambda)_{\partial\mathcal{T}_H} + (\mu, u^0)_{\partial\mathcal{T}_H} &= -(\mu, \gamma\tilde{T}g)_{\partial\mathcal{T}_H} && \text{for all } \mu \in \Lambda(\mathcal{T}_H), \\ (\lambda, v^0)_{\partial\mathcal{T}_H} &= -(\rho g, v^0)_{\mathcal{T}_H} && \text{for all } v^0 \in \mathbb{P}^0(\mathcal{T}_H). \end{aligned} \tag{9}$$

From now on we drop the trace operator γ.

We use the unknowns u^0 and λ to reconstruct the u as follows:

$$u = u^0 + \tilde{u} = u^0 + T\lambda + \tilde{T}g. \tag{10}$$

Unlike the HMM [5] and DEM [7], the methods we describe below approximate $\Lambda(\mathcal{T}_H)$ by multiscale basis functions with lowest global energy property and that decay exponentially, achieving optimal energy approximation without requiring regularity of the solution of the problem.

3 Primal Hybrid Finite Element Methods

Let $\mathcal{F}_h$ be a partition of the faces of elements in $\mathcal{T}_H$, refining them in the sense that every (coarse) face of the elements in $\mathcal{T}_H$ can be written as a union of faces of $\mathcal{F}_h$. Let $\Lambda_h \subset \Lambda(\mathcal{T}_H)$ be the space of piecewise constants on $\mathcal{F}_h$, i.e.,

$$\Lambda_h = \{\mu_h \in \Lambda(\mathcal{T}_H) : \mu_h|_{F_h} \text{ is constant on each face } F_h \in \mathcal{F}_h\}.$$

For simplicity, we do not discretize $H^1(\tau)$ and $H(\mathrm{div};\tau)$ for $\tau \in \mathcal{T}_H$. We remark that the methods develop here extend easily when we discretize $H(\mathrm{div};\tau)$ by simplices or cubical elements with lowest order Raviart–Thomas spaces or discretize $H^1(\tau)$ fine enough to resolve the heterogeneities of $\rho(x)$ and to satisfy inf-sup conditions with respect to the space Λ_h.

We then pose the problem of finding $u_h^0 \in \mathbb{P}^0(\mathcal{T}_H)$ and $\lambda_h \in \Lambda_h$ such that

$$\begin{aligned}
(\mu_h, T\lambda_h)_{\partial\mathcal{T}_H} + (\mu_h, u_h^0)_{\partial\mathcal{T}_H} &= -(\mu_h, \tilde{T}g)_{\partial\mathcal{T}_H} && \text{for all } \mu_h \in \Lambda_h, \\
(\lambda_h, v^0)_{\partial\mathcal{T}_H} &= -(\rho g, v_h^0)_{\mathcal{T}_H} && \text{for all } v_h^0 \in \mathbb{P}^0(\mathcal{T}_H).
\end{aligned} \tag{11}$$

We note that T restricted to τ, denoted by $T^\tau : \Lambda_h^\tau \to \widetilde{H}^1(\tau)$ solves

$$(\rho\,\boldsymbol{\nabla}(T^\tau \mu_h^\tau), \boldsymbol{\nabla}\, v)_\tau = (\mu_h^\tau, v)_{\partial\tau} \quad \text{for all } v \in \widetilde{H}^1(\tau)$$

and note that $\rho\,\boldsymbol{\nabla}(T^\tau \mu_h^\tau)\cdot \boldsymbol{n}^\tau = \mu_h$ on $\partial\tau$. Note also that $(\mu_h, T\mu_h)_{\partial\mathcal{T}_H} = 0$ implies $T\mu_h = 0$ and $\mu_h = 0$. As (11) is finite dimensional, it is well-posed since it is injective. We define our approximation as in (10), by

$$u_h = u_h^0 + T\lambda_h + \tilde{T}g. \tag{12}$$

Simple substitutions yield u_h, λ_h solve (5) if $\Lambda(\mathcal{T}_H)$ is replaced by Λ_h, i.e.,

$$\begin{aligned}
(\rho\,\boldsymbol{\nabla}\, u_h, \boldsymbol{\nabla}\, v)_{\mathcal{T}_H} - (\lambda_h, v)_{\partial\mathcal{T}_H} &= (g, v)_{\mathcal{T}_H} && \text{for all } v \in H^1(\mathcal{T}_H), \\
(\mu_h, u_h)_{\partial\mathcal{T}_H} &= 0 && \text{for all } \mu_h \in \Lambda_h.
\end{aligned}$$

We also assume that Λ_h is chosen fine enough so that

$$|u - u_h|^2_{H^1_\rho(\mathcal{T}_H)} = \big(\lambda - \lambda_h, T(\lambda - \lambda_h)\big)_{\mathcal{T}_H} \le \widetilde{\mathcal{H}}^2 \|g\|^2_{L^2_\rho(\Omega)},$$

where $\widetilde{\mathcal{H}}$ represents a "target precision" the method should achieve. For instance, one could choose $\widetilde{\mathcal{H}} = H$ or $\widetilde{\mathcal{H}} = h^s$ for some $0 < s \le 1$. It must be mentioned that λ_h is never computed, only an approximation of order $\widetilde{\mathcal{H}}$.

Above, and in what follows, c denotes an arbitrary constant that does not depend on H, $\widetilde{\mathcal{H}}$, h, ρ. For details and proofs, see [1]. See also [2] for a related multiscale conforming method.

4 Adaptive FETI-DP Spectral Decomposition I

Let $\tau \in \mathcal{T}_H$, F a face of $\partial\tau$, and let $F_\tau^c = \partial\tau \backslash F$. Define

$$\Lambda_h^\tau = \{\mu_h|_{\partial\tau} : \mu_h \in \Lambda_h\}, \Lambda_h^F = \{\mu_h|_F : \mu_h \in \Lambda_h^\tau\}, \Lambda_h^{F_\tau^c} = \{\mu_h|_{F_\tau^c} : \mu_h \in \Lambda_h^\tau\}.$$

Denote $\mu_h^\tau = \{\mu_h^F, \mu_h^{F_\tau^c}\}$ with $\mu_h^\tau \in \Lambda_h^\tau$, $\mu_h^F \in \Lambda_h^F$ and $\mu_h^{F_\tau^c} \in \Lambda_h^{F_\tau^c}$, and define

$$\begin{aligned} T_{FF}^\tau : \Lambda_h^F \to (\Lambda_h^F)', &\qquad T_{F^cF}^\tau : \Lambda_h^F \to (\Lambda_h^{F_\tau^c})' \\ T_{FF^c}^\tau : \Lambda_h^{F_\tau^c} \to (\Lambda_h^F)', &\qquad T_{F^cF^c}^\tau : \Lambda_h^{F_\tau^c} \to (\Lambda_h^{F_\tau^c})', \end{aligned}$$

and note that $\qquad (\mu_h, T^\tau \mu_h)_{\partial\tau} = (\mu_h^F, T_{FF}^\tau \mu_h^F)_F +$

$$(\mu_h^F, T_{FF^c}^\tau \mu_h^{F_\tau^c})_F + (\mu_h^{F_\tau^c}, T_{F^cF}^\tau \mu_h^F)_{F_\tau^c} + (\mu_h^{F_\tau^c}, T_{F^cF^c}^\tau \mu_h^{F_\tau^c})_{F_\tau^c}.$$

It follows from the properties of T^τ that T_{FF}^τ and $T_{F^cF^c}^\tau$ are symmetric and positive definite matrices, and follows by Schur complement arguments that

$$\begin{aligned} (\mu_h^F, T_{FF}^\tau \mu_h^F)_F &= (\{\mu_h^F, 0\}, T^\tau\{\mu_h^F, 0\})_{\partial\tau} \\ &\geq \min_{\nu_h^{F_\tau^c} \in \Lambda_h^{F_\tau^c}} (\{\mu_h^F, \nu_h^{F_\tau^c}\}, T^\tau\{\mu_h^F, \nu_h^{F_\tau^c}\})_{\partial\tau} = (\mu_h^F, \hat{T}_{FF}^\tau \mu_h^F)_F, \end{aligned} \tag{13}$$

where

$$\hat{T}_{FF}^\tau = T_{FF}^\tau - T_{FF^c}^\tau (T_{F^cF^c}^\tau)^{-1} T_{F^cF}^\tau$$

and the minimum is attained at $\nu_h^{F_\tau^c} = -(T_{F^cF^c}^\tau)^{-1} T_{F^cF}^\tau \mu_h^F$.

To take into account high-contrast coefficients, we consider the following generalized eigenvalue problem: Find $(\alpha_i^F, \mu_{i,h}^F) \in (\mathbb{R}, \Lambda_h^F)$ such that:

1. If the face F is shared by elements τ and τ' we solve

$$(\nu_h^F, (T_{FF}^\tau + T_{FF}^{\tau'})\mu_{h,i}^F)_F = \alpha_i^F (\nu_F, (\hat{T}_{FF}^\tau + \hat{T}_{FF}^{\tau'})\mu_{h,i}^F)_F, \quad \forall \nu_h^F \in \Lambda_h^F.$$

2. If the face F is on the boundary $\partial\Omega$ we solve

$$(\nu_h^F, T_{FF}^\tau \mu_{h,i}^F)_F = \alpha_i^F (\nu_h^F, \hat{T}_{FF}^\tau \mu_{h,i}^F)_F, \quad \forall \nu_h^F \in \Lambda_h^F.$$

The use of such generalized eigenvalue problems is known in the domain decomposition community as "adaptive selection of primal constraints". It is used to make preconditioners robust with respect to coefficients; see [6] and references therein. Here, we apply this technique to design robust discretizations; see [8, 2] on related work on component mode synthesis.

Now we decompose $\Lambda_h^F := \Lambda_h^{F,\triangle} \oplus \Lambda_h^{F,\Pi}$ where

$$\Lambda_h^{F,\triangle} := \operatorname{span}\{\mu_{h,i}^F : \alpha_i^F < \alpha_*\}, \qquad \Lambda_h^{F,\Pi} := \operatorname{span}\{\mu_{h,i}^F : \alpha_i^F \geq \alpha_*\}.$$

From (13) we know that $\alpha_i^F \geq 1$. The parameter α_* is defined by the user and it controls how fast is the exponential decay of the multiscale basis functions. We point out that the dimension of the space $\Lambda_h^{F,\Pi}$ is related to the number of connected subregions on $\bar{\tau} \cup \bar{\tau}'$ with large coefficients surrounded by regions with small coefficients. Finally, let $\Lambda_h = \Lambda_h^{\Pi} \oplus \Lambda_h^{\triangle}$, where

$$\begin{aligned}\Lambda_h^{\Pi} &:= \{\mu_h \in \Lambda_h : \mu_h|_F \in \Lambda_h^{F,\Pi} \text{ for all } F \in \partial\mathcal{T}_H\},\\ \Lambda_h^{\triangle} &:= \{\mu_h \in \Lambda_h : \mu_h|_F \in \Lambda_h^{F,\triangle} \text{ for all } F \in \partial\mathcal{T}_H\}.\end{aligned} \tag{14}$$

5 NLSD-Nonlocalized Spectral Decomposition Method I

Define the operator $P : H^1(\Omega) \to \Lambda_h^{\triangle}$ such that for $w \in H^1(\mathcal{T}_H)$,

$$(\mu_h^{\triangle}, TPw)_{\partial\mathcal{T}_H} = (\mu_h^{\triangle}, w)_{\partial\mathcal{T}_H} \quad \text{for all } \mu_h^{\triangle} \in \Lambda_h^{\triangle}. \tag{15}$$

Let us decompose $\lambda_h = \lambda_h^{\Pi} + \lambda_h^{\triangle}$. We first eliminate $\lambda_h^{\triangle}$ from the first equation of (11) to obtain

$$\lambda_h^{\triangle} = -P(u_h^0 + T\lambda_h^{\Pi} + \tilde{T}g), \tag{16}$$

hence

$$u_h = (I - TP)u_h^0 + T(I - PT)\lambda_h^{\Pi} + (I - TP)\tilde{T}g). \tag{17}$$

Then using algebraic manipulations with (11) and (15) we find $u_h^0 \in \mathbb{P}^0(\mathcal{T}_H)$ and $\lambda_h^{\Pi} \in \Lambda_h^{\Pi}$ satisfy:

$$\begin{aligned}(\hat{\mu}_h^{\Pi}, T\hat{\lambda}_h^{\Pi})_{\partial\mathcal{T}_H} + (\hat{\mu}_h^{\Pi}, \hat{u}_h^0)_{\partial\mathcal{T}_H} &= -(\hat{\mu}_h^{\Pi}, \widehat{\tilde{T}g})_{\partial\mathcal{T}_H} \quad \text{for all } \mu_h^{\Pi} \in \Lambda_h^{\Pi}\\ (\hat{\lambda}_h^{\Pi}, \hat{v}_h^0)_{\partial\mathcal{T}_H} - (Pu_0^h, v_0^h)_{\partial\mathcal{T}_H} &= -(\rho\hat{g}, \hat{v}_h^0)_{\mathcal{T}_H} \quad \text{for all } v_h^0 \in \mathbb{P}^0(\mathcal{T}_H),\end{aligned} \tag{18}$$

where the hat functions are non-local multiscale functions defined by

$$\hat{\lambda}_h^{\Pi} = (I - PT)\lambda_h^{\Pi}, \qquad \hat{\mu}_h^{\Pi} = (I - PT)\mu_h^{\Pi}, \qquad \hat{u}_h^0 = (I - TP)u_h^0,$$

$$\hat{v}_h^0 = (I - TP)v_h^0, \qquad \widehat{\tilde{T}g} = (I - TP)\tilde{T}g \quad \text{and} \quad \hat{g} = (I - P\tilde{T})g.$$

We note that the idea of performing global static condensation goes back to the Multiscale Variational Finite Element Method [9]. Recent variations of this method called Localized Orthogonal Decomposition Methods were introduced and analyzed in [11] and references therein. Some theoretical progresses for high-contrast were made in [9] for a class of coefficients and by using overlapping spectral decomposition introduced in [12]. Here in this paper no condition on the coefficient is imposed and the theoretical results are based on non-overlapping decomposition techniques.

5.1 NLSD Method II

In the splitting (17), the non-local term TPu_h^0 adds theoretical difficulties and more complexity on the implementation. We now introduce the Adaptive FETI-DP Spectral Decomposition II such that $Pu_h^0 = 0$. Indeed, first decompose $\Lambda_h = \Lambda_h^{RT} \oplus \tilde{\Lambda}_h^f$, where Λ_h^{RT} $(\tilde{\Lambda}_h^f)$ is the space of constant (average zero) functions on each face F of $\mathcal{T}_H$. Further decompose $\tilde{\Lambda}_h^f = \tilde{\Lambda}_h^{f,\Pi} \oplus \tilde{\Lambda}_h^{f,\triangle}$ by solving the same generalized eigenvalue problem before however on $\tilde{\Lambda}_h^{f,F}$ rather than on Λ_h^F. Denote $\Lambda_h^\Pi = \Lambda_h^{RT} \oplus \tilde{\Lambda}_h^{f,\Pi}$ and $\Lambda_h^\triangle = \tilde{\Lambda}_h^{f,\triangle}$. Repeat the same algebraic steps as in Section 5 and use that $(\mu_h^\triangle, v_h^0)_{\partial\mathcal{T}_H} = 0$. This method is analyzed in [1].

6 LSD-Localized Spectral Decomposition Method II

We next show that the exponential decay of the multiscale basis functions is independently of the coefficient contrast. Hence, instead of building global multiscale basis functions we actually build local basis functions. Lemma 1 implies exponential decay of functions, such as $PT\mu_h^\Pi$ and Pv_h^0 when μ_h^Π and v_h^0 has local support, and Lemma 2 shows $T(P-P^j)v$ decreases exponentially.

For $K \in \mathcal{T}_H$, define $\mathcal{T}_0(K) = \emptyset$, $\mathcal{T}_1(K) = \{K\}$, and for $j = 1, 2, \dots$ let

$$\mathcal{T}_{j+1}(K) = \{\tau \in \mathcal{T}_H : \overline{\tau} \cap \overline{\tau}_j \neq \emptyset \text{ for some } \tau_j \in \mathcal{T}_j(K)\}.$$

Lemma 1. *Let $v \in H^1(\mathcal{T}_H)$ such that $\operatorname{supp} v \subset K$, and $\mu_h^\triangle = Pv$. Then*

$$|T\mu_h^\triangle|^2_{H^1_\rho(\mathcal{T}_H \backslash \mathcal{T}_{j+1}(K))} \leq e^{-\frac{[(j+1)/2]}{1+d^2\alpha_*}} |T\mu_h^\triangle|^2_{H^1_\rho(\mathcal{T}_H)}.$$

We now localize Pv since it decays exponentially when v has local support. For each fixed K, j, let $\Lambda_h^{\triangle,K,j} \subset \Lambda_h^\triangle$ be the set of functions of $\Lambda_h^\triangle$ which vanish on faces of elements in $\mathcal{T}_H \backslash \mathcal{T}_j(K)$. We introduce the operator $P^{K,j}$: $H^1(\mathcal{T}_H) \to \Lambda_h^{\triangle,K,j}$ such that, for $v \in H^1(\mathcal{T}_H)$,

$$(\mu_h^\triangle, TP^{K,j}v)_{\partial\mathcal{T}_H} = (\mu_h^\triangle, v)_{\partial\mathcal{T}_H} \quad \text{for all } \mu_h^\triangle \in \Lambda_h^{\triangle,K,j}.$$

For $v \in H^1(\mathcal{T}_H)$ let v_K be equal to v on K and zero otherwise. We define then $P^j v \in \Lambda_h^\triangle$ by

$$P^j v = \sum_{K \in \mathcal{T}_H} P^{K,j} v_K. \tag{19}$$

Lemma 2. *Let $v \in H^1(\mathcal{T}_H)$ and P defined by (15) and P^j by (19). Then*

$$|T(P-P^j)v|^2_{H^1_\rho(\mathcal{T}_H)} \leq cj^{2d} d^4 \alpha_*^2 e^{-\frac{[(j-3)/2]}{1+d^2\alpha_*}} |v|^2_{H^1_\rho(\mathcal{T}_H)}.$$

We define the LSD methods by (18), (16) and (17) with P_j instead of P. Denote the solution by u_h^j. The follow lemma shows the localization error.

Theorem 1. *For the LSD II method, if* $j = c\left(4d^2\alpha_* \log(C_P/\widetilde{\mathcal{H}})\right)$ *then*

$$|u_h - u_h^j|_{H^1_\rho(\mathcal{T}_H)} \leq c\widetilde{\mathcal{H}}\|g\|_{L^2_\rho(\Omega)}.$$

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