IMA Journal of Numerical Analysis (2007) **27**, 172−197
doi:10.1093/imanum/drl013
Advance Access publication on May 15, 2006

A stabilized finite-element method for the Stokes problem including element and edge residuals

RODOLFO ARAYA† AND GABRIEL R. BARRENECHEA‡
Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile

AND

FRÉDÉRIC VALENTIN§
Departamento de Matemática Aplicada, Laboratório Nacional de Computação Científica Av. Getúlio Vargas, 333, 25651-070 Petrópolis–RJ, Brazil

[Received on 29 July 2005; revised on 7 March 2006]

A new stabilized finite-element method is presented for the Stokes problem. The method is of a Douglas–Wang type, and includes a positive jump term controlling the residual of the Cauchy stress tensor on the internal edges of the triangulation. A priori error estimates are obtained in the natural norms of the unknowns and an a posteriori error estimator is proposed, analysed and tested through numerical experiments.

Keywords: Stokes equation; stabilized method; a posteriori analysis; jump term.

1. Introduction

The use of equal-order interpolation for the pressure and the velocity for the Stokes problem, or the use of piecewise linear elements for the velocity and piecewise constant elements for the pressure, do not lead to stable finite element methods since these choices do not satisfy the discrete Babuska–Brezzi (or inf-sup) condition (see Girault & Raviart, 1986, and the references therein). In order to overcome this problem, several stabilized finite-element methods have been proposed in the last two decades, including those of Brezzi & Pitkäranta (1984), the first consistent SUPG/SD-method (Hughes *et al.*, 1986), GLS methods (Hughes & Franca, 1987; Franca & Stenberg, 1991), bubble condensation-based methods (Baiocchi *et al.*, 1993; Barrenechea & Valentin, 2002), projection methods (Codina & Blasco, 2000; Dohrmann & Bochev, 2004), etc. For an overview on stabilized finite-element methods for the Stokes problem, see Franca *et al.* (1993) and Barth *et al.* (2004), and the recent book Elman *et al.* (2005). Among all the possible methods, the absolutely stabilized finite-element method (or Douglas–Wang method), Douglas & Wang (1989) presents a number of advantages, both from a theoretical and a practical point of view. Indeed, the possibility of having a stable method for any value of the stabilization parameter is desirable, since, in general, the value of this parameter is bounded above by a constant related to an inverse inequality (cf Hughes *et al.*, 1986; Franca & Stenberg, 1991).

†Email: raraya@ing-mat.udec.cl
‡Corresponding author. Email: gbarrene@ing-mat.udec.cl
§Email: valentin@lncc.br

The original Douglas–Wang method includes the possibility of using discontinuous pressure spaces, where a pressure jump term on the inter-element boundaries is added to stabilize the pressure. This jump term is the same as the one in the GLS method (see Hughes & Franca, 1987; Franca & Stenberg, 1991). In recent work (cf Araya *et al.*, 2006), the authors have proposed a multiscale enrichment of the velocity space leading to a family of stabilized finite-element methods containing jump terms on the internal edges of the triangulation. One of the proposed methods involved the jump of the Cauchy stress tensor, which are the jump terms appearing in a natural way in residual-based a posteriori error estimators, but the stabilization parameter involved was subject to a restriction in order to obtain optimal order error estimates.

The aim of this work is to present a new method, preserving all the good features of the original Douglas–Wang method, in particular, the stability independent of the stabilization parameter, but containing jump terms including the residual of the Cauchy stress tensor. The proposed method leads to a positive formulation controlling also the jump of the Cauchy stress tensor in the inter-element boundaries, a fact that will be exploited to obtain optimal-order error estimates in all the terms involved in the formulation. Based on previous considerations, we propose an a posteriori error estimator of a residual type for this method, which is shown to be equivalent to the discretization error.

The plan of the paper is as follows: in Section 2, we present the model problem and the finite-element method to be used. An a priori error analysis is carried out in Section 3, and an a posteriori error estimator is proposed and justified in Section 4. Numerical experiments confirming the a priori error estimations and the performance of the a posteriori error estimator are presented in Section 5, and some final remarks and conclusions are given in Section 6.

2. The model problem and the finite-element method

Let Ω be an bounded open domain in $\mathbb{R}^2$ with polygonal boundary and let $\mathbf{f} \in L^2(\Omega)^2$. Our Stokes problem reads: find $(\mathbf{u}, p) \in H^1(\Omega)^2 \times L^2_0(\Omega)$ such that

$$\begin{aligned} -\nu\,\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\nu \in \mathbb{R}^+$ is the fluid viscosity and $L^2_0(\Omega) := \{q \in L^2(\Omega)\colon (q,1)_\Omega = 0\}$, where $(\cdot,\cdot)_D$ stands for the inner product in $L^2(D)$ (or in $L^2(D)^2$ or $L^2(D)^{2\times 2}$, when necessary), and we denote by $\|\cdot\|_{s,D}$ ($|\cdot|_{s,D}$) the norm (semi-norm) in $H^s(D)$ (or $H^s(D)^2$, if necessary). As usual, $H^0(D) = L^2(D)$ and $|\cdot|_{0,D} = \|\cdot\|_{0,D}$.

The usual variational formulation for problem (2.1) is given by: find $(\mathbf{u}, p) \in \mathbf{V}\times Q := H^1_0(\Omega)^2 \times L^2_0(\Omega)$ such that

$$\mathbf{B}((\mathbf{u},p),(\mathbf{v},q)) = \mathbf{F}(\mathbf{v},q) \quad \forall (\mathbf{v},q)\in \mathbf{V}\times Q, \tag{2.2}$$

where

$$\mathbf{B}((\mathbf{u},p),(\mathbf{v},q)) := \nu(\nabla\mathbf{u},\nabla\mathbf{v})_\Omega - (p,\nabla\cdot\mathbf{v})_\Omega - (q,\nabla\cdot\mathbf{u})_\Omega, \tag{2.3}$$

$$\mathbf{F}(\mathbf{v},q) := (\mathbf{f},\mathbf{v})_\Omega. \tag{2.4}$$

Now, it is well-known (cf Girault & Raviart, 1986) that (2.2) has a unique solution $(\mathbf{u}, p) \in \mathbf{V}\times Q$. Moreover, it may be shown that there exists $C > 0$, independent of ν, such that

$$\sqrt{\nu}|\mathbf{v}|_{1,\Omega} + \frac{1}{\sqrt{\nu}}\|q\|_{0,\Omega} \leqslant C \sup_{(\mathbf{w},t)\in\mathbf{V}\times Q-\{\mathbf{0}\}} \frac{\mathbf{B}((\mathbf{v},q),(\mathbf{w},t))}{\sqrt{\nu}|\mathbf{w}|_{1,\Omega} + \frac{1}{\sqrt{\nu}}\|t\|_{0,\Omega}} \quad \forall(\mathbf{v},q)\in\mathbf{V}\times Q. \tag{2.5}$$

Let $\{\mathscr{T}_h\}_{h>0}$ be a family of regular partitions (meshes) of Ω. For simplicity, we will restrict ourselves to triangular meshes, with elements K with boundary ∂K, $h_K := \operatorname{diam}(K)$ and $h := \max\{h_K : K \in \mathscr{T}_h\}$ (even though most of the results below are also valid for quadrilateral meshes). Let also $\mathscr{E}_\Omega$ be the set of internal edges of the triangulation and, for $F \in \mathscr{E}_\Omega$, let $h_F := |F|$. With these notations, for $k \geqslant 1$ and $l \geqslant 0$, we introduce the following finite-element spaces:

$$V_k := \{v \in C^0(\overline{\Omega}) : v|_K \in \mathbb{P}^k(K)\, \forall K \in \mathscr{T}_h\}, \tag{2.6}$$

where $\mathbb{P}^k$ denotes the set of polynomials of total degree less than or equal to k, and denote by $\mathbf{V}_h := [V_k \cap H_0^1(\Omega)]^2$ the finite-element space for the velocity, and by

$$Q_h := \{q \in L_0^2(\Omega) : q|_K \in \mathbb{P}^l(K)\, \forall K \in \mathscr{T}_h\}, \tag{2.7}$$

the finite element space for the pressure, and state our discrete problem as follows: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\mathbf{B}_\tau((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = \mathbf{F}_\tau(\mathbf{v}_h, q_h) \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h, \tag{2.8}$$

where

$$\mathbf{B}_\tau((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) := \mathbf{B}((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) - \sum_{K \in \mathscr{T}_h} \tau_K(-\nu \Delta \mathbf{u}_h + \nabla p_h, \nu \Delta \mathbf{v}_h + \nabla q_h)_K$$
$$- \sum_{F \in \mathscr{E}_\Omega} \tau_F([\![-\nu \partial_{\mathbf{n}} \mathbf{u}_h + p_h \mathbf{I} \cdot \mathbf{n}]\!], [\![\nu \partial_{\mathbf{n}} \mathbf{v}_h + q_h \mathbf{I} \cdot \mathbf{n}]\!])_F, \tag{2.9}$$

$$\mathbf{F}_\tau(\mathbf{v}_h, q_h) := (\mathbf{f}, \mathbf{v}_h)_\Omega - \sum_{K \in \mathscr{T}_h} \tau_K(\mathbf{f}, \nu \Delta \mathbf{v}_h + \nabla q_h)_K, \tag{2.10}$$

where $[\![v]\!]_F$ stands for the jump of v across F, and the stabilization parameters are given by

$$\tau_K := \alpha \frac{h_K^2}{\nu}, \tag{2.11}$$

$$\tau_F := \beta \frac{h_F}{\nu}, \tag{2.12}$$

with $\alpha, \beta > 0$.

REMARK 2.1 The present method exhibits some differences compared with existing stabilized finite-element methods that include discontinuous pressure interpolations. For example, in Kechkar & Silvester (1992), pressure jump terms are included to stabilize the lowest order pair, but a subdivision of the domain into macro elements was necessary, making the adaptive refinement procedure complicated. Also, in Dohrmann & Bochev (2004) and Bochev *et al.* (2006), projection terms are added to the formulation in order to stabilize the pressure, preserving locality of the method without the need to implement jump terms, but, at the same time, the implementation of such methods for higher-order pressure interpolations is not yet clear.

REMARK 2.2 In the case in which continuous approximations of the pressure are used (i.e. if $Q_h := V_l \cap L_0^2(\Omega)$ with V_l defined in (2.6)), the bilinear form $\mathbf{B}_\tau$ reduces to

$$\mathbf{B}_\tau((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) := \mathbf{B}((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) - \sum_{K \in \mathscr{T}_h} \tau_K(-\nu \Delta \mathbf{u}_h + \nabla p_h, \nu \Delta \mathbf{v}_h + \nabla q_h)_K$$
$$- \sum_{F \in \mathscr{E}_\Omega} \tau_F([\![-\nu \partial_{\mathbf{n}} \mathbf{u}_h]\!], [\![\nu \partial_{\mathbf{n}} \mathbf{v}_h]\!])_F,$$

with jump terms similar to those in Araya *et al.* (2006). However, this does not mean that for continuous pressure elements the two methods coincide, unless we use piecewise linear elements for the velocity, due to the change of sign in the $\Delta \mathbf{v}_h$ term.

3. Stability and a priori error analysis

From now on, C will denote a positive constant independent of h and ν, but possibly depending on α and β (unless otherwise stated), and which may change its value in different occurrences.

Defining the mesh-dependent norm

$$\|(\mathbf{v}_h, q_h)\|_h := \left[\nu |\mathbf{v}_h|^2_{1,\Omega} + \sum_{K \in \mathscr{T}_h} \tau_K |q_h|^2_{1,K} + \sum_{F \in \mathscr{E}_\Omega} \tau_F \|[\![-\nu \partial_{\mathbf{n}} \mathbf{v}_h + q_h \mathbf{I} \cdot \mathbf{n}]\!]\|^2_{0,F} \right]^{\frac{1}{2}}, \tag{3.1}$$

we have the following stability result.

LEMMA 3.1 Let $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$. Then, the bilinear form $\mathbf{B}_\tau$ satisfies

$$\mathbf{B}_\tau((\mathbf{v}_h, q_h), (\mathbf{v}_h, -q_h)) \geqslant C \|(\mathbf{v}_h, q_h)\|^2_h, \tag{3.2}$$

where $C > 0$ is independent of h, ν and β.

Proof. First, the definition of $\mathbf{B}_\tau$ leads to

$$\begin{aligned} \mathbf{B}_\tau((\mathbf{v}_h, q_h), (\mathbf{v}_h, -q_h)) = \nu |\mathbf{v}_h|^2_{1,\Omega} &+ \sum_{K \in \mathscr{T}_h} \tau_K \| - \nu \Delta \mathbf{v}_h + \nabla q_h \|^2_{0,K} \\ &+ \sum_{F \in \mathscr{E}_\Omega} \tau_F \|[\![-\nu \partial_{\mathbf{n}} \mathbf{v}_h + q_h \mathbf{I} \cdot \mathbf{n}]\!]\|^2_{0,F}. \end{aligned}$$

Next, using the following inverse inequality (cf Ern & Guermond, 2004):

$$|\mathbf{v}_h|_{2,K} \leqslant C_k h_K^{-1} |\mathbf{v}_h|_{1,K} \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{3.3}$$

where the constant $C_k > 0$ depends only on k, $2ab \leqslant \gamma a^2 + \gamma^{-1} b^2$ with $\gamma > 1$, and the definition of τ_K, we arrive at

$$\begin{aligned} \nu |\mathbf{v}_h|^2_{1,\Omega} &+ \sum_{K \in \mathscr{T}_h} \tau_K \| - \nu \Delta \mathbf{v}_h + \nabla q_h \|^2_{0,K} \\ &= \nu |\mathbf{v}_h|^2_{1,\Omega} + \sum_{K \in \mathscr{T}_h} \tau_K [\|\nu \Delta \mathbf{v}_h\|^2_{0,K} - 2(\nu \Delta \mathbf{v}_h, \nabla q_h)_K + |q_h|^2_{1,K}] \\ &\geqslant \nu |\mathbf{v}_h|^2_{1,\Omega} + \sum_{K \in \mathscr{T}_h} \tau_K \left[(1-\gamma) \|\nu \Delta \mathbf{v}_h\|^2_{0,K} + \left(1 - \frac{1}{\gamma}\right) |q_h|^2_{1,K} \right] \\ &\geqslant \sum_{K \in \mathscr{T}_h} (1 + 2\alpha(1-\gamma) C_k^2) \nu |\mathbf{v}_h|^2_{1,K} + \left(1 - \frac{1}{\gamma}\right) \sum_{K \in \mathscr{T}_h} \tau_K |q_h|^2_{1,K} \\ &\geqslant C \left(\nu |\mathbf{v}_h|^2_{1,\Omega} + \sum_{K \in \mathscr{T}_h} \tau_K |q_h|^2_{1,K} \right), \end{aligned}$$

where γ has been chosen so that $1 < \gamma < (1 + \frac{C_k^{-2}\alpha^{-1}}{2})$, and the result follows from the definition of $\|\cdot\|_h$. □

REMARK 3.1 It is worth remarking that the most characteristic feature of the Douglas–Wang method, namely the fact that it is stable for any positive value of stabilization parameters α and β, is also valid for our method, and hence, the method is stable independent of the constants involved in its definition. Unlike the proof given in Douglas & Wang (1989), this property is valid for a norm not containing the Laplacian of the velocity. This fact was first remarked in Franca *et al.* (1992) for continuous pressure interpolations.

REMARK 3.2 We further remark that we can obtain a separate control on both the jump of $\partial_{\mathbf{n}}\mathbf{v}_h$ and q_h across the inter-element boundaries, but this imposes a restriction on the value of β. In fact, using the Cauchy–Schwarz inequality, the mesh regularity, a local trace inequality (see (3.7) below) and the inverse inequality (3.3), we arrive at

$$\begin{aligned}
&\sum_{F\in\mathscr{E}_\Omega} \tau_F \|[\![-\nu\partial_{\mathbf{n}}\mathbf{v}_h + q_h\mathbf{I}\cdot\mathbf{n}]\!]\|_{0,F}^2 \\
&\quad = \sum_{F\in\mathscr{E}_\Omega} \tau_F [\|[\![-\nu\partial_{\mathbf{n}}\mathbf{v}_h]\!]\|_{0,F}^2 - 2([\![\nu\partial_{\mathbf{n}}\mathbf{v}_h]\!],[\![q_h]\!])_F + \|[\![q_h]\!]\|_{0,F}^2] \\
&\quad \geqslant \sum_{F\in\mathscr{E}_\Omega} \tau_F \left[\|[\![\nu\partial_{\mathbf{n}}\mathbf{v}_h]\!]\|_{0,F}^2 - \gamma_1\|[\![\nu\partial_{\mathbf{n}}\mathbf{v}_h]\!]\|_{0,F}^2 + \left(1-\frac{1}{\gamma_1}\right)\|[\![q_h]\!]\|_{0,F}^2\right] \\
&\quad \geqslant -C_1\beta\gamma_1\nu|\mathbf{v}_h|_{1,\Omega}^2 + \sum_{F\in\mathscr{E}_\Omega} \tau_F \left[\|[\![\nu\partial_{\mathbf{n}}\mathbf{v}_h]\!]\|_{0,F}^2 + \left(1-\frac{1}{\gamma_1}\right)\|[\![q_h]\!]\|_{0,F}^2\right],
\end{aligned}$$

where $\gamma_1 > 0$ and hence, recalling the proof of Lemma 3.1, we have

$$\begin{aligned}
&\mathbf{B}_\tau((\mathbf{v}_h, q_h),(\mathbf{v}_h, -q_h)) \\
&\quad = \nu|\mathbf{v}_h|_{1,\Omega}^2 + \sum_{K\in\mathscr{T}_h} \tau_K \|-\nu\Delta\mathbf{v}_h + \nabla q_h\|_{0,K}^2 + \sum_{F\in\mathscr{E}_\Omega} \tau_F \|[\![-\nu\partial_{\mathbf{n}}\mathbf{v}_h + q_h\mathbf{I}\cdot\mathbf{n}]\!]\|_{0,F}^2 \\
&\quad \geqslant (1 + 2\alpha(1-\gamma)C_k^2 - C_1\beta\gamma_1)\nu|\mathbf{v}_h|_{1,\Omega}^2 + \left(1-\frac{1}{\gamma}\right)\sum_{K\in\mathscr{T}_h} \tau_K |q_h|_{1,K}^2 \\
&\qquad + \sum_{F\in\mathscr{E}_\Omega} \tau_F \left[\|[\![\nu\partial_{\mathbf{n}}\mathbf{v}_h]\!]\|_{0,F}^2 + \left(1-\frac{1}{\gamma_1}\right)\|[\![q_h]\!]\|_{0,F}^2\right] \\
&\quad \geqslant C_* \left\{\nu|\mathbf{v}_h|_{1,\Omega}^2 + \sum_{K\in\mathscr{T}_h} \tau_K |q_h|_{1,K}^2 + \sum_{F\in\mathscr{E}_\Omega} \tau_F [\|[\![\nu\partial_{\mathbf{n}}\mathbf{v}_h]\!]\|_{0,F}^2 + \|[\![q_h]\!]\|_{0,F}^2]\right\},
\end{aligned}$$

provided that β is bounded above by a suitable constant. This fact is more precise than the bound we actually obtain, but it imposes an undesirable restriction on the stabilization parameter β. Finally, we remark that a positivity result, involving only the jump of the pressure, could be obtained independent of β.

The next result states the consistency of the proposed method.

LEMMA 3.2 Let $(\mathbf{u}, p) \in [H^2(\Omega) \cap H_0^1(\Omega)]^2 \times [H^1(\Omega) \cap L_0^2(\Omega)]$ be the weak solution of (2.1) and $(\mathbf{u}_h, p_h)$ the solution of (2.8). Then,

$$\mathbf{B}_\tau((\mathbf{u} - \mathbf{u}_h, p - p_h), (\mathbf{v}_h, q_h)) = 0 \quad \text{for all } (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h. \tag{3.4}$$

Proof. The result follows from the definition of the continuous problem and noting that

$$[\![-\nu \partial_{\mathbf{n}} \mathbf{u} + p\mathbf{I} \cdot \mathbf{n}]\!] = \mathbf{0},$$

a.e. across all the internal edges. □

3.1 *A priori error analysis*

3.1.1 *Preliminaries.* First, for $K \in \mathcal{T}_h$ and $F \in \mathcal{E}_\Omega$, we denote by $\mathcal{N}(K)$ and $\mathcal{N}(F)$ the set of nodes of K and F, respectively, and by $\mathcal{E}(K)$ the set of sides of K. Also, for $K \in \mathcal{T}_h$ and $F \in \mathcal{E}_\Omega$ we define the following neighbourhoods:

$$\omega_K := \bigcup_{\mathcal{E}(K) \cap \mathcal{E}(K') \neq \emptyset} K', \quad \tilde{\omega}_K := \bigcup_{\mathcal{N}(K) \cap \mathcal{N}(K') \neq \emptyset} K',$$

$$\omega_F := \bigcup_{F \in \mathcal{E}(K')} K', \quad \tilde{\omega}_F := \bigcup_{K' \cap F \neq \emptyset} K'.$$

In order to perform the numerical analysis of (2.8), we will consider the Lagrange interpolation operator $I_h \colon C^0(\overline{\Omega}) \to V_k$ (if $\mathbf{v} = (v_1, v_2) \in C^0(\overline{\Omega})^2$, we denote $I_h(\mathbf{v}) = (I_h(v_1), I_h(v_2))$) to approximate the velocity. Then, it is well-known (cf Ern & Guermond, 2004) that for all $K \in \mathcal{T}_h$ and all $F \in \mathcal{E}(K)$, we have

$$|v - I_h(v)|_{m,K} \leqslant C h_K^{s-m} |v|_{s,K}, \tag{3.5}$$

$$|v - I_h(v)|_{t,F} \leqslant C h_F^{s-t-1/2} |v|_{s,K}, \tag{3.6}$$

for all $v \in H^s(K)$, and $0 \leqslant m \leqslant 2$, $0 \leqslant t \leqslant 1$, $2 \leqslant s \leqslant k+1$. Let us remark that in order to obtain the second estimate mentioned above, we have used the following local trace theorem (for a proof, see Thomée, 1997): there exists $C > 0$, independent of h, such that

$$\|v\|_{0,\partial K}^2 \leqslant C \left(\frac{1}{h_K} \|v\|_{0,K}^2 + h_K |v|_{1,K}^2 \right), \tag{3.7}$$

for all $v \in H^1(K)$.

For the pressure approximation, we have to consider two separate cases:
First case: $l = 0$: To treat this case, we denote by $\Pi_h \colon L^2(\Omega) \to Q_h$ the $L^2(\Omega)$-projection operator onto Q_h. This operator satisfies (cf Ern & Guermond, 2004)

$$\|q - \Pi_h(q)\|_{0,\Omega} \leqslant C h |q|_{1,\Omega}, \tag{3.8}$$

if $q \in H^1(\Omega)$. Hence, using the local trace theorem (3.7) and the regularity of the mesh, we obtain

$$\left[\sum_{F\in\mathscr{E}_\Omega} h_F \|[\![q-\Pi_h(q)]\!]\|_{0,F}^2\right]^{\frac{1}{2}} \leqslant Ch|q|_{1,\Omega}, \tag{3.9}$$

for all $q \in H^1(\Omega)$.

Second case: $l \geqslant 1$: To treat this case, we will consider the Clément interpolation operator (cf Clément, 1975; Ern & Guermond, 2004) $\mathscr{C}_h\colon L^2(\Omega) \to V_l$, satisfying for all $K \in \mathscr{T}_h$ and all $F \in \mathscr{E}_\Omega$,

$$|q-\mathscr{C}_h(q)|_{m,K} \leqslant Ch_K^{s-m}|q|_{s,\tilde{\omega}_K}, \tag{3.10}$$

$$\|q-\mathscr{C}_h(q)\|_{0,F} \leqslant Ch_F^{s-\frac{1}{2}}|q|_{s,\tilde{\omega}_F}, \tag{3.11}$$

for all $q \in H^s(\Omega)$, and all $0 \leqslant m \leqslant 1$, $1 \leqslant s \leqslant l$.

LEMMA 3.3 Let, for $l \geqslant 1$, $s = l$ and for $l = 0$, $s = 1$. Let $(\mathbf{v}, q) \in H^{k+1}(\Omega)^2 \times H^s(\Omega)$ and let us denote $\tilde{q}_h := \mathscr{C}_h(q) - \frac{(\mathscr{C}_h(q),1)_\Omega}{|\Omega|} \in Q_h$ if $l \geqslant 1$ and $\tilde{q}_h := \Pi_h(q)$ if $l = 0$. Then,

$$\sum_{F\in\mathscr{E}_\Omega} \tau_F^{-1}\|\mathbf{v}-I_h(\mathbf{v})\|_{0,F}^2 + \sum_{K\in\mathscr{T}_h} [\tau_K^{-1}\|\mathbf{v}-I_h(\mathbf{v})\|_{0,K}^2 + \nu h_K^2\|\Delta(\mathbf{v}-I_h(\mathbf{v}))\|_{0,K}^2]$$
$$+\frac{1}{\nu}\|q-\tilde{q}_h\|_{0,\Omega}^2 + \|(\mathbf{v}-I_h(\mathbf{v}), q-\tilde{q}_h)\|_h^2 \leqslant C\left(h^{2k}\nu|\mathbf{v}|_{k+1,\Omega}^2 + \frac{h^{2s}}{\nu}|q|_{s,\Omega}^2\right).$$

Proof. For both cases, $l = 0$ and $l \geqslant 1$, the first three terms are bounded using (3.5)–(3.6). Now, for $l = 0$ the proof follows using the definition of $\|\cdot\|_h$ and (3.8)–(3.9). For $l \geqslant 1$, we remark that $\|q-\tilde{q}_h\|_{0,\Omega} \leqslant \|q-\mathscr{C}_h(q)\|_{0,\Omega}$ and $|q-\tilde{q}_h|_{1,\Omega} = |q-\mathscr{C}_h(q)|_{1,\Omega}$, and (3.10)–(3.11) hold, and the proof follows from the definition of the norm. □

3.1.2 *A convergence result.*

THEOREM 3.1 Let $s = l$ if $l \geqslant 1$ and $s = 1$ if $l = 0$. Let us suppose that $(\mathbf{u}, p) \in [H^{k+1}(\Omega) \cap H_0^1(\Omega)]^2 \times [H^s(\Omega) \cap L_0^2(\Omega)]$ is the solution of (2.1) and $(\mathbf{u}_h, p_h)$ is the solution of (2.8), and let us denote $(\mathbf{e}^{\mathbf{u}}, e^p) := (\mathbf{u}-\mathbf{u}_h, p-p_h)$. Then, the following error estimates hold:

$$\|(\mathbf{e}^{\mathbf{u}}, e^p)\|_h \leqslant C\left(h^k\sqrt{\nu}|\mathbf{u}|_{k+1,\Omega} + \frac{h^s}{\sqrt{\nu}}|p|_{s,\Omega}\right), \tag{3.12}$$

$$\|e^p\|_{0,\Omega} \leqslant C(h^k\nu|\mathbf{u}|_{k+1,\Omega} + h^s|p|_{s,\Omega}). \tag{3.13}$$

Proof. We first prove the result for $l \geqslant 1$. Let us denote $(\eta^{\mathbf{u}}, \eta^p) := (\mathbf{u}-I_h(\mathbf{u}), p-(\mathscr{C}_h(p)-\frac{(\mathscr{C}_h(p),1)_\Omega}{|\Omega|}))$ and $(\mathbf{e}_h^{\mathbf{u}}, e_h^p) := (\mathbf{u}_h-I_h(\mathbf{u}), p_h-(\mathscr{C}_h(p)-\frac{(\mathscr{C}_h(p),1)_\Omega}{|\Omega|}))$. From Lemma 3.1, the consistency of the method (cf Lemma 3.2), integration by parts and the Cauchy–Schwarz inequality we have

$$
\begin{aligned}
C\|(\mathbf{e}_h^{\mathbf{u}}, e_h^p)\|_h^2 &\leqslant \mathbf{B}_\tau((\mathbf{e}_h^{\mathbf{u}}, e_h^p), (\mathbf{e}_h^{\mathbf{u}}, -e_h^p)) = \mathbf{B}_\tau((\eta^{\mathbf{u}}, \eta^p), (\mathbf{e}_h^{\mathbf{u}}, -e_h^p)) \\
&= \nu(\nabla\eta^{\mathbf{u}}, \nabla\mathbf{e}_h^{\mathbf{u}})_\Omega - (\eta^p, \nabla\cdot\mathbf{e}_h^{\mathbf{u}})_\Omega + \sum_{K\in\mathscr{T}_h}[-(\nabla e_h^p, \eta^{\mathbf{u}})_K + (e_h^p\mathbf{I}\cdot\mathbf{n}, \eta^{\mathbf{u}})_{\partial K}] \\
&\quad - \sum_{K\in\mathscr{T}_h}\tau_K(-\nu\Delta\eta^{\mathbf{u}} + \nabla\eta^p, \nu\Delta\mathbf{e}_h^{\mathbf{u}} - \nabla e_h^p)_K \\
&\quad + \sum_{F\in\mathscr{E}_\Omega}\tau_F([\![-\nu\partial_{\mathbf{n}}\eta^{\mathbf{u}} + \eta^p\mathbf{I}\cdot\mathbf{n}]\!], [\![-\nu\partial_{\mathbf{n}}\mathbf{e}_h^{\mathbf{u}} + e_h^p\mathbf{I}\cdot\mathbf{n}]\!])_F \\
&\leqslant \Bigg\{\sum_{K\in\mathscr{T}_h}\nu|\eta^{\mathbf{u}}|_{1,K}^2 + \frac{1}{\nu}\|\eta^p\|_{0,K}^2 + \tau_K^{-1}\|\eta^{\mathbf{u}}\|_{0,K}^2 + \tau_K\|-\nu\Delta\eta^{\mathbf{u}} + \nabla\eta^p\|_{0,K}^2 \\
&\qquad + \sum_{F\in\mathscr{E}_\Omega}\tau_F\|[\![-\nu\partial_{\mathbf{n}}\eta^{\mathbf{u}} + \eta^p\mathbf{I}\cdot\mathbf{n}]\!]\|_{0,F}^2 + \tau_F^{-1}\|\eta^{\mathbf{u}}\|_{0,F}^2\Bigg\}^{\frac{1}{2}} \\
&\quad \times \Bigg\{\sum_{K\in\mathscr{T}_h}\nu|\mathbf{e}_h^{\mathbf{u}}|_{1,K}^2 + \nu\|\nabla\cdot\mathbf{e}_h^{\mathbf{u}}\|_{0,K}^2 + \tau_K|e_h^p|_{1,K}^2 + \tau_K\|\nu\Delta\mathbf{e}_h^{\mathbf{u}} - \nabla e_h^p\|_{0,K}^2 \\
&\qquad + \sum_{F\in\mathscr{E}_\Omega}\tau_F\|[\![-\nu\partial_{\mathbf{n}}\mathbf{e}_h^{\mathbf{u}} + e_h^p\mathbf{I}\cdot\mathbf{n}]\!]\|_{0,F}^2 + \tau_F\|[\![e_h^p\mathbf{I}\cdot\mathbf{n}]\!]\|_{0,F}^2\Bigg\}^{\frac{1}{2}} \\
&\leqslant \Bigg\{\|(\eta^{\mathbf{u}}, \eta^p)\|_h^2 + \frac{1}{\nu}\|\eta^p\|_{0,\Omega}^2 \\
&\qquad + \sum_{K\in\mathscr{T}_h}[\tau_K^{-1}\|\eta^{\mathbf{u}}\|_{0,K}^2 + \tau_K\|\nu\Delta\eta^{\mathbf{u}}\|_{0,K}^2] + \sum_{F\in\mathscr{E}_\Omega}\tau_F^{-1}\|\eta^{\mathbf{u}}\|_{0,F}^2\Bigg\}^{\frac{1}{2}} \\
&\quad \times \Bigg\{\sum_{K\in\mathscr{T}_h}\nu|\mathbf{e}_h^{\mathbf{u}}|_{1,K}^2 + \nu\|\nabla\cdot\mathbf{e}_h^{\mathbf{u}}\|_{0,K}^2 + \tau_K|e_h^p|_{1,K}^2 + \tau_K\|\nu\Delta\mathbf{e}_h^{\mathbf{u}} - \nabla e_h^p\|_{0,K}^2 \\
&\qquad + \sum_{F\in\mathscr{E}_\Omega}\tau_F\|[\![-\nu\partial_{\mathbf{n}}\mathbf{e}_h^{\mathbf{u}} + e_h^p\mathbf{I}\cdot\mathbf{n}]\!]\|_{0,F}^2 + \tau_F\|[\![e_h^p\mathbf{I}\cdot\mathbf{n}]\!]\|_{0,F}^2\Bigg\}^{\frac{1}{2}}.
\end{aligned}
\tag{3.14}
$$

Now, from the inverse inequality (3.3) and the definition of τ_K, we obtain

$$
\sum_{K\in\mathscr{T}_h}\tau_K\|\nu\Delta\mathbf{e}_h^{\mathbf{u}} - \nabla e_h^p\|_{0,K}^2 \leqslant C\sum_{K\in\mathscr{T}_h}[\nu|\mathbf{e}_h^{\mathbf{u}}|_{1,K}^2 + \tau_K\|\nabla e_h^p\|_{0,K}^2] \leqslant C\|(\mathbf{e}_h^{\mathbf{u}}, e_h^p)\|_h^2.
$$

On the other hand, using the mesh regularity, (3.7) once again, and (3.3) we obtain

$$\begin{aligned}
\sum_{F\in\mathscr{E}_\Omega} \tau_F \|[\![e_h^p \mathbf{I}\cdot\mathbf{n}]\!]\|_{0,F}^2 &\leqslant 2\sum_{F\in\mathscr{E}_\Omega} \tau_F \|[\![-\nu\partial_{\mathbf{n}}\mathbf{e}_h^{\mathbf{u}} + e_h^p\mathbf{I}\cdot\mathbf{n}]\!]\|_{0,F}^2 + 2\sum_{F\in\mathscr{E}_\Omega} \tau_F\|[\![-\nu\partial_{\mathbf{n}}\mathbf{e}_h^{\mathbf{u}}]\!]\|_{0,F}^2 \\
&\leqslant 2\|(\mathbf{e}_h^{\mathbf{u}}, e_h^p)\|_h^2 + C\sum_{K\in\mathscr{T}_h} \nu h_K \|\partial_{\mathbf{n}}\mathbf{e}_h^{\mathbf{u}}\|_{0,\partial K}^2 \\
&\leqslant C\left[\|(\mathbf{e}_h^{\mathbf{u}}, e_h^p)\|_h^2 + \sum_{K\in\mathscr{T}_h} \nu h_K (h_K^{-1}|\mathbf{e}_h^{\mathbf{u}}|_{1,K}^2 + h_K|\mathbf{e}_h^{\mathbf{u}}|_{2,K}^2)\right] \\
&\leqslant C\left[\|(\mathbf{e}_h^{\mathbf{u}}, e_h^p)\|_h^2 + \sum_{K\in\mathscr{T}_h} \nu h_K h_K^{-1}|\mathbf{e}_h^{\mathbf{u}}|_{1,K}^2\right] \\
&\leqslant C\|(\mathbf{e}_h^{\mathbf{u}}, e_h^p)\|_h^2.
\end{aligned}$$

Hence, the last term in (3.14) may be bounded by $C\|(\mathbf{e}_h^{\mathbf{u}}, e_h^p)\|_h$, and then applying Lemma 3.3 we arrive at

$$\|(\mathbf{e}_h^{\mathbf{u}}, e_h^p)\|_h \leqslant C\left(\sqrt{\nu}h^k|\mathbf{u}|_{k+1,\Omega} + \frac{h^l}{\sqrt{\nu}}|p|_{l,\Omega}\right), \tag{3.15}$$

and the first estimate follows from $(\mathbf{e}^{\mathbf{u}}, e^p) = (\eta^{\mathbf{u}}, \eta^p) - (\mathbf{e}_h^{\mathbf{u}}, e_h^p)$, the triangle inequality and applying Lemma 3.3 again.

For the estimate on the pressure, from the continuous inf-sup condition (see Girault & Raviart, 1986), there exists $\mathbf{w}\in H_0^1(\Omega)^2$ such that $\nabla\cdot\mathbf{w} = p - p_h$ in Ω and $|\mathbf{w}|_{1,\Omega} \leqslant C\|p-p_h\|_{0,\Omega}$. Let $\mathbf{w}_h = \mathcal{C}_h(\mathbf{w}) \in [V_1\cap H_0^1(\Omega)]^2$ be the $\mathbb{P}^1$-Clément interpolant of $\mathbf{w}$. Then, applying the consistency of the method and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
\|p-p_h\|_{0,\Omega}^2 &= (\nabla\cdot\mathbf{w}, p-p_h)_\Omega = (\nabla\cdot(\mathbf{w}-\mathbf{w}_h), p-p_h)_\Omega + (\nabla\cdot\mathbf{w}_h, p-p_h)_\Omega \\
&= \sum_{K\in\mathscr{T}_h} [-(\mathbf{w}-\mathbf{w}_h, \nabla(p-p_h))_K + (\mathbf{w}-\mathbf{w}_h, (p-p_h)\mathbf{I}\cdot\mathbf{n})_{\partial K}] \\
&\quad + \nu(\nabla(\mathbf{u}-\mathbf{u}_h), \nabla\mathbf{w}_h)_\Omega - \sum_{F\in\mathscr{E}_\Omega} \tau_F([\![-\nu\partial_{\mathbf{n}}(\mathbf{u}-\mathbf{u}_h) + (p-p_h)\mathbf{I}\cdot\mathbf{n}]\!], [\![\nu\partial_{\mathbf{n}}\mathbf{w}_h]\!])_F \\
&\leqslant \left[\sum_{K\in\mathscr{T}_h} \frac{h_K^2}{\nu}|p-p_h|_{1,K}^2 + \sum_{F\in\mathscr{E}_\Omega} \tau_F\|[\![(p-p_h)\mathbf{I}\cdot\mathbf{n}]\!]\|_{0,F}^2 + \nu|\mathbf{u}-\mathbf{u}_h|_{1,\Omega}^2\right. \\
&\quad \left. + \sum_{F\in\mathscr{E}_\Omega} \tau_F\|[\![-\nu\partial_{\mathbf{n}}(\mathbf{u}-\mathbf{u}_h) + (p-p_h)\mathbf{I}\cdot\mathbf{n}]\!]\|_{0,F}^2\right]^{\frac{1}{2}} \\
&\quad \times\left[\sum_{K\in\mathscr{T}_h} \frac{\nu}{h_K^2}\|\mathbf{w}-\mathbf{w}_h\|_{0,K}^2 + \sum_{F\in\mathscr{E}_\Omega} \tau_F^{-1}\|\mathbf{w}-\mathbf{w}_h\|_{0,F}^2 + \nu|\mathbf{w}_h|_{1,\Omega}^2\right. \\
&\quad \left. + \sum_{F\in\mathscr{E}_\Omega} \tau_F\|[\![\nu\partial_{\mathbf{n}}\mathbf{w}_h]\!]\|_{0,F}^2\right]^{\frac{1}{2}}.
\end{aligned} \tag{3.16}$$

Now, using the local trace theorem (3.7), (3.10) and (3.11) we obtain

$$\left[\sum_{K\in\mathscr{T}_h}\frac{\nu}{h_K^2}\|\mathbf{w}-\mathbf{w}_h\|_{0,K}^2+\sum_{F\in\mathscr{E}_\Omega}\tau_F^{-1}\|\mathbf{w}-\mathbf{w}_h\|_{0,F}^2+\nu|\mathbf{w}_h|_{1,\Omega}^2+\sum_{F\in\mathscr{E}_\Omega}\tau_F\|[\![\nu\partial_{\mathbf{n}}\mathbf{w}_h]\!]\|_{0,F}^2\right]^{\frac{1}{2}}$$
$$\leqslant C\sqrt{\nu}|\mathbf{w}|_{1,\Omega}\leqslant C\sqrt{\nu}\|p-p_h\|_{0,\Omega}.$$

Hence, dividing in (3.16) by $\|p-p_h\|_{0,\Omega}$, using the definition of the norm $\|\cdot\|_h$ and (3.12), we have

$$\|p-p_h\|_{0,\Omega}\leqslant C\sqrt{\nu}\left[\|(\mathbf{u}-\mathbf{u}_h,p-p_h)\|_h^2+\sum_{F\in\mathscr{E}_\Omega}\tau_F\|[\![\nu\partial_{\mathbf{n}}(\mathbf{u}-\mathbf{u}_h)]\!]\|_{0,F}^2\right]^{\frac{1}{2}}$$
$$\leqslant C\sqrt{\nu}\left[\nu h^{2k}|\mathbf{u}|_{k+1,\Omega}^2+\frac{h^{2l}}{\nu}|p|_{l,\Omega}^2+\sum_{F\in\mathscr{E}_\Omega}\tau_F\|[\![\nu\partial_{\mathbf{n}}(\mathbf{u}-\mathbf{u}_h)]\!]\|_{0,F}^2\right]^{\frac{1}{2}}.$$

It only remains to bound last term above. Applying the definition of τ_F, mesh regularity, the trace inequality (3.7), the inverse estimate (3.3) and (3.15), we arrive at

$$\sum_{F\in\mathscr{E}_\Omega}\tau_F\|[\![\nu\partial_{\mathbf{n}}(\mathbf{u}-\mathbf{u}_h)]\!]\|_{0,F}^2$$
$$\leqslant C\sum_{K\in\mathscr{T}_h}\frac{\beta h_K}{\nu}\|\nu\partial_{\mathbf{n}}(\mathbf{u}-\mathbf{u}_h)\|_{0,\partial K}^2$$
$$\leqslant C\sum_{K\in\mathscr{T}_h}\frac{\beta h_K\nu^2}{\nu}[h_K^{-1}|\mathbf{u}-\mathbf{u}_h|_{1,K}^2+h_K|\mathbf{u}-\mathbf{u}_h|_{2,K}^2]$$
$$\leqslant C\left[\nu|\mathbf{u}-\mathbf{u}_h|_{1,\Omega}^2+\nu\sum_{K\in\mathscr{T}_h}h_K^2(|\eta^{\mathbf{u}}|_{2,K}^2+|\mathbf{e}_h^{\mathbf{u}}|_{2,K}^2)\right]$$
$$\leqslant C\left[\nu|\mathbf{u}-\mathbf{u}_h|_{1,\Omega}^2+\nu\sum_{K\in\mathscr{T}_h}h_K^2(h_K^{2k-2}|\mathbf{u}|_{k+1,K}^2+h_K^{-2}|\mathbf{e}_h^{\mathbf{u}}|_{1,K}^2)\right]$$
$$\leqslant C\left[\nu|\mathbf{u}-\mathbf{u}_h|_{1,\Omega}^2+\nu h^{2k}|\mathbf{u}|_{k+1,\Omega}^2+\frac{1}{\nu}h^{2l}|p|_{l,\Omega}^2\right],$$

and the result follows from (3.12).

Finally, the proof for $l=0$ follows by defining $\eta^p := p-\Pi_h(p)$ and applying the same arguments as above. □

REMARK 3.3 Theorem 3.1 gives an optimal convergence result for the mesh-dependent norm (3.1) containing, in particular a control on the $H^1(\Omega)$ semi-norm of the velocity error, as well as the following bound on the jump of the stress tensor

$$\left[\sum_{F\in\mathscr{E}_\Omega}\tau_F\|[\![-\nu\partial_{\mathbf{n}}(\mathbf{u}-\mathbf{u}_h)+(p-p_h)\mathbf{I}\cdot\mathbf{n}]\!]\|_{0,F}^2\right]^{\frac{1}{2}}\leqslant C\left(h^k\sqrt{\nu}|\mathbf{u}|_{k+1,\Omega}+\frac{h^s}{\sqrt{\nu}}|p|_{s,\Omega}\right), \quad (3.17)$$

which is exclusive to this formulation. Also, since (3.1) defines a norm which does not involve the natural norm of the pressure, a convergence result for the pressure in its natural norm (i.e. the $L^2(\Omega)$ norm) is given.

COROLLARY 3.1 Let us suppose the same regularity assumptions as in Theorem 3.1. Then, there exists $C > 0$ such that

$$\sqrt{\nu}|\mathbf{u}-\mathbf{u}_h|_{1,\Omega} + \frac{1}{\sqrt{\nu}}\|p-p_h\|_{0,\Omega} + \left\{\sum_{F\in\mathscr{E}_\Omega} \tau_F[\|[\![\nu\partial_n \mathbf{e}^{\mathbf{u}}]\!]\|_{0,F}^2 + \|[\![e^p]\!]\|_{0,F}^2]\right\}^{\frac{1}{2}} \leqslant C\left(h^k\sqrt{\nu}|\mathbf{u}|_{k+1,\Omega} + \frac{h^s}{\sqrt{\nu}}|p|_{s,\Omega}\right),$$

where $s = l$ if $l \geqslant 1$ and $s = 1$ if $l = 0$.

Proof. By virtue of Theorem 3.1, it remains to bound the term containing the jumps. First, using mesh regularity, the local trace inequality (3.7) and Theorem 3.1 we arrive at

$$\begin{aligned}\sum_{F\in\mathscr{E}_\Omega} \tau_F\|[\![e^p]\!]\|_{0,F}^2 &\leqslant C\sum_{K\in\mathscr{T}_h}\frac{h_K}{\nu}\|e^p\|_{0,\partial K}^2\\ &\leqslant C\sum_{K\in\mathscr{T}_h}\frac{h_K}{\nu}[h_K^{-1}\|e^p\|_{0,K}^2 + h_K|e^p|_{1,K}^2]\\ &\leqslant C\left[\frac{1}{\nu}\|e^p\|_{0,K}^2 + \sum_{K\in\mathscr{T}_h}\tau_K|e^p|_{1,K}^2\right]\\ &\leqslant C\left(h^{2k}\nu|\mathbf{u}|_{k+1,\Omega}^2 + \frac{h^{2s}}{\nu}|p|_{s,\Omega}^2\right).\end{aligned} \tag{3.18}$$

The bound on the remaining jump term arises from the triangle inequality, (3.17) and (3.18). □

REMARK 3.4 We end this section by remarking the fact that if we suppose a more regular pressure we may obtain a higher order error estimate. Indeed, if we fix l and suppose that $p \in H^{l+1}(\Omega)$, then, the interpolation inequalities (3.10)–(3.11) read

$$|q-\mathscr{C}_h(q)|_{m,K} \leqslant Ch_K^{l+1-m}|q|_{l+1,\tilde{\omega}_K},$$
$$\|q-\mathscr{C}_h(q)\|_{0,F} \leqslant Ch_F^{l+1-\frac{1}{2}}|q|_{l+1,\tilde{\omega}_F},$$

for all $q \in H^{l+1}(\Omega)$, and hence Lemma 3.3 reads

$$\sum_{F\in\mathscr{E}_\Omega}\tau_F^{-1}\|\mathbf{v}-I_h(\mathbf{v})\|_{0,F}^2 + \sum_{K\in\mathscr{T}_h}[\tau_K^{-1}\|\mathbf{v}-I_h(\mathbf{v})\|_{0,K}^2 + \nu h_K^2\|\Delta(\mathbf{v}-I_h(\mathbf{v}))\|_{0,K}^2] + \frac{1}{\nu}\|q-\tilde{q}_h\|_{0,\Omega}^2 + \|(\mathbf{v}-I_h(\mathbf{v}), q-\tilde{q}_h)\|_h^2 \leqslant C\left(h^{2k}\nu|\mathbf{v}|_{k+1,\Omega}^2 + \frac{h^{2l+2}}{\nu}|q|_{l+1,\Omega}^2\right),$$

leading to the following error estimates:

$$\|(\mathbf{e^u}, e^p)\|_h \leqslant C\left(h^k\sqrt{\nu}|\mathbf{u}|_{k+1,\Omega} + \frac{h^{l+1}}{\sqrt{\nu}}|p|_{l+1,\Omega}\right), \tag{3.19}$$

$$\|e^p\|_{0,\Omega} \leqslant C(h^k\nu|\mathbf{u}|_{k+1,\Omega} + h^{l+1}|p|_{l+1,\Omega}), \tag{3.20}$$

which are optimal if $l+1=k$.

4. A posteriori error analysis

In this section, we propose and analyse a residual-based a posteriori error estimator for the method introduced in Section 3. Throughout this section we will suppose, for simplicity of the presentation, that $\mathbf{f}$ is a piecewise polynomial function.

Since we are interested in treating problems without any extra regularity, we will suppose that $(\mathbf{u}, p) \in \mathbf{V} \times Q$ is the solution of (2.2) rather than the solution of the strong formulation (2.1) of the problem.

4.1 *Preliminaries: bubble functions*

For all $K \in \mathscr{T}_h$, we define the 'element bubble function' b_K by

$$b_K := 27 \prod_{x\in\mathscr{N}(K)} \lambda_x, \tag{4.1}$$

where λ_x denotes the barycentric coordinate associated with node x. Let $\hat{K}$ be the standard reference element, with vertices $\hat{n}_1 = (1,0)$, $\hat{n}_2 = (0,1)$ and $\hat{n}_3 = (0,0)$. Set

$$b_{\hat{F}} := 4\hat{\lambda}_3\hat{\lambda}_1 \quad \text{on } \hat{K},$$

where $\hat{F} := \{(t,0) \in \mathbb{R}^2 \colon 0 \leqslant t \leqslant 1\}$. Next, let $F \in \mathscr{E}_\Omega$, let us suppose that $\omega_F = K_1 \cup K_2$ and let $G_{F,i}$ be the (orientation-preserving) affine transformation defined in Fig. 1 such that $G_{F,i}(\hat{K}) = K_i$ and

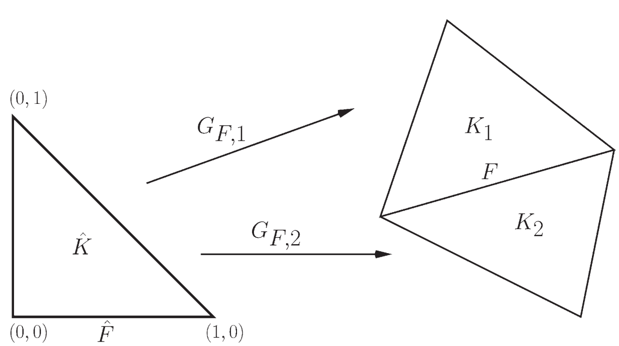

FIG. 1. Affine transformation $G_{F,i}$, $i = 1, 2$.

$G_{F,i}(\hat{F}) = F, i = 1, 2$. We then define the bubble function associated with F by

$$b_F := \begin{cases} b_{\hat{F}} \circ G_{F,i}^{-1} & \text{on } K_i, i = 1, 2, \\ 0 & \text{on } \Omega \backslash \omega_F. \end{cases} \tag{4.2}$$

Let $\hat{\Pi} := \{(x, 0) : x \in \mathbb{R}\}$ and let $\hat{Q} : \mathbb{R}^2 \to \hat{\Pi}$ be the orthogonal projection from $\mathbb{R}^2$ to $\hat{\Pi}$. We introduce the lifting operator $\hat{P}_{\hat{F}} : \mathbb{P}^k(\hat{F}) \to \mathbb{P}^k(\hat{K})$ by

$$\hat{s} \longmapsto \hat{P}_{\hat{F}}(\hat{s}) = \hat{s} \circ \hat{Q}.$$

Let $K_i \subseteq \omega_F$. We define the lifting operator $P_{F,K_i} : \mathbb{P}^k(F) \to \mathbb{P}^k(K_i)$ by

$$P_{F,K_i}(s) = \hat{P}_{\hat{F}}(s \circ G_{F,i}) \circ G_{F,i}^{-1}.$$

Using these notations, we can define a lifting operator $P_F : \mathbb{P}^k(F) \to \mathbb{P}^k(\omega_F)$ by

$$s \in \mathbb{P}^k(F) \longmapsto P_F(s) := \begin{cases} P_{F,K_1}(s) & \text{in } K_1, \\ P_{F,K_2}(s) & \text{in } K_2, \end{cases}$$

and, for $\mathbf{s} = (s_1, s_2) \in \mathbb{P}^k(F)^2$, we denote

$$\mathbf{P}_F(\mathbf{s}) = (P_F(s_1), P_F(s_2)). \tag{4.3}$$

4.2 *A residual error estimator*

For each $K \in \mathscr{T}_h$ and each $F \in \mathscr{E}_\Omega$, we define the residuals

$$\mathbf{R}_K := (\mathbf{f} + \nu \Delta \mathbf{u}_h - \nabla p_h)|_K, \tag{4.4}$$

$$\mathbf{R}_F := [\![-\nu \partial_{\mathbf{n}} \mathbf{u}_h + p_h \mathbf{I} \cdot \mathbf{n}]\!]_F. \tag{4.5}$$

REMARK 4.1 From the definition of these residuals, (element-wise) integration by parts leads to

$$\mathbf{B}((\mathbf{e}^{\mathbf{u}}, e^p), (\mathbf{v}, q)) = \sum_{K \in \mathscr{T}_h} (\mathbf{R}_K, \mathbf{v})_K + \sum_{F \in \mathscr{E}_\Omega} (\mathbf{R}_F, \mathbf{v})_F + (q, \nabla \cdot \mathbf{u}_h)_\Omega, \tag{4.6}$$

for all $(\mathbf{v}, q) \in \mathbf{V} \times Q$. Next, from the definition of $\mathbf{R}_K$, $\mathbf{R}_F$ and the stabilized method (2.8), we see that, for all $\mathbf{v}_h \in [V_1 \cap H_0^1(\Omega)]^2$,

$$\begin{aligned} \mathbf{B}((\mathbf{e}^{\mathbf{u}}, e^p), (\mathbf{v}_h, 0)) &= (\mathbf{f}, \mathbf{v}_h)_\Omega - \mathbf{B}((\mathbf{u}_h, p_h), (\mathbf{v}_h, 0)) \\ &= (\mathbf{f}, \mathbf{v}_h)_\Omega - \mathbf{B}_\tau((\mathbf{u}_h, p_h), (\mathbf{v}_h, 0)) - \sum_{F \in \mathscr{E}_\Omega} \tau_F (\mathbf{R}_F, [\![\nu \partial_{\mathbf{n}} \mathbf{v}_h]\!])_F \\ &= - \sum_{F \in \mathscr{E}_\Omega} \tau_F (\mathbf{R}_F, [\![\nu \partial_{\mathbf{n}} \mathbf{v}_h]\!])_F \leqslant \sum_{F \in \mathscr{E}_\Omega} \tau_F \|\mathbf{R}_F\|_{0,F} \|[\![\nu \partial_{\mathbf{n}} \mathbf{v}_h]\!]\|_{0,F}. \end{aligned} \tag{4.7}$$

It is worth remarking that (4.7) has been obtained without using the consistency of the method, and hence not assuming any extra regularity of the solution.

Our residual-based error estimator is given by

$$\eta := \left\{ \sum_{K\in\mathscr{T}_h} \eta_K^2 \right\}^{\frac{1}{2}}, \tag{4.8}$$

where, in each $K \in \mathscr{T}_h$, we have defined

$$\eta_K^2 := \tau_K \|\mathbf{R}_K\|_{0,K}^2 + \frac{1}{2} \sum_{F\in\mathscr{E}(K)\cap\mathscr{E}_\Omega} \tau_F \|\mathbf{R}_F\|_{0,F}^2 + \nu \|\nabla\cdot\mathbf{u}_h\|_{0,K}^2. \tag{4.9}$$

REMARK 4.2 A similar estimator may be found in Verfürth (1989, 1991), applied to specific discretizations. In fact, the equivalence result below may be seen as a generalization of the corresponding result in the above references. For a detailed survey of existing a posteriori error estimators for the Stokes problem, see Ainsworth & Oden (2000) (and the references therein) and Kay & Silvester (1999), where an a posteriori error analysis of a stabilized finite-element method for the Stokes problem is performed.

THEOREM 4.1 Let $(\mathbf{e}^{\mathbf{u}}, e^p) := (\mathbf{u}-\mathbf{u}_h, p-p_h)$, where $(\mathbf{u}, p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ is the solution of (2.2) and $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ is the solution of (2.8). Then, there exist two positive constants, C_1 and C_2, independent on h and ν, such that

$$\sqrt{\nu}|\mathbf{e}^{\mathbf{u}}|_{1,\Omega} + \frac{1}{\sqrt{\nu}}\|e^p\|_{0,\Omega} \leqslant C_1\eta,$$

$$C_2\eta_K \leqslant \sqrt{\nu}|\mathbf{e}^{\mathbf{u}}|_{1,\omega_K} + \frac{1}{\sqrt{\nu}}\|e^p\|_{0,\omega_K}.$$

Proof. **Upper bound:** Let $(\mathbf{v}, q) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ and let $\mathbf{v}_h := \mathscr{C}_h(\mathbf{v}) \in [V_1 \cap H_0^1(\Omega)]^2$ be the $\mathbb{P}^1$-Clément interpolant of $\mathbf{v}$. First, (4.6) applied to $(\mathbf{v}-\mathbf{v}_h, q)$ leads to

$$\begin{aligned}
\mathbf{B}((\mathbf{e}^{\mathbf{u}}, e^p), (\mathbf{v}-\mathbf{v}_h, q)) &= \sum_{K\in\mathscr{T}_h} (\mathbf{R}_K, \mathbf{v}-\mathbf{v}_h)_K + \sum_{F\in\mathscr{E}_\Omega} (\mathbf{R}_F, \mathbf{v}-\mathbf{v}_h)_F + (q, \nabla\cdot\mathbf{u}_h)_\Omega \\
&\leqslant \sum_{K\in\mathscr{T}_h} \|\mathbf{R}_K\|_{0,K}\|\mathbf{v}-\mathbf{v}_h\|_{0,K} + \sum_{F\in\mathscr{E}_\Omega} \|\mathbf{R}_F\|_{0,F}\|\mathbf{v}-\mathbf{v}_h\|_{0,F} \\
&\quad + \|q\|_{0,\Omega}\|\nabla\cdot\mathbf{u}_h\|_{0,\Omega}.
\end{aligned} \tag{4.10}$$

Then, from (4.10), (4.7) and (3.10)–(3.11), we arrive at

$$\begin{aligned}
&\mathbf{B}((\mathbf{e}^{\mathbf{u}}, e^p), (\mathbf{v}, q)) \\
&\quad = \mathbf{B}((\mathbf{e}^{\mathbf{u}}, e^p), (\mathbf{v}-\mathbf{v}_h, q)) + \mathbf{B}((\mathbf{e}^{\mathbf{u}}, e^p), (\mathbf{v}_h, 0)) \\
&\quad \leqslant \sum_{K\in\mathscr{T}_h} \|\mathbf{R}_K\|_{0,K}\|\mathbf{v}-\mathbf{v}_h\|_{0,K} + \sum_{F\in\mathscr{E}_\Omega} \|\mathbf{R}_F\|_{0,F}\|\mathbf{v}-\mathbf{v}_h\|_{0,F} \\
&\qquad + \|q\|_{0,\Omega}\|\nabla\cdot\mathbf{u}_h\|_{0,\Omega} + \sum_{F\in\mathscr{E}_\Omega} \tau_F\|\mathbf{R}_F\|_{0,F}\|[\![\nu\partial_{\mathbf{n}}\mathbf{v}_h]\!]\|_{0,F}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\left[\sum_{K\in\mathscr{T}_h} h_K\|\mathbf{R}_K\|_{0,K}|\mathbf{v}|_{1,\tilde{\omega}_K}+\sum_{F\in\mathscr{E}_\Omega} h_F^{\frac12}\|\mathbf{R}_F\|_{0,F}|\mathbf{v}|_{1,\tilde{\omega}_F}\right.\\
&\qquad\left.+\|q\|_{0,\Omega}\|\nabla\cdot\mathbf{u}_h\|_{0,\Omega}+\sum_{F\in\mathscr{E}_\Omega}\tau_F\|\mathbf{R}_F\|_{0,F}\|[\![\nu\partial_{\mathbf{n}}\mathbf{v}_h]\!]\|_{0,F}\right]\\
&\leqslant C\left\{\sum_{K\in\mathscr{T}_h}\tau_K\|\mathbf{R}_K\|_{0,K}^2+\sum_{F\in\mathscr{E}_\Omega}\tau_F\|\mathbf{R}_F\|_{0,F}^2+\nu\|\nabla\cdot\mathbf{u}_h\|_{0,\Omega}^2\right\}^{\frac12}\\
&\quad\times\left\{\sum_{K\in\mathscr{T}_h}\nu|\mathbf{v}|_{1,\tilde{\omega}_K}^2+\sum_{F\in\mathscr{E}_\Omega}\nu|\mathbf{v}|_{1,\tilde{\omega}_F}^2+\frac{1}{\nu}\|q\|_{0,\Omega}^2+\sum_{F\in\mathscr{E}_\Omega}\tau_F\nu^2\|[\![\partial_{\mathbf{n}}\mathbf{v}_h]\!]\|_{0,F}^2\right\}^{\frac12}. \qquad (4.11)
\end{aligned}
$$

Next, applying the mesh regularity and the local trace theorem (3.7), we obtain

$$
\begin{aligned}
&\sum_{K\in\mathscr{T}_h}\nu|\mathbf{v}|_{1,\tilde{\omega}_K}^2+\sum_{F\in\mathscr{E}_\Omega}\nu|\mathbf{v}|_{1,\tilde{\omega}_F}^2+\sum_{F\in\mathscr{E}_\Omega}\tau_F\nu^2\|[\![\partial_{\mathbf{n}}\mathbf{v}_h]\!]\|_{0,F}^2\\
&\leqslant C\left[\nu|\mathbf{v}|_{1,\Omega}^2+\sum_{K\in\mathscr{T}_h}\nu h_K\|\partial_{\mathbf{n}}\mathbf{v}_h\|_{0,\partial K}^2\right]\\
&\leqslant C\left[\nu|\mathbf{v}|_{1,\Omega}^2+\nu\sum_{K\in\mathscr{T}_h}h_K[h_K^{-1}|\mathbf{v}_h|_{1,K}^2+h_K|\mathbf{v}_h|_{2,K}^2]\right]\\
&\leqslant C\nu|\mathbf{v}|_{1,\Omega}^2. \qquad (4.12)
\end{aligned}
$$

Hence, from (2.5), (4.11) and (4.12), we arrive at

$$
\begin{aligned}
\sqrt{\nu}|\mathbf{e}^{\mathbf{u}}|_{1,\Omega}+\frac{1}{\sqrt{\nu}}\|e^p\|_{0,\Omega}&\leqslant C\sup_{(\mathbf{v},q)\in\mathbf{V}\times Q-\{0\}}\frac{\mathbf{B}((\mathbf{e}^{\mathbf{u}},e^p),(\mathbf{v},q))}{\sqrt{\nu}|\mathbf{v}|_{1,\Omega}+\frac{1}{\sqrt{\nu}}\|q\|_{0,\Omega}}\\
&\leqslant C\sup_{(\mathbf{v},q)\in\mathbf{V}\times Q-\{0\}}\frac{\eta\left\{\nu|\mathbf{v}|_{1,\Omega}^2+\frac{1}{\nu}\|q\|_{0,\Omega}^2\right\}^{\frac12}}{\sqrt{\nu}|\mathbf{v}|_{1,\Omega}+\frac{1}{\sqrt{\nu}}\|q\|_{0,\Omega}}\\
&\leqslant C_1\eta. \qquad (4.13)
\end{aligned}
$$

Lower bound: For all $K\in\mathscr{T}_h$ and $F\in\mathscr{E}_\Omega$, let b_K and b_F be the element and edge bubble functions defined in (4.1) and (4.2), respectively. Let also $\mathbf{b}_K := b_K\mathbf{R}_K$. Then, from Araya *et al.* (2005) (see also Verfürth, 1998 for the original result in the scalar case) we know that

$$C\|\mathbf{R}_K\|_{0,K}^2\leqslant(\mathbf{R}_K,\mathbf{b}_K)_K\leqslant\|\mathbf{R}_K\|_{0,K}^2, \qquad (4.14)$$

$$C\|\mathbf{R}_F\|_{0,F}^2\leqslant(\mathbf{R}_F,b_F\mathbf{R}_F)_F\leqslant\|\mathbf{R}_F\|_{0,F}^2. \qquad (4.15)$$

On the other hand, since $\mathbf{b}_K \in H_0^1(K)^2$, using the inverse inequality (3.3) and the definition of τ_K, we obtain

$$
\begin{aligned}
(\mathbf{R}_K, \mathbf{b}_K)_K &= \sum_{K' \in \mathscr{T}_h} (\mathbf{R}_{K'}, \mathbf{b}_K)_{K'} = \mathbf{B}((\mathbf{u} - \mathbf{u}_h, p - p_h), (\mathbf{b}_K, 0)) \\
&= \nu(\nabla(\mathbf{u} - \mathbf{u}_h), \nabla \mathbf{b}_K)_K - (p - p_h, \nabla \cdot \mathbf{b}_K)_K \\
&\leqslant C \left\{ \nu |\mathbf{u} - \mathbf{u}_h|_{1,K}^2 + \frac{1}{\nu} \|p - p_h\|_{0,K}^2 \right\}^{\frac{1}{2}} \sqrt{\nu} |\mathbf{b}_K|_{1,K} \\
&\leqslant C \left\{ \nu |\mathbf{u} - \mathbf{u}_h|_{1,K}^2 + \frac{1}{\nu} \|p - p_h\|_{0,K}^2 \right\}^{\frac{1}{2}} \sqrt{\nu} h_K^{-1} \|\mathbf{b}_K\|_{0,K} \\
&\leqslant C \left\{ \nu |\mathbf{u} - \mathbf{u}_h|_{1,K}^2 + \frac{1}{\nu} \|p - p_h\|_{0,K}^2 \right\}^{\frac{1}{2}} \tau_K^{-\frac{1}{2}} \|\mathbf{R}_K\|_{0,K},
\end{aligned}
$$

which, combined with (4.14) leads to

$$
\tau_K^{\frac{1}{2}} \|\mathbf{R}_K\|_{0,K} \leqslant C \left[\sqrt{\nu} |\mathbf{u} - \mathbf{u}_h|_{1,K} + \frac{1}{\sqrt{\nu}} \|p - p_h\|_{0,K} \right]. \tag{4.16}
$$

On the other hand, since $\mathbf{u} \in H_0^1(\Omega)^2$ we have $\nabla \cdot \mathbf{u} \in L_0^2(\Omega)$ and then from (2.2) we obtain that $\nabla \cdot \mathbf{u} = 0$ a.e. in Ω. Using this fact, (4.14) (applied to $\nabla \cdot \mathbf{u}_h$) and $b_K \in H_0^1(K)$, we arrive at

$$
\begin{aligned}
\|\nabla \cdot \mathbf{u}_h\|_{0,K}^2 &= (\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{u}_h)_K \\
&\leqslant C(\nabla \cdot \mathbf{u}_h, b_K \nabla \cdot \mathbf{u}_h)_K \\
&= C(\nabla \cdot \mathbf{u}_h, b_K \nabla \cdot \mathbf{u}_h)_\Omega \\
&= C(\nabla \cdot (\mathbf{u}_h - \mathbf{u}), b_K \nabla \cdot \mathbf{u}_h)_\Omega \\
&= C(\nabla \cdot (\mathbf{u}_h - \mathbf{u}), b_K \nabla \cdot \mathbf{u}_h)_K \\
&\leqslant C |\mathbf{u} - \mathbf{u}_h|_{1,K} \|b_K \nabla \cdot \mathbf{u}_h\|_{0,K} \\
&\leqslant C |\mathbf{u} - \mathbf{u}_h|_{1,K} \|\nabla \cdot \mathbf{u}_h\|_{0,K},
\end{aligned}
$$

and hence

$$
\nu \|\nabla \cdot \mathbf{u}_h\|_{0,K}^2 \leqslant C \nu |\mathbf{u} - \mathbf{u}_h|_{1,K}^2. \tag{4.17}
$$

Finally, let $F \in \mathscr{E}_\Omega$. Then, using (4.15), the definition of $\mathbf{P}_F$ (cf (4.3)), (4.7) and since $b_F \in H_0^1(\omega_F)$, we obtain

$$
\begin{aligned}
\|\mathbf{R}_F\|_{0,F}^2 &\leqslant C(\mathbf{R}_F, b_F \mathbf{R}_F)_F \\
&\leqslant C \left[\sum_{K \subseteq \omega_F} (\mathbf{R}_K, b_F \mathbf{P}_F(\mathbf{R}_F))_K - \mathbf{B}((\mathbf{u} - \mathbf{u}_h, p - p_h), (b_F \mathbf{P}_F(\mathbf{R}_F), 0)) \right]
\end{aligned}
$$

$$\leqslant C\left[\sum_{K\subseteq\omega_F}\|\mathbf{R}_K\|_{0,K}\|b_F\mathbf{P}_F(\mathbf{R}_F)\|_{0,K}+\left\{\sqrt{\nu}|\mathbf{u}-\mathbf{u}_h|_{1,\omega_F}+\frac{1}{\sqrt{\nu}}\|p-p_h\|_{0,\omega_F}\right\}\right.$$
$$\left.\times\sqrt{\nu}|b_F\mathbf{P}_F(\mathbf{R}_F)|_{1,\omega_F}\right].$$

Next, from Araya *et al.* (2005) we know that

$$\|b_F\mathbf{P}_F(\mathbf{R}_F)\|_{0,K}\leqslant Ch_K|b_F\mathbf{P}_F(\mathbf{R}_F)|_{1,K}, \tag{4.18}$$

$$|b_F\mathbf{P}_F(\mathbf{R}_F)|_{1,\omega_F}\leqslant Ch_F^{-\frac{1}{2}}\|\mathbf{R}_F\|_{0,F}, \tag{4.19}$$

and hence applying (4.16) and the definition of τ_F, we arrive at

$$\|\mathbf{R}_F\|_{0,F}^2\leqslant C\left\{\sum_{K\subseteq\omega_F}\frac{h_K^2}{\nu}\|\mathbf{R}_K\|_{0,K}^2+\nu|\mathbf{u}-\mathbf{u}_h|_{1,\omega_F}^2+\frac{1}{\nu}\|p-p_h\|_{0,\omega_F}^2\right\}^{1/2}\sqrt{\nu}|b_F\mathbf{P}_F(\mathbf{R}_F)|_{1,\omega_F}$$
$$\leqslant C\left\{\nu|\mathbf{u}-\mathbf{u}_h|_{1,\omega_F}^2+\frac{1}{\nu}\|p-p_h\|_{0,\omega_F}^2\right\}^{1/2}\tau_F^{-1/2}\|\mathbf{R}_F\|_{0,F},$$

and the estimate follows from last inequality (summing over all $F\in\mathscr{E}(K)$), (4.16) and (4.17). □

REMARK 4.3 The last result states the equivalence between the norm of the error $(\mathbf{e}^{\mathbf{u}}, e^p)$ and the error estimator η. It is worth remarking that we have not used at any moment any extra regularity hypothesis on $(\mathbf{u}, p)$, and that the only dependence of the equivalence on ν is given by the norm of the error we are considering; the equivalence constants are independent of γ. In that sense, Theorem 4.1 gives a result which is independent of ν, as will be confirmed by the numerical experiments in Section 5. We finally remark that the above result does not depend on the choice of the specific discretization, i.e. it is valid for all polynomial degrees k and l.

5. Numerical validations

In this section, we present three sets of numerical validations concerning both the a priori error analysis of the method (2.8) and the performance of the a posteriori error estimator (4.8). Convergence tests have been performed for both $[\mathbb{P}^1]^2\times\mathbb{P}^0$ and $[\mathbb{P}^1]^2\times\mathbb{P}^1$ pairs, and, following Araya *et al.* (2006), we have set $\beta=\frac{1}{12}$.

5.1 *An analytical solution*

For this test case, the domain is taken as the square $\Omega=(0,1)\times(0,1)$ and $\mathbf{f}$ is such that the exact solution of our Stokes problem is given by

$$u_1(x,y)=-256x^2(x-1)^2y(y-1)(2y-1),$$
$$u_2(x,y)=-u_1(y,x),$$
$$p(x,y)=150(x-0.5)(y-0.5).$$

We first depict in Figs 2–3 the convergence history for method (2.8) using $\nu = 1$. The results reproduce our theoretical results showing an O(h) order of convergence for $|\mathbf{u}-\mathbf{u}_h|_{1,\Omega}$, $\|p-p_h\|_{0,\Omega}$ and

$$|[\![\sigma \cdot \mathbf{n} - \sigma_h \cdot \mathbf{n}]\!]|_h := \left\{ \sum_{F\in\mathscr{E}_\Omega} \tau_F \|[\![-\nu\partial_{\mathbf{n}}(\mathbf{u}-\mathbf{u}_h) + (p-p_h)\mathbf{I}\cdot\mathbf{n}]\!]\|_{0,F}^2 \right\}^{\frac{1}{2}}.$$

In order to test the a posteriori error estimator (4.8), in Fig. 4 we depict the error in the norm

$$\|(\mathbf{e^u}, e^p)\|_{E,\Omega} := \sqrt{\nu}|\mathbf{e^u}|_{1,\Omega} + \frac{1}{\sqrt{\nu}}\|e^p\|_{0,\Omega}, \tag{5.1}$$

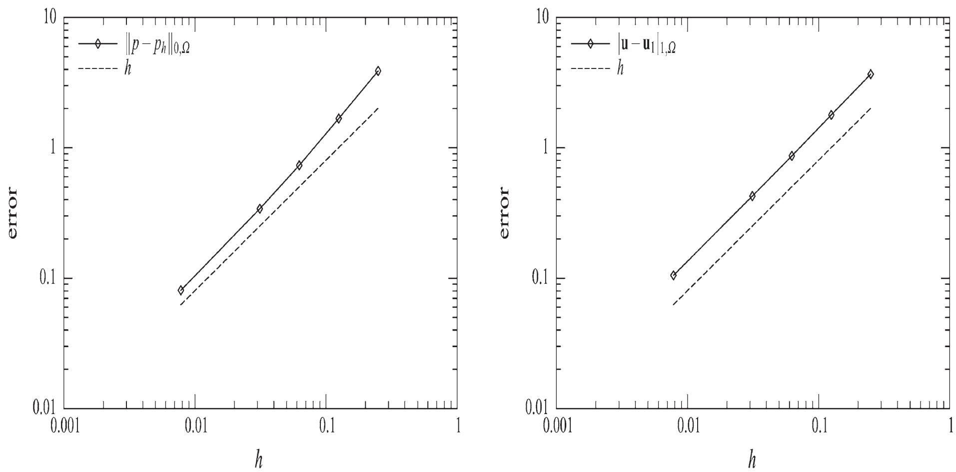

FIG. 2. Convergence history for method (2.8).

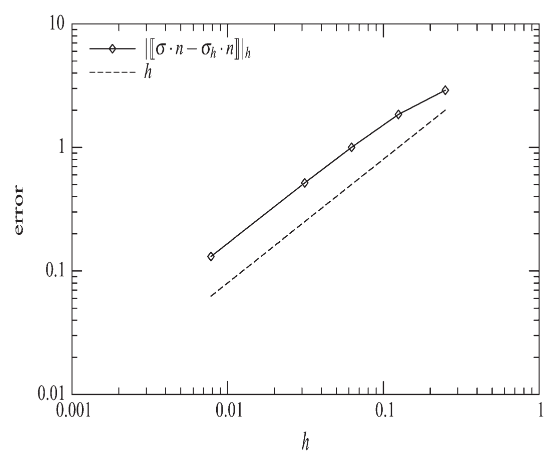

FIG. 3. Convergence history for method (2.8).

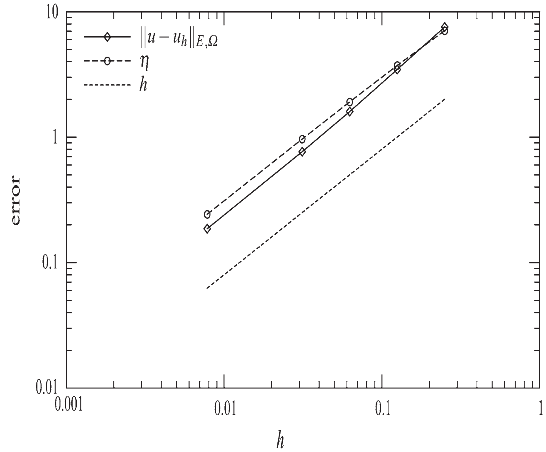

FIG. 4. Error versus η as $h \to 0$.

TABLE 1 *Effectivity index as* $h \to 0$

h	$\|(\mathbf{e}^{\mathbf{u}}, e^p)\|_{E,\Omega}$	η	θ
0.25	7.5627	7.0478	0.9319
0.125	3.4651	3.7197	1.0734
6.25×10^{-2}	1.5993	1.9052	1.1912
3.125×10^{-2}	0.7659	0.9623	1.2564
7.8125×10^{-3}	0.1860	0.2421	1.3016

as well as the estimator (4.8) as $h \to 0$, using $\nu = 1$. We can observe that the two values are in good agreement, which is confirmed in Table 1 where we show the effectivity index

$$\theta := \frac{\eta}{\|(\mathbf{e}^{\mathbf{u}}, e^p)\|_{E,\Omega}}, \tag{5.2}$$

which remains bounded, and tends towards a constant as $h \to 0$. Also, in order to study the sensitivity of the effectivity index as $\nu \to 0$, we depict in Fig. 5 the behaviour of η and $\|(\mathbf{e}^{\mathbf{u}}, e^p)\|_{E,\Omega}$ for a fixed mesh (with a fixed $h \approx 10^{-2}$) and for $\nu = 1, 0.1, \ldots, 10^{-6}$. We observe that, as was predicted by Theorem 4.1, the estimator η follows the same pattern as $\|(\mathbf{e}^{\mathbf{u}}, e^p)\|_{E,\Omega}$, and hence, the effectivity index is unaffected by ν, as may be seen in Table 2.

Finally, we present a brief comparison with the classical Douglas–Wang method (Douglas & Wang, 1989) which considers only pressure jumps. In Fig. 6, we depict the convergence history of the error $\|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_{E,\Omega}$ (left) and $|[\![\sigma \cdot \mathbf{n} - \sigma_h \cdot \mathbf{n}]\!]|_h$ (right) for both methods. We observe that the inclusion of the new jump terms makes the new method a competitive alternative to the classical Douglas–Wang method without complicating the implementation, and that it is slightly more precise in the norm of the jumps.

5.1.1 *Convergence validation using equal order* $[\mathbb{P}^1]^2 \times \mathbb{P}^1$ *elements.* In this section, we perform the convergence validation using the equal-order $[\mathbb{P}^1]^2 \times \mathbb{P}^1$ finite-element spaces. The solution and

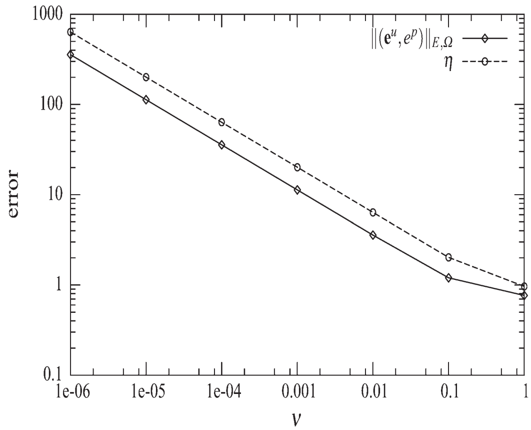

FIG. 5. Sensitivity of the estimator to ν.

TABLE 2 *Effectivity index as* $\nu \to 0$

ν	1	0.1	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
θ	1.25	1.68	1.77	1.78	1.78	1.78	1.78

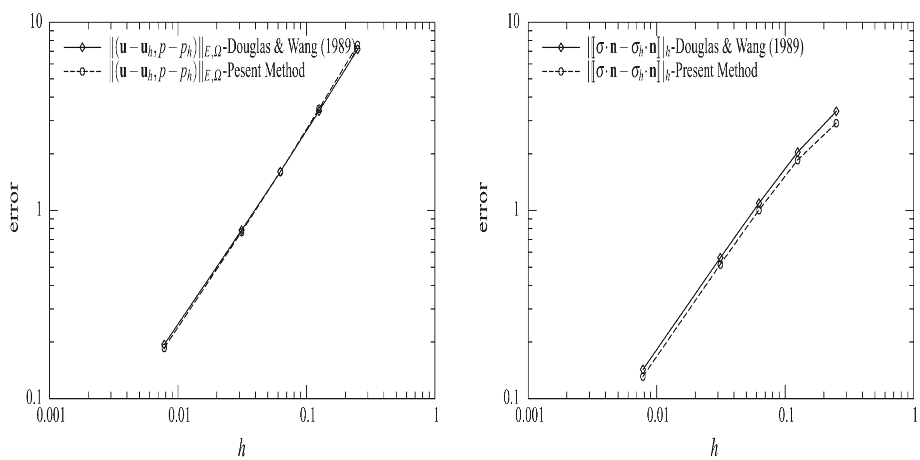

FIG. 6. Numerical comparison with Douglas & Wang (1989) for this test case.

the domain are exactly the same as in Section 5.1. In Figs 7 and 8, we observe the convergence of the method (2.8) as $h \to 0$ showing that the error behaves as predicted by Theorem 3.1.

5.2 *The lid-driven cavity problem*

For this case, we use the same domain as in Section 5.1, we set $\mathbf{f} = \mathbf{0}$, and the boundary conditions are given by $\mathbf{u} = (1, 0)^t$ on $(0, 1) \times \{1\}$, $\mathbf{u} = \mathbf{0}$ on $(0, 1) \times \{0\}$ and $\mathbf{u}(x, y) = (g(y), 0)^t$ on $[\{0\} \times (0, 1)] \cup$

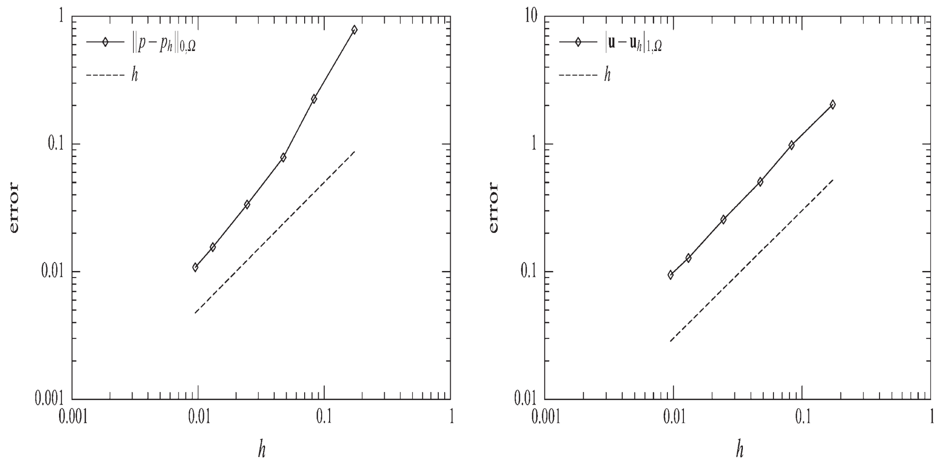

FIG. 7. Convergence history for $\|p - p_h\|_{0,\Omega}$ and $|\mathbf{u} - \mathbf{u}_h|_{1,\Omega}$.

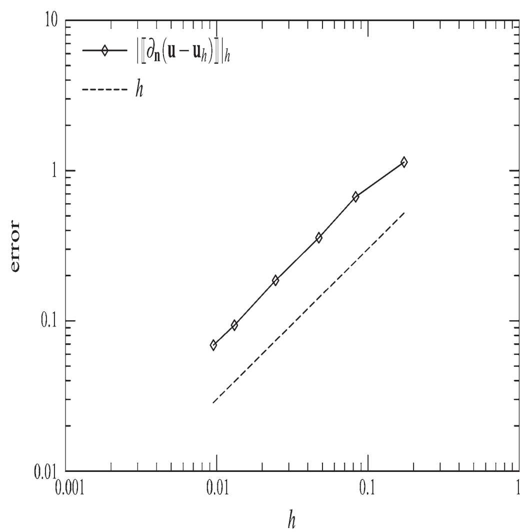

FIG. 8. Convergence history for $|[\![\partial_{\mathbf{n}}(\mathbf{u} - \mathbf{u}_h)]\!]|_h$.

$[\{1\} \times (0, 1)]$, where the function g is given by

$$g(y) := \begin{cases} 0 & \text{if } y \leqslant 0.995, \\ \left(\frac{y-0.995}{0.005}\right)^4 & \text{if } 0.995 < y \leqslant 1. \end{cases}$$

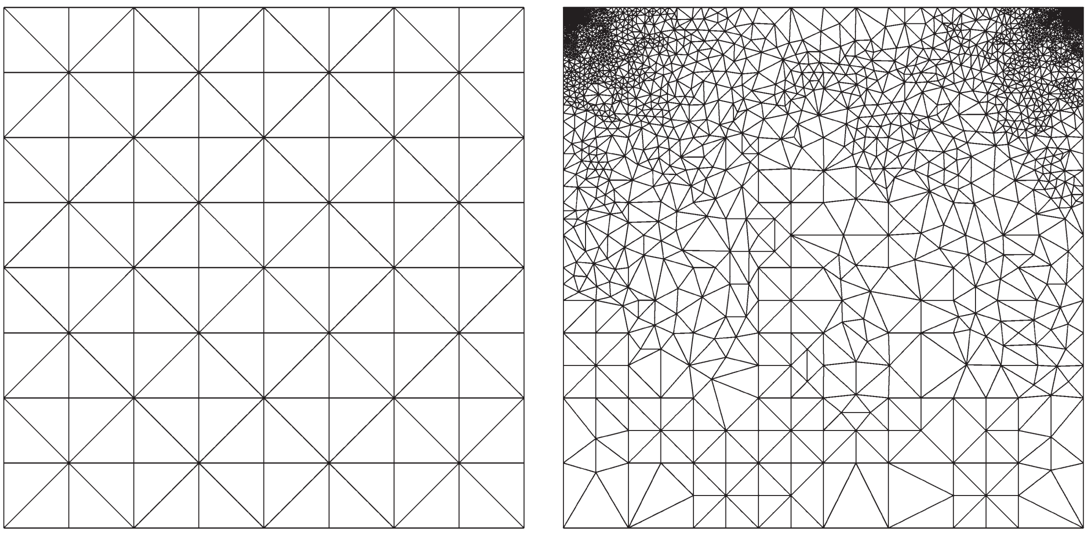

FIG. 9. Initial (left) and final adapted (right) meshes for the cavity problem.

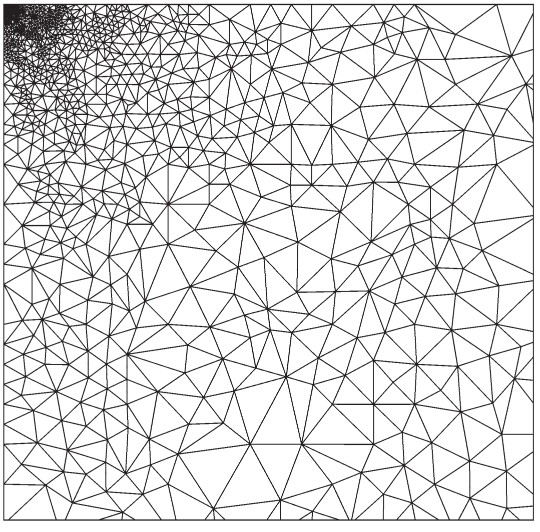

FIG. 10. Zoom into $[0, 0.1] \times [0.9, 1]$ for the adapted mesh.

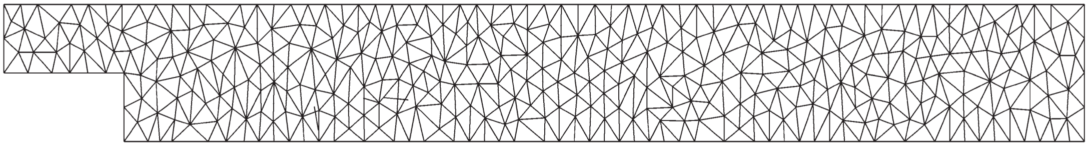

FIG. 11. Initial mesh for the step problem.

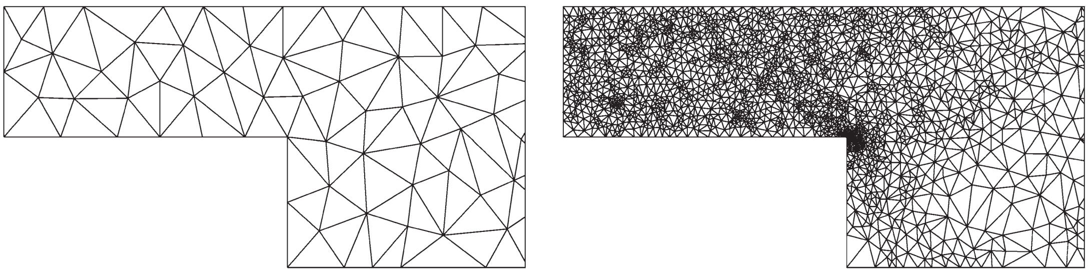

FIG. 12. Zooms into the original and adapted meshes for the step problem.

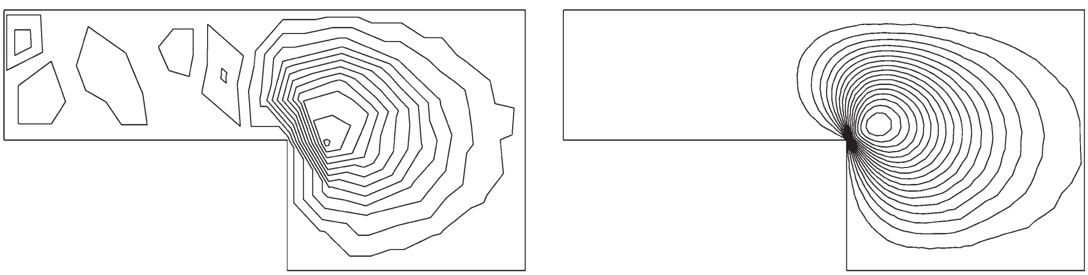

FIG. 13. Vertical velocity in the original and adapted meshes.

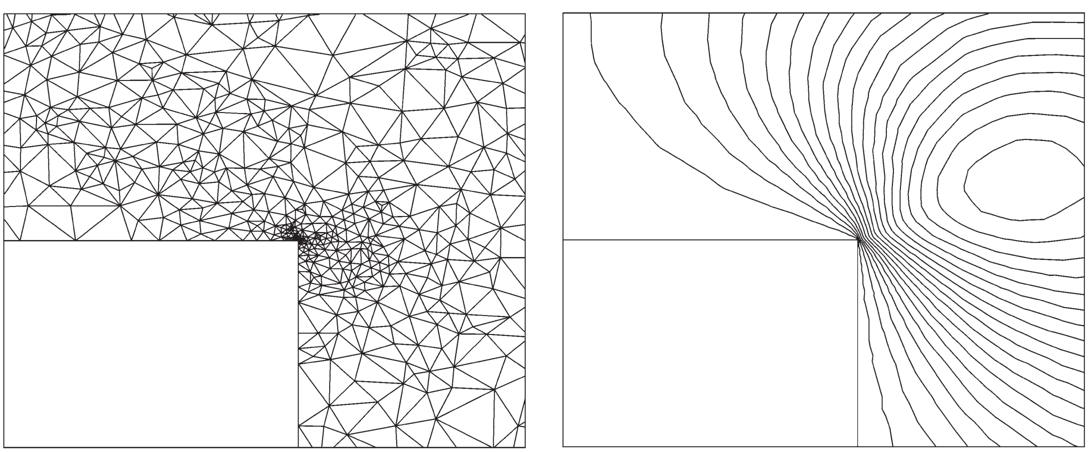

FIG. 14. Zooms into the adapted mesh and the solution in the step problem. We observe the concentration of the mesh points in the region close to the singularity.

For this test (and the following one), we are interested in assessing the adaptive procedure suggested by our estimator η, since we do not know the exact solution. This adaptive procedure is given by:

(i) Solve (2.8) in an initial mesh $\mathscr{T}_0$ and compute $\eta_K \; \forall K \in \mathscr{T}_0$.
(ii) If $K \in \mathscr{T}_0$ satisfies $\eta_K \geqslant \delta \max_{K \in \mathscr{T}_0} \eta_K$ (where $0 < \delta < 1$), then K is subdivided.
(iii) This process is repeated until a prescribed tolerance is attained.

The meshes are generated using 'Triangle', an adaptive mesh generator proposed in Shewchuk (2002), and from now on all the computations are performed using piecewise linear interpolation for the velocity and piecewise constants for the pressure.

In this problem, two (numerical) singularities arise at the top corners due to the large gradient in the boundary condition. This fact is captured by our estimator by refining mostly in those corners as can be seen in Fig. 9, where the initial and final adapted meshes may be found. In addition, and in order to show more precisely the local character of this refinement, we depict in Fig. 10 a zoom into the subdomain $[0, 0.1] \times [0.9, 1]$ for the final adapted mesh.

5.3 *The backward facing step problem*

This test case is posed on the backward facing step configuration. The step is located at $(x, y) = (2.5, 0)$, the entry of the channel is at $x = 0$ and the exit of the channel at $x = 22$. The channel width is 1 at entry and 2 at exit. The boundary conditions are inflow and outflow parabolic profiles set so that problem (2.1) with $\mathbf{f} = \mathbf{0}$ has a unique solution. In this case, a singularity arises at the step from the re-entrant corner. Hence, we can expect the meshes to be locally refined around the corner. In Fig. 11 we depict the initial mesh, and in Fig. 12 we show a zoom into the original and the final adapted meshes (the isovalues of the vertical component of the velocity are depicted in Fig. 13 for both meshes). We can observe the local behaviour of the adaptation of the mesh, a fact which is more clear in Fig. 14 where a close zoom into the final adapted mesh is depicted.

6. Concluding remarks

An extension of the well-known Douglas–Wang stabilized finite-element method for the Stokes problem has been proposed and analysed. The proposed method includes jump terms containing the residual of the Cauchy stress tensor on the internal edges of the triangulation that are well controlled by the method as was seen in the error analysis and the numerical experiments. Finally, the presence of the new jump terms allowed us to propose in a natural way a residual-based a posteriori error estimator which is robust with respect to the viscosity ν, was theoretically justified for all pairs of elements and was successfully tested in Section 5.

Acknowledgements

RA was partially supported by FONDECYT Project No. 1040595. GRB was partially supported by Comisión Nacional de Investigación Científica y Technológica (CONICYT)-Chile through FONDECYT Project No. 1030674 and FONDAP Program on Applied Mathematics. A part of this work was done during the stay of GRB at LNCC, Petrópolis, Brazil, in the framework of the joint Chile(CONICYT)-Brazil(CNPq) Project No. 2003-4-173 (Chile)-690221/02-9 (Brazil).

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