Topology of Scalar and Vector Fields

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Abstract. .

1 Introduction

In this Chapter, we review mathematical elements in scalar and vector fields. We start with basic elements in linear algebra (see [1, 2] for details in this area). Next, in section 3-4 we define scalar and vector fields in $\mathbb{R}^n$.

2 Background in Linear Algebra

Let the N-dimensional vector space V over $\Re$ composed by n-uplas:

$$\mathbf{v} = \begin{pmatrix} v(0) \\ v(1) \\ . \\ . \\ . \\ v(N-1) \end{pmatrix}. \tag{1}$$

and B a basis of V given by N linearly independent vectors:

$$B = \{\mathbf{e}_i \in V; \quad , \quad i = 0, 1, ..., N-1\}. \tag{2}$$

Therefore, we know that any vector $\mathbf{v} \in V$ can be written as a linear combination of elements in B, that means:

$$\mathbf{v} = \sum_{i=0}^{N-1} \alpha_i \mathbf{e}_i. \tag{3}$$

We call the array composed by the coefficients α_i the representation of $\mathbf{v}$ in the basis B, which we indicate by:

$$[\mathbf{v}]_B = \begin{pmatrix} \alpha(0) \\ \alpha(1) \\ . \\ . \\ . \\ \alpha(N-1) \end{pmatrix}. \tag{4}$$

Moreover, we know that such representation is unique. If B is the canonical basis given by:

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ . \\ . \\ . \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ . \\ . \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{pmatrix} \right\}, \tag{5}$$

them we have $[\mathbf{v}]_B \equiv \mathbf{v}$. In this case, we will drop the subscript "B" in expression (4).

In this context, we can consider functions $T : V \to \Re$, that satisfy the property:

$$T(\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2) = \alpha_1 T(\mathbf{u}_1) + \alpha_2 T(\mathbf{u}_2), \tag{6}$$

$\alpha_1, \alpha_2 \in \Re$ and $\mathbf{u}_1, \mathbf{u}_2 \in V$. The linear function T is called a *linear form.*

The space of linear forms is a vector space over $\Re$, called the dual of V, which is denoted by V^*. Let the dual basis

$$B = \left\{ \mathbf{e}^i \in V^*; \quad , \quad i = 0, 1, ..., N-1 \right\}. \tag{7}$$

where $\mathbf{e}^i$ is the dual of the vector $\mathbf{e}_i$; defined by:

$$\mathbf{e}^i(\mathbf{e}_j) = \delta(i - j)$$

Observe that $(V^*)^* \equiv V$. Therefore, we can see $\mathbf{u} \in \mathbf{V}$ as a linear form $\mathbf{u} : V^* \to \Re$;

$$\mathbf{u}(T) = T(\mathbf{u})$$

3 Scalar and Vector Fields in $\mathbb{R}^n$

The function

$$f : \mathbb{R}^n \to \mathbb{R},$$

is a scalar field in $\mathbb{R}^n$.

If V is vector space, the function

$$\boldsymbol{F} : \mathbb{R}^n \to V,$$

is a vector field from $\mathbb{R}^n$ to the vector space V. A specific and important case of vector field is:

$$\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3,$$

4 Topology of Scalar Fields

Given a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ we say that a point $\mathbf{x}_0 \in \mathbb{R}^n$ is a critical point of f if $D_{\mathbf{x}} f(\mathbf{x}_0) = \mathbf{0}$; that means:

$$\frac{\partial f}{\partial x_1}(\mathbf{x}_0) = \frac{\partial f}{\partial x_2}(\mathbf{x}_0) = \ldots = \frac{\partial f}{\partial x_n}(\mathbf{x}_0) = 0. \tag{8}$$

Also, we can compute the Hessian $H_{\mathbf{x}} f(\mathbf{x}_0)$ as:

$$[H_{\mathbf{x}} f(\mathbf{x}_0)]_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0), \quad 1 \le i, j \le n, \tag{9}$$

and the dimension of the null space of the matrix $[H_{\mathbf{x}} f(\mathbf{x}_0)]$ is called the corank of $H_{\mathbf{x}} f$ at $\mathbf{x}_0 \in \mathbb{R}^n$.

We say that f has a nondegenerate critical point at $\mathbf{x}_0 \in \mathbb{R}^n$ if $D_{\mathbf{x}} f(\mathbf{x}_0) = \mathbf{0}$ and if $H_{\mathbf{x}} f(\mathbf{x}_0)$, defined by expression (9), is non-singular (nondegenerate quadratic form). On the other hand, if $H_{\mathbf{x}} f(\mathbf{x}_0)$ is degenerate we say the point $\mathbf{x}_0 \in \mathbb{R}^n$ that satisfies $D_{\mathbf{x}} f(\mathbf{x}_0) = \mathbf{0}$ is a degenerate critical point.

Taylor Series for a crtical point $\mathbf{x}_0 \in \mathbb{R}^n$:

$$f(\boldsymbol{x}) = f(\mathbf{x}_0) + \frac{1}{2!}(\boldsymbol{x} - \mathbf{x}_0)^T H_{\mathbf{x}} f(\mathbf{x}_0)(\boldsymbol{x} - \mathbf{x}_0) + O\left(||\boldsymbol{x} - \mathbf{x}_0||^2\right). \tag{10}$$

From expression (10) we have tree possibilities for the critical point $\mathbf{x}_0 \in \mathbb{R}^n$:

- Local maximum: $D_{\mathbf{x}} f(\mathbf{x}_0) = \mathbf{0}$ and $H_{\mathbf{x}} f(\mathbf{x}_0)$ positive definite
- Local minimum: $D_{\mathbf{x}} f(\mathbf{x}_0) = \mathbf{0}$ and $H_{\mathbf{x}} f(\mathbf{x}_0)$ negative definite
- Seddle point: $D_{\mathbf{x}} f(\mathbf{x}_0) = \mathbf{0}$ and $H_{\mathbf{x}} f(\mathbf{x}_0)$ non-singular but with positive and negative eigenvalues
- Degenerate point: $D_{\mathbf{x}} f(\mathbf{x}_0) = \mathbf{0}$ and $H_{\mathbf{x}} f(\mathbf{x}_0)$ singular

5 Topology of Vector Fields

Consider the vector field:

$$\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n,$$

where:

$$\boldsymbol{F}(\boldsymbol{x}) = A\boldsymbol{x},$$

with A a real and constant matrix.

In this case, the solution of the differential equation:

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x}, \tag{11}$$

subject to:

$$\boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{12}$$

is [3]:

$$\boldsymbol{x}(t) = \exp(At)\,\boldsymbol{x}_0.$$

We shall remember that:

$$\exp(At) = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \ldots$$

Therefore, if $(\boldsymbol{v}_i, \lambda_i)$, $i = 1, 2, \ldots, n$ are the distint eigenvectors/eigenvalues of matrix A, then, we can write:

$$\boldsymbol{x}_0 = \sum_{i=1}^{n} \alpha_i \boldsymbol{v}_i, \tag{13}$$

and:

$$\boldsymbol{x}(t) = \exp(At)\,\boldsymbol{x}_0 = \exp(At)\left[\sum_{i=1}^{n} \alpha_i \boldsymbol{v}_i\right]$$

$$= \sum_{i=1}^{n} \alpha_i \exp(At)\,\boldsymbol{v}_i$$

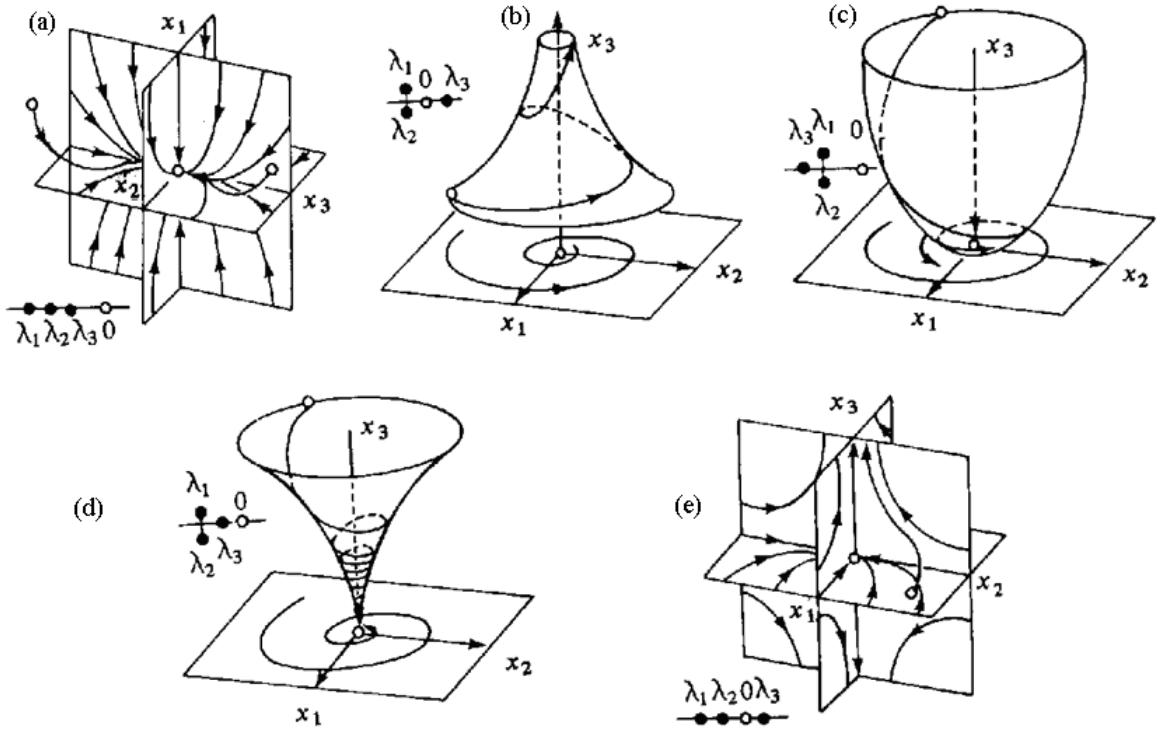

Figure 1: Hyperbolic linear systems in $\mathbb{R}^3$.

$$= \sum_{i=1}^{n} \alpha_i \left[\boldsymbol{v}_i + tA\boldsymbol{v}_i + \frac{1}{2!}t^2A^2\boldsymbol{v}_i + \frac{1}{3!}t^3A^3\boldsymbol{v}_i + \dots \right]$$

$$= \sum_{i=1}^{n} \alpha_i \left[\boldsymbol{v}_i + t\lambda_i\boldsymbol{v}_i + \frac{1}{2!}t^2\lambda_i^2\boldsymbol{v}_i + \frac{1}{3!}t^3\lambda_i^3\boldsymbol{v}_i + \dots \right]$$

$$= \sum_{i=1}^{n} \alpha_i \left[I + t\lambda_i + \frac{1}{2!}t^2\lambda_i^2 + \frac{1}{3!}t^3\lambda_i^3 + \dots \right] \boldsymbol{v}_i$$

$$= \sum_{i=1}^{n} \alpha_i \exp\left(\lambda_i t\right) \boldsymbol{v}_i. \tag{14}$$

Therefore, we can express the solution of the initial value problem using the eigenvalues/eigenvectors of the matrix A and solving equation (13). Consequently the eigendecomposition of A defines the topology of the general solution of the equation (11).

If λ_i has real part non-null for $i = 1, 2, \dots, n$, we say that the singular point $\boldsymbol{x} = \boldsymbol{0}$ is hyperbolic. For $n = 3$ we have the cases pictured in Figure 1 for hyperbolic systems [3].

Now, consider the vector field $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$, with $\boldsymbol{F}$ smooth and:

$$\boldsymbol{F}\left(\boldsymbol{x}\right) = \left(F_1\left(x_1, x_2, \dots, x_n\right), F_2\left(x_1, x_2, \dots, x_n\right), \dots, F_n\left(x_1, x_2, \dots, x_n\right)\right). \tag{15}$$

Then, we define the Jacobian matrix:

$$D\boldsymbol{F}\left(\boldsymbol{x}\right) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \dots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \dots & \frac{\partial F_2}{\partial x_n} \\ & \dots & & \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \dots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}.$$

Definition: A hyperbolic singular point of $\boldsymbol{F}$ is a point $\boldsymbol{p} \in \mathbb{R}^n$ such that: $\boldsymbol{F}(\boldsymbol{p}) = \mathbf{0}$ and all the eigenvalues of $D\boldsymbol{F}(\boldsymbol{p})$ have real part non-null.

Hartman's Theorem. Let a smooth vector field $\boldsymbol{F} : \Re^n \to \Re^n$and $\boldsymbol{p} \in \mathbb{R}^n$ a hyperbolic singular point. Then, there are neighbourhoods $V(\boldsymbol{p}) \subset \mathbb{R}^n$ and $W(\mathbf{0}) \subset \mathbb{R}^n$ such that $\boldsymbol{F} : V(\boldsymbol{p}) \longrightarrow \mathbb{R}^n$ can be represented by the solutions of:

$$\frac{d\boldsymbol{x}}{dt} = D\boldsymbol{F}(\boldsymbol{p})\,\boldsymbol{x},$$

in the neighbourhood $W(\mathbf{0})$.

6 Complex Eigenvalues

For real matriz A and a complex eigenvector $\boldsymbol{v} = \boldsymbol{v}_1 + i\boldsymbol{v}_2$ with complex eigenvalue $\lambda = \gamma + i\beta$ we have:

$$A\boldsymbol{v} = \lambda\boldsymbol{v} \quad \Longrightarrow \quad A\overline{\boldsymbol{v}} = \overline{\lambda}\overline{\boldsymbol{v}}.$$

Therefore, $\overline{\boldsymbol{v}} = \boldsymbol{v}_1 - i\boldsymbol{v}_2$ is an eigenvector of A with eigenvalue $\overline{\lambda} = \gamma - i\beta$. For instance, if $A \in \mathbb{R}^{2\times 2}$, with complex eigenvalues, then from expression (14), the general solution of (11) is:

$$\boldsymbol{x}(t) = \alpha_1 \exp\left[(\gamma + i\beta)\,t\right](\boldsymbol{v}_1 + i\boldsymbol{v}_2) + \alpha_2 \exp\left[(\gamma - i\beta)\,t\right](\boldsymbol{v}_1 - i\boldsymbol{v}_2)\,. \tag{16}$$

Let us define the function:

$$\boldsymbol{\varphi}(t) = \exp\left[(\gamma + i\beta)\,t\right](\boldsymbol{v}_1 + i\boldsymbol{v}_2) = \exp\left[\gamma t\right](\cos(\beta t) + i\sin(\beta t))(\boldsymbol{v}_1 + i\boldsymbol{v}_2)\,,$$

$$\Longleftrightarrow$$

$$\boldsymbol{\varphi}(t) = \exp\left[\gamma t\right]\left[(\cos(\beta t)\,\boldsymbol{v}_1 - \sin(\beta t)\,\boldsymbol{v}_2) + i\,(\sin(\beta t)\,\boldsymbol{v}_1 + \cos(\beta t)\,\boldsymbol{v}_2)\right].$$

With this expression we can build another function:

$$\overline{\boldsymbol{\varphi}}(t) = \exp\left[\gamma t\right]\left[(\cos(\beta t)\,\boldsymbol{v}_1 - \sin(\beta t)\,\boldsymbol{v}_2) - i\,(\sin(\beta t)\,\boldsymbol{v}_1 + \cos(\beta t)\,\boldsymbol{v}_2)\,,\right]$$

So, we can construct the functions

$$\boldsymbol{x}_1(t) = \frac{1}{2}\left(\boldsymbol{\varphi}(t) + \overline{\boldsymbol{\varphi}}(t)\right) = Re\left(\boldsymbol{\varphi}(t)\right), \tag{17}$$

$$\boldsymbol{x}_2(t) = \frac{1}{2i}\left(\boldsymbol{\varphi}(t) + \overline{\boldsymbol{\varphi}}(t)\right) = Im\left(\boldsymbol{\varphi}(t)\right). \tag{18}$$

We can show that:

$$\boldsymbol{x}_1(t) = \exp\left[\gamma t\right](\cos(\beta t)\,\boldsymbol{v}_1 - \sin(\beta t)\,\boldsymbol{v}_2)\,,$$

$$\boldsymbol{x}_2(t) = \exp\left[\gamma t\right](\sin(\beta t)\,\boldsymbol{v}_1 + \cos(\beta t)\,\boldsymbol{v}_2)\,.$$

It is possible to verify that $\{\boldsymbol{x}_1(t), \boldsymbol{x}_2(t)\}$ is a linerarly independent set of solutions of expression (11). Consequently, in this case, the general solution of the system (11) can be written as:

$$\boldsymbol{x}(t) = c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t)\,. \tag{19}$$

If we write $c_1 = \rho\cos\omega$ and $c_2 = \rho\sin\omega$, then we can re-write expression (19) as:

$$\boldsymbol{x}(t) = \rho\cos\omega\exp\left[\gamma t\right](\cos(\beta t)\,\boldsymbol{v}_1 - \sin(\beta t)\,\boldsymbol{v}_2) + \rho\sin\omega\exp\left[\gamma t\right](\sin(\beta t)\,\boldsymbol{v}_1 + \cos(\beta t)\,\boldsymbol{v}_2)\,,$$

that renders:

$$\boldsymbol{x}(t) = \rho \exp[\gamma t] (\cos\omega \cos(\beta t) + \sin\omega \sin(\beta t)) \boldsymbol{v}_1 +$$

$$\rho \exp[\gamma t] (\sin\omega \cos(\beta t) - \cos\omega \sin(\beta t)) \boldsymbol{v}_2,$$

which finally gives:

$$\boldsymbol{x}(t) = \rho \exp[\gamma t] (\cos(\omega - \beta t) \boldsymbol{v}_1 + \sin(\omega - \beta t) \boldsymbol{v}_2). \tag{20}$$

Case 1. Closed trajectories: $\gamma = 0$

$$\boldsymbol{x}(t) = \rho (\cos(\omega - \beta t) \boldsymbol{v}_1 + \sin(\omega - \beta t) \boldsymbol{v}_2). \tag{21}$$

Case 2. $\gamma < 0$

$$\lim_{t \longrightarrow +\infty} \boldsymbol{x}(t) = (0, 0).$$

Case 3. $\gamma > 0$

$$\lim_{t \longrightarrow +\infty} \boldsymbol{x}(t) = \boldsymbol{\infty}.$$

References

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