# CONVERGENCE ANALYSIS OF A MULTISCALE FINITE ELEMENT METHOD FOR SINGULARLY PERTURBED PROBLEMS

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Abstract. In this paper we perform an error analysis for a multiscale finite element method for singularly perturbed reaction–diffusion equation. Such method is based on enriching the usual piecewise linear finite element trial spaces with local solutions of the original problem, but do not require these functions to vanish on each element edge. Bubbles are the choice for the test functions allowing static condensation, thus our method is of Petrov–Galerkin type. We perform convergence analysis in different asymptotic regimes, and we show uniform convergence in an appropriate norm with respect to the small parameter. Numerical results show that the new method is able to compute solutions even on coarse meshes.

## 1. Introduction

It is well known that standard Galerkin method is inadequate to solve singularly perturbed problems. Indeed, the method is not uniformly stable and the solution presents spurious oscillations in the presence of boundary layers (e.g., see [24] and references therein).

Specially refined meshes, such as Shishkin meshes (see [22, 27], and references therein) can ameliorate this situation. Nonetheless, such strategy becomes increasingly complex for complicate geometries, and can be prohibitive to treat realistic three-dimensional problems. Adaptivity is another possibility and consists on associating *a posteriori* estimators to the Galerkin method in order to built refined meshes (see for instance [1, 3], and references therein).

Previous papers [11, 12, 18, 19, 28] carried out more stable and accurate formulations based on stabilized methods for the reaction–diffusion model, using coarse meshes. The stabilized methods are based on modified variational formulations, but still employ piecewise polynomials. These modifications involve nontrivial mesh-dependent stability parameters, and also depend on the residuals of the governing differential equation.

Partial justification of these ideas were made possible by relating stabilized methods to the Galerkin method using piecewise polynomials enriched with "bubble" functions, as illustrated

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in [4, 5, 6]. To systematically treat various singularly perturbed problems, residual-free bubbles were introduced in [8, 13, 14, 15, 16, 17]. These bubbles are functions with local support which solve, exactly or not, differential equations at the element level, involving the differential operator of the problem. The right hand sides of these local problems are the residuals due to the polynomial part of the solution. The other ingredient is the requirement that the bubble part vanishes on element boundaries for second order problems. Convergence results for linear and bilinear elements can be found in [7, 23, 26]. It turns out that such construction for the reaction–diffusion problem yields a poor approximation. Assuming the bubble part of the trial solution to be zero across element edges introduces inaccuracies.

In a previous work, Franca, Madureira and Valentin [10] have explored a new strategy, without the zero boundary value restriction on each element, conjugate with a Petrov-Galerkin method. They let the test space to be enriched with residual-free bubble functions, but the functions in the trial space have boundary values determined by edge restrictions of the governing differential operator. Such restrictions yield ordinary differential equations that can be solved a priori. Even more importantly, the modification is computable at the element level. Related ideas were proposed by [25] in the context of spline theory, and by Hou, Wu and Cai [20, 21] for PDEs with oscillatory coefficients.

The present work is devoted to develop error estimates for the multiscale finite element method proposed in [10]. We perform convergence analysis in two different asymptotic regimes, and we point out sufficient conditions to obtain uniform convergence with respect to the small parameter in an appropriate norm. Moreover, we show that we recover the standard Galerkin energy norm error estimates when the mesh is fine enough.

The paper is organized as follows. In Section 2 we revisit our Petrov-Galerkin formulation. In Section 3 we derive error estimates in different asymptotic regimes, and next in Section 4 we perform numerical tests. Finally in the Appendix we present some auxiliary results.

## 2. The Multiscale Method

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with polygonal boundary $\partial\Omega$. We consider $u \in H_0^1(\Omega)$ the solution of the reaction diffusion equation

$$\mathcal{L}u := -\varepsilon\Delta u + \sigma u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $\varepsilon$ and $\sigma$ are positive constants. We assume $f$ piecewise linear, thus (1) is well-posed.

The usual weak formulation of problem (1) consists on finding $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v), \qquad \text{for all } v \text{ in } H_0^1(\Omega), \tag{2}$$

where the bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is given by

$$a(u,v) := \varepsilon(\nabla u, \nabla v) + \sigma(u,v). \tag{3}$$

As usual $(\cdot,\cdot)_D$ denotes the inner product in $L^2(D)$ where $D$ is a open set of $\Omega$. The norm induce by such inner product is denoted by $\|.\|_{0,D}$. To simplify the notation, we write $(\cdot,\cdot)$ and $\|.\|_0$ when $D = \Omega$. Similarly the $L^\infty(D)$ norm is denoted by $\|.\|_{\infty,D}$. The weak problem (2) is well-posed thanks to the coercivity of the bounded bilinear form $a(\cdot,\cdot)$ over $H_0^1(\Omega)$ and the Lax–Milgram Theorem.

Let $\mathcal{T}_h$ be a regular triangulation of domain $\Omega$ into elements $K$ with boundary $\partial K$ such that

$$\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} \overline{K},$$

where the intersection of two elements is either a vertex, or an edge, or empty. We define $\mathcal{V}_h$ as the set of edges $Z$ belonging to $\mathcal{T}_h$, we denote by $h_K$ a characteristic length of $K \in \mathcal{T}_h$, and we set $h = \max_{K\in\mathcal{T}_h}\{h_K\}$. By $\Omega_{layer}$, we denote the set of elements in $\mathcal{T}_h$ which boundaries have nontrivial intersection with $\partial\Omega$, and we define

$$\Omega^0 = \Omega/\overline{\Omega}_{layer},$$

and $h_l = \max_{K\in\Omega_{layer}} \{h_K\}$. In the sequel $C, C_0, C_1, C_2, \ldots$ will denote generic positive constants, independent of $h$, $\varepsilon$ or $\sigma$, but whose value may vary in each occurrence. Moreover, we write $b \simeq d$ meaning that

$$b \le Cd \text{ and } d \le Cb.$$

The space of piecewise linear polynomials $\mathbb{P}^1(K)$ is used to approximate the exact solution. We denote by $V_h$ the standard finite element space

$$V_h := \{v_h \in H^1(\Omega) \,|\, v_{h|K} \in \mathbb{P}^1(K) \text{ for all K} \in \mathcal{T}_h\}, \tag{4}$$

and

$$V_h^0 := V_h \cap H_0^1(\Omega),$$

and the Galerkin scheme associated to the continuous problem reads: *find* $u_g \in V_h^0$ *such that*

$$a(u_g, v_1) = (f, v_1), \qquad \text{for all } v_1 \text{ in } V_h^0. \tag{5}$$

It is well known that the Galerkin method (5) is unable to approximate the solution if $\varepsilon \ll \sigma h^2$. To overcome such limitation, we have proposed in [10] a method based on enriching the standard finite element space. The idea is to add special functions, also called multiscale functions, to the usual polynomial spaces to stabilize and improve the accuracy of the Galerkin method. For the sake of completeness, we describe the main steps to obtain our multiscale method.

We start by introducing some new notation. We denote by $H^1_0(\mathcal{T}_h)$ and $H^1(\mathcal{T}_h)$ the spaces of functions on $\Omega$ whose restrictions to each element $K$ belongs to $H^1_0(K)$ and $H^1(K)$, respectively. Given an edge $Z$ belonging to $\mathcal{V}_h$, let $\mathbb{P}^1(Z)$ the space of linear polynomials on $Z$, and let us introduce the operators $\mathcal{B}^i_K : \mathbb{P}^1(Z) \to L^2(Z)$ defined in the following way: given a base function $q_i$ of $\mathbb{P}^1(Z)$ we associate $w_i = \mathcal{B}^i_K q_i \in L^2(Z)$ such that

$$\mathcal{L}^i_{\partial K} w_i := -\varepsilon \partial_{ss} w_i + \overline{\sigma}_i w_i = q_i \quad \text{on } Z, \quad w_i = 0 \quad \text{at the nodes.} \tag{6}$$

The coefficient $\overline{\sigma}_i$ is set as a positive parameter which can depend on $|K|$ and $|Z|$, and on the node $i$. Such dependence will be specified later (see equation (28)), and we denote by $s$ a variable that parametrize $Z$ by arc-length. We point out that (6) is well-posed. A similar boundary condition was used in Hou, Wu and Cai [20, 21] for elliptic problems with oscillatory coefficients. Now, let $\mathcal{M}^i_K : \mathbb{P}^1(K) \to H^1(K)$ be the linear operator defined as follows: given $v_i$ a base function of $\mathbb{P}^1(K)$ let $b_i = \mathcal{M}^i_K v_i \in H^1(K)$ be the solution of the problem

$$\mathcal{L} b_i = v_i \text{ in } K, \quad b_i = \mathcal{B}^i_K\left(\frac{\overline{\sigma}_i}{\sigma} v_i\right) \text{ on each } Z \in \partial K, \tag{7}$$

where $\mathcal{B}^i_K$ are the local operators defined in (6). Since $b_i|_Z \in L^2(Z)$ problem (7) is clearly well-posed in each $K \in \mathcal{T}_h$. Therefore, using (7) we introduce the operator $\mathcal{M}_K : \mathbb{P}^1(K) \to H^1(K)$ defined by

$$\mathcal{M}_K \, p_h := \sum_i \mathcal{M}^i_K(b_i) \, p_i, \quad p_h \in \mathbb{P}^1(K), \tag{8}$$

where $p_i$ represents the coefficients of $p_h$. Furthermore, we denote by $E_h$ the subspace of $H^1(\mathcal{T}_h)$, called *multiscale space*, such that $E_h \cap V^0_h = \{0\}$ and defined by

$$E_h := \{ v_e \in H^1(\mathcal{T}_h) \,|\, v_{e|K} = \mathcal{M}_K \, v_1 \text{ for all } v_1 \in V_h \}, \tag{9}$$

where $\mathcal{M}_K$ is the operator (8). Hence, we introduce the trial subspace $U_h$ of $H^1(\mathcal{T}_h)$ defined by

$$U_h := V^0_h \oplus E_h, \tag{10}$$

thus an element $v_h$ of $U_h$ may be uniquely written as

$$v_h := v_1 + v_e,$$

where $v_1 \in V_h^0$ and $v_e \in E_h$. The space $E_h$ is a finite dimensional space and $\dim(E_h) = \dim(V_h)$. We note from (7) that the functions belonging to $E_h$ may be *a priori* discontinuous across the edges of triangles. The continuity is enforced only at the nodes of the triangulation. Therefore, the method is nonconforming. Our approximation of the exact solution in the enriched space (10) is defined by the solution of the following Petrov-Galerkin problem: *find* $u_h \in U_h$ *such that*

$$a_h(u_h, v_h) = (f, v_h), \quad \text{for all } v_h \in V_h^0 \oplus H_0^1(\mathcal{T}_h) \tag{11}$$

where

$$a_h(u, v) := \sum_{K \in \mathcal{T}_h} a(u, v)_K,$$

and

$$a(u, v)_K := \varepsilon(\nabla u, \nabla v)_K + \sigma(u, v)_K.$$

From (11) we immediately have that the corresponding $u_h \in U_h$ satisfies

$$a_h(u_h, v_1) = (f, v_1) \quad \text{for all } v_1 \in V_h^0, \tag{12}$$

$$a(u_h, v_b^K)_K = (f, v_b^K)_K \quad \text{for all } v_b^K \in H_0^1(K). \tag{13}$$

We postpone to Section 3 the discussion of well-posedness of (11). By integrating (13) by parts, we immediately have that the enriched part of the solution $u_h$, denoted by $u_e \in E_h$, is the strong solution of the local problem

$$\mathcal{L}u_e = f - \mathcal{L}u_1 \quad \text{in each } K \in \mathcal{T}_h, \tag{14}$$

and hence, from (14) we impose

$$u_e = \mathcal{M}_K(f - \mathcal{L}u_1). \tag{15}$$

It follows by construction and by (12) that (11) is equivalent to the finite dimensional problem: *find* $u_1 \in V_h^0$ *such that*

$$a_h((\mathcal{I} - \mathcal{M}_K\mathcal{L})u_1, v_1) = (f, v_1) - a_h(\mathcal{M}_K f, v_1) \quad \text{for all } v_1 \in V_h^0, \tag{16}$$

where $\mathcal{I}$ is the identity operator.

2.1. **Corresponding discrete formulation.** Let us rewrite (15) in terms of basis functions. We assume that

$$E_h = \text{span}\{\phi_i\}_{i\in I} \text{ and } V_h = \text{span}\{\psi_i\}_{i\in I}, \tag{17}$$

where $\psi_i$ are the usual hat functions. Then, $f$ and $u_1$ are given by

$$u_1 = \sum_{i\in I_0} \psi_i u_i, \quad f = \sum_{j\in I} \psi_j f_j,$$

where $u_i$ , $i \in I_0$, and $f_j$, $j \in I$, are the nodal values of $u$ and $f$, respectively. Here $I$ and $I_0$ are the set of indexes of total and internal nodal points, respectively. It follows from (15), and from the linearity of the operators $\mathcal{L}$ and $\mathcal{L}_{\partial K}$ that

$$u_e = \sum_{i\in I_0} \phi_i u_i - \sum_{i\in I} \phi_i \frac{f_i}{\sigma}, \tag{18}$$

where the basis functions $\phi_i \in E_h$, $i \in I$, satisfy

$$\mathcal{L}\phi_i = -\sigma\psi_i \quad \text{in } K, \tag{19}$$

$$\phi_i = \mu_i \quad \text{on each } Z \in \partial K, \tag{20}$$

for all $K \in \mathcal{T}_h$. From (6) and (18), $\mu_i$ is the solution of the boundary value problem

$$\mathcal{L}^i_{\partial K}\mu_i = -\overline{\sigma}_i\psi_i \text{ on } Z \quad \text{and} \quad \mu_i = 0 \text{ at the nodes.} \tag{21}$$

It is convenient to present such problem in terms of the unknown $\lambda_i \in U_h$, $i \in I$, be defined by

$$\lambda_j := \psi_j + \phi_j = (\mathcal{I} - \sigma\mathcal{M}_K)\psi_j \quad \text{for all } j \in I. \tag{22}$$

Hence, from the definition (22) the function $\lambda_i$, $i \in I$, satisfies

$$\mathcal{L}\lambda_i = 0 \quad \text{in } K, \tag{23}$$

$$\lambda_i = \rho_i \quad \text{on each } Z \in \partial K, \tag{24}$$

where $\rho_i$, $i \in I$, satisfies the ordinary differential problem

$$\mathcal{L}^i_{\partial K}\rho_i = 0 \text{ on } Z \quad \text{and} \quad \rho_i = \psi_i \text{ at the nodes.} \tag{25}$$

Thus the discrete version of the weak formulation (16) reads

$$\sum_{j\in I_0} a(\lambda_j, \psi_i) u_j = \sum_{j\in I} \left[a(\lambda_j, \psi_i) - \varepsilon(\nabla\psi_j, \nabla\psi_i)\right] \frac{f_j}{\sigma} \quad \text{for all } i \in I_0. \tag{26}$$

*Remark* 1. Numerical experiments indicate that the modified scheme type

$$\sum_{j\in I_0} a(\lambda_j,\psi_i)u_j = \sum_{j\in I}(\lambda_j,\psi_i)\frac{f_j}{\sigma} \quad \text{for all } i\in I_0, \tag{27}$$

also yields accurate numerical approximations. Nevertheless, we do not believe that we can derive (27) using the procedure described above. Thus, we do not advocate this approach.

Let $K$ be an element of the triangulation $\mathcal{T}_h$, and $Z$ an edge of its boundary $\partial K$. The dependence of coefficients $\overline{\sigma}_i$ in terms of the shape of elements $K$ is given by setting

$$\overline{\sigma}_i := \frac{4|K|^2}{|Z|^2|\overline{Z}_i|^2}\sigma, \tag{28}$$

where $\overline{Z}_i$ denotes the corresponding edge of $K$ opposed to the node $i$. Moreover, we define

$$\gamma_K^i = \left(\frac{\partial\psi_i}{\partial x}|_K\right)^2 + \left(\frac{\partial\psi_i}{\partial y}|_K\right)^2 = \frac{|\overline{Z}_i|^2}{4|K|^2} \simeq h_K^{-2} \quad \text{for all } i\in I. \tag{29}$$

Thanks to the definitions (28), (29) the analytical solution of (23), (24) is given by

$$\lambda_i(x,y) = \frac{\sinh\left(\sqrt{\frac{\sigma}{\gamma_K^i\varepsilon}}\psi_i(x,y)\right)}{\sinh\left(\sqrt{\frac{\sigma}{\gamma_K^i\varepsilon}}\right)} \quad \text{for all } i\in I. \tag{30}$$

By taking a particular node $k\in I$, and look at all elements connected to this node, then the equation (30) illustrate the nodal shape functions $\lambda_k$. Fixing $\sigma=1$, we obtain for $\varepsilon = 1$, $10^{-2}$, $10^{-4}$, the shape functions $\lambda_k$, depicted in Figures 1 and 2. Note that as $\varepsilon$ approaches zero, the usual pyramid is squeezed in its domain of influence in the neighborhood around node $k$. Note that the support of $\lambda_k$ coincide with the support of the piecewise linear function $\psi_k$. Since the elements in the patch are all identical, the shape functions $\lambda_k$ depicted in Figures 1, 2 are continuous. As we pointed out before this is not true in general.

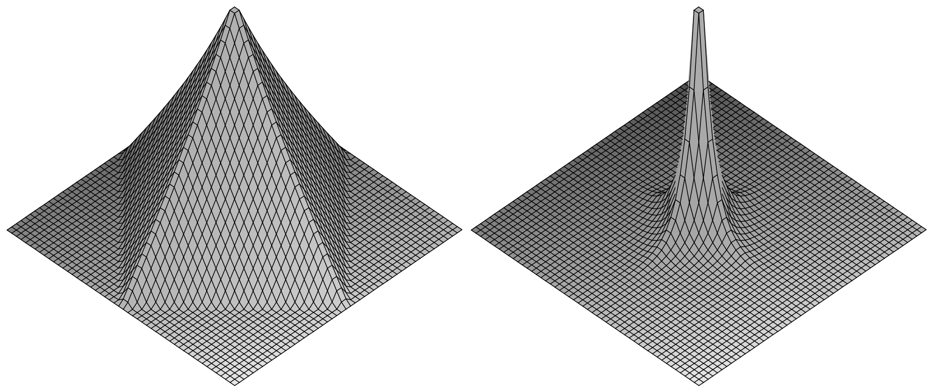

Figure 1. The function $\lambda_k$ for $\varepsilon = 1$ (left) and $\varepsilon = 10^{-2}$ (right).

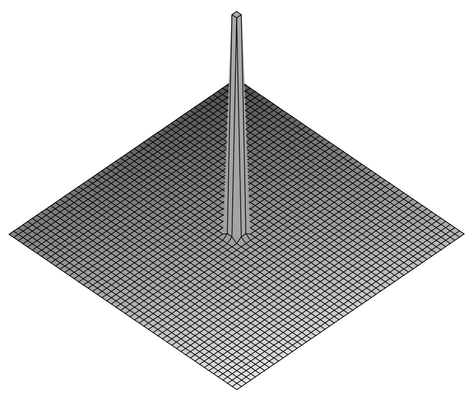

Figure 2. The function $\lambda_k$ for $\varepsilon = 10^{-4}$.

## 3. Error Analysis

We are now concerned with the error analysis of the multiscale method (16) in both $\varepsilon$ and $h$ asymptotic limits. For simplicity we perform the error analysis of the method by setting $\gamma_K^i$ independent of $i \in I$. With such assumption we assume an equilateral triangulation of the domain. The general case is similar, but involves a quite cumbersome symbolic computation (see Lemma 1 below). We start by recalling that the multiscale method (16) reads: *find* $u_1 \in V_h^0$ *such that*

$$a_e(u_1, v_1) = f_e(v_1), \quad \text{for all } v_1 \in V_h^0, \tag{31}$$

where the modified bilinear and linear forms are

$$a_e(u, v) := a(u, v) - a_h(\mathcal{M}_K \mathcal{L} u, v), \quad \text{and} \quad f_e(v) := (f, v) - a_h(\mathcal{M}_K f, v). \tag{32}$$

We first observe that the method (31) is consistent since $\mathcal{M}_K(\mathcal{L}u - f) = 0$, see definition (7). We shall show that the problem (31), and consequently (16), is well-posed. Before presenting the main coercivity result, we need the following estimates.

**Lemma 1.** Let the linear operator $\mathcal{M}_K$ be defined by (8). Then, there exist $C^1_\lambda$, $C^2_\lambda$, $C^1_\rho$, $C^2_\rho$, $C_\theta$, $C_\xi$, $C_\gamma$, and $C_\zeta$ positive constants depending only on the inner angles of $K$, and such that

*i)* $C^1_\lambda \lambda^K_{min} \|v_1\|^2_{0,K} \le ((\mathcal{I} - \sigma\mathcal{M}_K)v_1, v_1)_K \le C^2_\lambda \lambda^K_{max} \|v_1\|^2_{0,K}, \quad \forall v_1 \in V_h,$

*ii)* $-C^1_\rho \rho^K_{min} h_K^{-2} \|v_1\|^2_{0,K} \le -(\sigma\nabla\mathcal{M}_K v_1), \nabla v_1)_K \le 0, \quad \forall v_1 \in V_h,$

*iii)* $0 \le (\nabla((\mathcal{I} - \sigma\mathcal{M}_K)v_1), \nabla v_1)_K \le C^2_\rho \rho^K_{max} h_K^{-2} \|v_1\|^2_{0,K}, \quad \forall v_1 \in V_h,$

*iv)* $\|\mathcal{M}_K v_1\|^2_{0,K} \le C_\theta \theta^K_{max} \sigma^{-2} \|v_1\|^2_{0,K}, \quad \forall v_1 \in V_h,$

*v)* $\|\nabla(\mathcal{M}_K v_1)\|^2_{0,K} \le C_\xi \xi^K_{max} (\sigma h_K)^{-2} \|v_1\|^2_{0,K}, \quad \forall v_1 \in V_h,$

*vi)* $\|(\mathcal{I} - \sigma\mathcal{M}_K)v_1\|^2_{0,K} \le C_\gamma \gamma^K_{max} \|v_1\|^2_{0,K}, \quad \forall v_1 \in V_h,$

*vii)* $\|\nabla((\mathcal{I} - \sigma\mathcal{M}_K)v_1)\|^2_{0,K} \le C_\zeta \zeta^K_{max} h_K^{-2} \|v_1\|^2_{0,K}, \quad \forall v_1 \in V_h,$

where the quantities constants $\lambda^K_{min}$, $\lambda^K_{max}$, $\rho^K_{min}$, $\rho^K_{max}$, $\theta^K_{max}$, $\xi^K_{max}$, $\gamma^K_{max}$ and $\zeta^K_{max}$ depend in a nontrivial way on $\varepsilon$, $\sigma$, $h_K$, and are given in the Appendix. Here $h_K = \frac{\gamma^i_K}{C_K}$ where $C_K = 6\frac{C^1_\rho}{C^1_\lambda}$.

*Proof.* Let $K$ be a triangle element of partition $\mathcal{T}_h$ with characteristic length $h_K$. Then,

$$K = T(\hat{K}),$$

where $T : (\xi, \eta) \to (x, y)$ is an affine transformation and $\hat{K}$ is the unit triangle reference element. Let $v_1$ be an element of $V_h$, and $v_1|_K = \sum_{i=1}^3 v_i\psi_i$. The basis functions defined on the reference element $\hat{K}$ are

$$\hat{\psi}_i := \psi_i \circ T(\xi, \eta), \quad \hat{\lambda}_i := \lambda_i \circ T(\xi, \eta), \quad \hat{\phi}_i := \phi_i \circ T(\xi, \eta),$$

and we have from definition (22) that

$$((\mathcal{I} - \sigma\mathcal{M}_K)v_1, v_1)_K = \sum_{i=1}^3 \sum_{j=1}^3 v_i v_j (\lambda_i, \psi_j)_K = 2|K| \sum_{i=1}^3 \sum_{j=1}^3 v_i v_j (\hat{\lambda}_i, \hat{\psi}_j)_{\hat{K}},$$

$$-(\sigma\nabla(\mathcal{M}_K v_1), \nabla v_1)_K = \sum_{i=1}^3 \sum_{j=1}^3 v_i v_j (\nabla\phi_i, \nabla\psi_j)_K \simeq \sum_{i=1}^3 \sum_{j=1}^3 v_i v_j (\nabla_{\xi\eta}\hat{\phi}_i, \nabla_{\xi\eta}\hat{\psi}_j)_{\hat{K}},$$

$$(\nabla((\mathcal{I} - \sigma\mathcal{M}_K)v_1), \nabla v_1)_K = \sum_{i=1}^3 \sum_{j=1}^3 v_i v_j (\nabla\lambda_i, \nabla\psi_j)_K \simeq \sum_{i=1}^3 \sum_{j=1}^3 v_i v_j (\nabla_{\xi\eta}\hat{\lambda}_i, \nabla_{\xi\eta}\hat{\psi}_j)_{\hat{K}},$$

$$\|\mathcal{M}_K\sigma v_1\|_{0,K}^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} v_i v_j(\phi_i,\phi_j)_K = 2|K|\sum_{i=1}^{3}\sum_{j=1}^{3} v_i v_j(\hat{\phi}_i,\hat{\phi}_j)_{\hat{K}},$$

$$\|\nabla\mathcal{M}_K\sigma v_1\|_{0,K}^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} v_i v_j(\nabla\phi_i,\nabla\phi_j)_K \simeq \sum_{i=1}^{3}\sum_{j=1}^{3} v_i v_j(\nabla_{\xi\eta}\hat{\phi}_i,\nabla_{\xi\eta}\hat{\phi}_j)_{\hat{K}},$$

$$\|(\mathcal{I}-\sigma\mathcal{M}_K)v_1\|_{0,K}^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} v_i v_j(\lambda_i,\lambda_j)_K = 2|K|\sum_{i=1}^{3}\sum_{j=1}^{3} v_i v_j(\hat{\lambda}_i,\hat{\lambda}_j)_{\hat{K}},$$

and

$$\|\nabla((\mathcal{I}-\sigma\mathcal{M}_K)v_1)\|_{0,K}^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} v_i v_j(\nabla\lambda_i,\nabla\lambda_j)_K \simeq \sum_{i=1}^{3}\sum_{j=1}^{3} v_i v_j(\nabla_{\xi\eta}\hat{\lambda}_i,\nabla_{\xi\eta}\hat{\lambda}_j)_{\hat{K}},$$

where $\nabla_{\xi\eta}$ represents the gradient in terms of local coordinates $\xi$ and $\eta$. In addition, it is well known that

$$h_K^2\sum_{i=1}^{3} v_i^2 \simeq \|v_1\|_{0,K}^2 \quad \text{for all } K\in\mathcal{T}_h.$$

Since the matrices $(\hat{\lambda}_i,\hat{\psi}_j)_{1\le i,j\le 3}$, $(\nabla_{\xi\eta}\hat{\phi}_i,\nabla_{\xi\eta}\hat{\psi}_j)_{1\le i,j\le 3}$, $(\nabla_{\xi\eta}\hat{\lambda}_i,\nabla_{\xi\eta}\hat{\psi}_j)_{1\le i,j\le 3}$, $(\hat{\phi}_i,\hat{\phi}_j)_{1\le i,j\le 3}$, $(\nabla_{\xi\eta}\hat{\phi}_i,\nabla_{\xi\eta}\hat{\phi}_j)_{1\le i,j\le 3}$, $(\hat{\lambda}_i,\hat{\lambda}_j)_{1\le i,j\le 3}$, and $(\nabla_{\xi\eta}\hat{\lambda}_i,\nabla_{\xi\eta}\hat{\lambda}_j)_{1\le i,j\le 3}$ are symmetric, thus diagonalizable, we have that

$$C_\lambda^1\lambda_{min}^K\|v_1\|_{0,K}^2 \le \sum_{i=1}^{3}\sum_{j=1}^{3} v_i v_j(\lambda_i,\psi_j)_K \le C_\lambda^2\lambda_{max}^K\|v_1\|_{0,K}^2,$$

$$-C_\rho^1\rho_{min}^K h_K^{-2}\|v_1\|_{0,K}^2 \le \sum_{i=1}^{3}\sum_{j=1}^{3} v_i v_j(\nabla\phi_i,\nabla\psi_j)_K \le 0,$$

$$0 \le \sum_{i=1}^{3}\sum_{j=1}^{3} v_i v_j(\nabla\lambda_i,\nabla\psi_j)_K \le C_\rho^2\rho_{max}^K h_K^{-2}\|v_1\|_{0,K}^2,$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} v_i v_j(\phi_i,\phi_j)_K \le C_\theta\theta_{max}^K\|v_1\|_{0,K}^2,$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} v_i v_j(\nabla\phi_i,\nabla\phi_j)_K \le C_\xi\xi_{max}^K h_K^{-2}\|v_1\|_{0,K}^2,$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} v_i v_j(\lambda_i, \lambda_j)_K \leq C_\gamma \gamma_{max}^K \|v_1\|_{0,K}^2,$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} v_i v_j(\nabla\lambda_i, \nabla\lambda_j)_K \leq C_\zeta \zeta_{max}^K h_K^{-2} \|v_1\|_{0,K}^2,$$

where the positive constants $\lambda_{min}^K$ and $\lambda_{max}^K$ are the minimum and maximum eigenvalue of matrix $(\hat{\lambda}_i, \hat{\psi}_j)_{1\leq i,j\leq 3}$. The positive constants $\rho_{max}^K$, $\theta_{max}^K$ and $\xi_{max}^K$ are the maximum eigenvalues of the matrices $(\nabla_{\xi\eta}\hat{\lambda}_i, \nabla_{\xi\eta}\hat{\psi}_j)_{1\leq i,j\leq 3}$, $(\hat{\phi}_i, \hat{\phi}_j)_{1\leq i,j\leq 3}$, and $(\nabla_{\xi\eta}\hat{\phi}_i, \nabla_{\xi\eta}\hat{\phi}_j)_{1\leq i,j\leq 3}$, respectively. The negative constant $-\rho_{min}^K$ denote the minimum eigenvalue of matrix $(\nabla_{\xi\eta}\hat{\phi}_i, \nabla_{\xi\eta}\hat{\psi}_j)_{1\leq i,j\leq 3}$, and the positive constants $\gamma_{max}^K$ and $\zeta_{max}^K$ are the maximum eigenvalues of the matrices $(\hat{\lambda}_i, \hat{\lambda}_j)_{1\leq i,j\leq 3}$, and $(\nabla_{\xi\eta}\hat{\lambda}_i, \nabla_{\xi\eta}\hat{\lambda}_j)_{1\leq i,j\leq 3}$, respectively. Their expressions are presented in the Appendix. □

We are ready to prove the existence and uniqueness of solution for the problem (31). Consider the local $h$-dependent norm

$$\|v\|_{E,K} := \sqrt{C_K\alpha_K\|v\|_{0,K}^2 + h_K^2\|\nabla v\|_{0,K}^2} \quad \text{for all } v \in H^1(\mathcal{T}_h), \tag{33}$$

where $\alpha_K$ is the positive constant given by

$$\alpha_K = \frac{\sigma h_K^2}{C_K\varepsilon}\lambda_{min}^K - \frac{\rho_{min}^K}{6}, \tag{34}$$

and we define $\alpha = \min_{K\in\mathcal{T}_h}\alpha_K$. The positiveness of $\alpha_K$ follows from the definition of the eigenvalues $\rho_{min}^K$ and $\lambda_{min}^K$, and is illustrated in Figure 12. As usual the associate global norm is given by

$$\|v\|_E := \sqrt{\sum_{K\in\mathcal{T}_h}\|v\|_{E,K}^2} \quad \text{for all } v \in H^1(\mathcal{T}_h), \tag{35}$$

and we have the following coercivity result.

**Lemma 2.** Let $\|.\|_{E,K}$ be the norm defined by (33). Then, the bilinear form $a_e : V_h \times V_h \to \mathbb{R}$ is coercive and

$$a_e(v_1, v_1) \geq C\sum_{K\in\mathcal{T}_h}\frac{\varepsilon}{h_K^2}\|v_1\|_{E,K}^2 \quad \text{for all } v_1 \in V_h.$$

*Proof.* From the definition of bilinear form (32), from (34), applying the items $(i)$ and $(ii)$ of Lemma 1, and since $C_K = 6\frac{C_\rho^1}{C_\lambda^1}$ we obtain that

$$
\begin{aligned}
a_e(v_1, v_1) &\geq \sum_{K\in\mathcal{T}_h} \sigma C_\lambda^1 \lambda_{min}^K \|v_1\|_{0,K}^2 + \varepsilon\|\nabla v_1\|_{0,K}^2 - \varepsilon\left(\nabla(\mathcal{M}_K \sigma v_1), \nabla v_1\right)_K \\
&\geq \sum_{K\in\mathcal{T}_h} \sigma C_\lambda^1 \lambda_{min}^K \|v_1\|_{0,K}^2 + \varepsilon\|\nabla v_1\|_{0,K}^2 - C_\rho^1 \frac{\varepsilon}{h_K^2} \rho_{min}^K \|v_1\|_{0,K}^2 \\
&= \sum_{K\in\mathcal{T}_h} \frac{\varepsilon}{h_K^2} \left[ C_K \left( \frac{\sigma h_K^2}{C_K \varepsilon} C_\lambda^1 \lambda_{min}^K - \frac{C_\rho^1 \rho_{min}^K}{C_K} \right) \|v_1\|_{0,K}^2 + h_K^2 \|\nabla v_1\|_{0,K}^2 \right] \\
&= \sum_{K\in\mathcal{T}_h} \frac{\varepsilon}{h_K^2} \left[ C_\lambda^1 C_K \alpha_K \|v_1\|_{0,K}^2 + h_K^2 \|\nabla v_1\|_{0,K}^2 \right],
\end{aligned}
$$

and the result follows redefining the constants. □

*Remark* 2. Existence and uniqueness of solutions for problem (31) follows from Lax-Milgram Theorem. Let $u_e \in E_h$ be uniquely defined by $u_e = \mathcal{M}_K(f - \sigma u_1)$ in $K$, where $u_1$ is the unique solution of (31). Then, $u_e + u_1$ belongs to $U_h$ and satisfies (16).

*Remark* 3. The following limits will be useful in the sequel

$$
\lim_{\varepsilon\to 0} \alpha_K = \frac{3}{4} \quad \text{and} \quad \lim_{h_K\to 0} \frac{C_K \varepsilon}{\sigma h_K^2} \alpha_K = \frac{1}{48}. \tag{36}
$$

The behavior of coefficients and eigenvalues mentioned here are illustrated by the Figure 12 in the Appendix.

3.1. **Case $\varepsilon \to 0$.** We study the behavior of the convergence error in the case that $\varepsilon \ll 1$. In this case we shall use the asymptotic properties of the exact solution $u$. As $\varepsilon$ goes to zero the exact solution converges, at least away from the boundary, to $f\sigma^{-1}$. We shall estimate the related error in the norm (35), and also bound $u - f\sigma^{-1}$ in the same norm. The final result, i.e., the estimate for $u - u_1$, follows from the triangle inequality. We start by noting that [2]

$$
\left\| u - \frac{f}{\sigma} \right\|_0 \leq \frac{C}{\sigma} \left( \varepsilon^{1/4} \|f\|_{0,\partial\Omega} + \varepsilon^{1/2} \|f\|_1 \right), \quad \left\| \nabla \left( u - \frac{f}{\sigma} \right) \right\|_0 \leq \frac{C}{\sigma} \left( \varepsilon^{-1/4} \|f\|_{0,\partial\Omega} + \|f\|_1 \right).
$$

Thus by the norm definition (33), and since $\alpha_K < 1$, we obtain the following estimate

$$
\left\| u - \frac{f}{\sigma} \right\|_E \leq \frac{C}{\sigma} \left[ \left( h\varepsilon^{-1/4} + \varepsilon^{1/4} \right) \|f\|_{0,\partial\Omega} + \left( h + \varepsilon^{1/2} \right) \|f\|_1 \right]. \tag{37}
$$

The estimate (37) indicates that we have to refine the entire domain in order to control the error when $\varepsilon$ tends to zero. Such estimate seems pessimist, and indeed, we can improve it. Let us define $\overline{f} \in V_h^0$ such that $\overline{f} = f$ in $\Omega^0$. We have the following result.

**Lemma 3.** Let $u$ be the solution of (1). Then, there exit $C_1$, $C_2$, and $C_3$ such that

$$i)\quad \left\| u - \frac{f}{\sigma} \right\|_0^2 \leq \frac{C_1}{\sigma^2} \left( h_l \left\| f \right\|_{\infty,\partial\Omega}^2 + \varepsilon \sigma^{-1} \left\| \nabla \overline{f} \right\|_0^2 \right),$$

$$ii)\quad \left\| \nabla \left( u - \frac{f}{\sigma} \right) \right\|_0^2 \leq \frac{C_2}{\sigma^2} \left[ (h_l \varepsilon^{-1} \sigma + 1) \left\| f \right\|_{\infty,\partial\Omega}^2 + \left\| \nabla \overline{f} \right\|_0^2 \right],$$

$$iii)\quad \sum_{K \in \mathcal{T}_h} h_K^2 \left\| \nabla \left( u - \frac{f}{\sigma} \right) \right\|_{0,K}^2 \leq \frac{C_3}{\sigma^2} \left[ (h_l^3 \varepsilon^{-1} \sigma + h_l^2) \left\| f \right\|_{\infty,\partial\Omega}^2 + h^2 \left\| \nabla \overline{f} \right\|_0^2 \right].$$

*Proof.* Let $\overline{u}$ be the solution of the problem

$$\mathcal{L} \overline{u} = \overline{f} \quad \text{in } \Omega, \quad \overline{u} = 0 \quad \text{on } \partial\Omega, \tag{38}$$

then $e = u - \overline{u}$ satisfies

$$\mathcal{L} e = f - \overline{f} \quad \text{in } \Omega, \quad e = 0 \quad \text{on } \partial\Omega, \tag{39}$$

and it follows from (39) that

$$\varepsilon \left\| \nabla e \right\|_0^2 + \sigma \left\| e \right\|_0^2 \leq \frac{C}{\sigma} \left\| f - \overline{f} \right\|_0^2 = \frac{C}{\sigma} \left\| f - \overline{f} \right\|_{0,\Omega_{layer}}^2 \leq \frac{C}{\sigma} h_l \left\| f \right\|_{\infty,\partial\Omega}^2, \tag{40}$$

and from (38) that

$$\varepsilon \left\| \nabla \left( \overline{u} - \frac{\overline{f}}{\sigma} \right) \right\|_0^2 + \sigma \left\| \overline{u} - \frac{\overline{f}}{\sigma} \right\|_0^2 \leq C \frac{\varepsilon}{\sigma^2} \left\| \nabla \overline{f} \right\|_0^2. \tag{41}$$

Moreover,

$$\left\| \frac{f - \overline{f}}{\sigma} \right\|_0^2 \leq \frac{C}{\sigma^2} h_l \left\| f \right\|_{\infty,\partial\Omega}^2, \quad \left\| \nabla \left( \frac{f - \overline{f}}{\sigma} \right) \right\|_0^2 \leq \frac{C}{\sigma^2} h_l \left\| \nabla \left( f - \overline{f} \right) \right\|_{\infty,\partial\Omega}^2 \leq \frac{C}{\sigma^2} \left\| f \right\|_{\infty,\partial\Omega}^2, \tag{42}$$

hence, by using

$$\left\| u - \frac{f}{\sigma} \right\|_0^2 \leq C \left[ \left\| e \right\|_0^2 + \left\| \overline{u} - \frac{\overline{f}}{\sigma} \right\|_0^2 + \left\| \frac{f - \overline{f}}{\sigma} \right\|_0^2 \right],$$

$$\left\| \nabla \left( u - \frac{f}{\sigma} \right) \right\|_0^2 \leq C \left[ \left\| \nabla e \right\|_0^2 + \left\| \nabla \left( \overline{u} - \frac{\overline{f}}{\sigma} \right) \right\|_0^2 + \left\| \nabla \left( \frac{f - \overline{f}}{\sigma} \right) \right\|_0^2 \right],$$

the items $(i)$ and $(ii)$ follows from (40), (41), and (42).

From (39) we obtain that

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} h_K^2 \left\| \nabla e \right\|_{0,K}^2 &\leq C \sum_{K \in \mathcal{T}_h} h_K^2 \varepsilon^{-1} \sigma^{-1} \left\| f - \overline{f} \right\|_{0,K}^2 = C \sum_{K \in \Omega_{layer}} h_K^2 \varepsilon^{-1} \sigma^{-1} \left\| f - \overline{f} \right\|_{0,K}^2 \\ &\leq C h_l^3 \varepsilon^{-1} \sigma^{-1} \left\| f \right\|_{\infty,\partial\Omega}^2, \end{aligned} \tag{43}$$

and from (41) we have that

$$\sum_{K\in\mathcal{T}_h} h_K^2 \left\|\nabla\left(\overline{u}-\frac{\overline{f}}{\sigma}\right)\right\|_{0,K}^2 \leq Ch^2 \left\|\nabla\left(\overline{u}-\frac{\overline{f}}{\sigma}\right)\right\|_0^2 \leq \frac{C}{\sigma^2}h^2 \,\|\nabla\overline{f}\|_0^2. \tag{44}$$

Hence, the item $(iii)$ follows by the triangular inequality

$$\sum_{K\in\mathcal{T}_h} h_K^2 \left\|\nabla\left(u-\frac{f}{\sigma}\right)\right\|_{0,K}^2 \leq C\sum_{K\in\mathcal{T}_h} h_K^2 \left[\|\nabla e\|_{0,K}^2 + \left\|\nabla\left(\overline{u}-\frac{\overline{f}}{\sigma}\right)\right\|_{0,K}^2 + \left\|\nabla\left(\frac{f-\overline{f}}{\sigma}\right)\right\|_{0,K}^2\right],$$

and using (42), (43), and (44). □

**Corollary 4.** Let $u$ be the solution of (1). Then, there exits constant $C$ such that

$$\left\|u-\frac{f}{\sigma}\right\|_E \leq \frac{C}{\sigma}\left[h_l^{1/2}\left(h_l\varepsilon^{-1/2}\sigma^{1/2}+1\right)\|f\|_{\infty,\partial\Omega} + \left(h+\varepsilon^{1/2}\sigma^{-1/2}\right)\|\nabla\overline{f}\|_0\right].$$

*Proof.* The result follows by the norm definition (33), since $\alpha_K < 1$ for all $K\in\mathcal{T}_h$, and from Lemma 3. □

We have the following estimate.

**Lemma 5.** Let $u_1$ be the solution of (31). There exists $C$ such that

$$\left\|\frac{f}{\sigma}-u_1\right\|_E \leq \frac{C}{\sigma}\left[h_l\left(h_l^{1/2}\varepsilon^{-1/2}\sigma^{1/2}+1\right)\left(1+\alpha^{-1/2}\right)\|f\|_{\infty,\partial\Omega} + h_l\|\nabla\overline{f}\|_0 + h\alpha^{-1/2}\,\|\nabla f\|_0\right].$$

*Proof.* Applying Lemma 2 we have that

$$C\sum_{K\in\mathcal{T}_h}\frac{\varepsilon}{h_K^2}\|\frac{f}{\sigma}-u_1\|_{E,K}^2 \leq a_e\left(\frac{f}{\sigma}-u_1,\frac{f}{\sigma}-u_1\right) = a_e\left(\frac{f}{\sigma}-u,\frac{f}{\sigma}-u_1\right)+a_e\left(u-u_1,\frac{f}{\sigma}-u_1\right)$$

$$= a\left(\frac{f}{\sigma}-u,\frac{f}{\sigma}-u_1\right) - a_h\left(\mathcal{M}_K\mathcal{L}\left(\frac{f}{\sigma}-u\right),\frac{f}{\sigma}-u_1\right) + a_e\left(u-u_1,\frac{f}{\sigma}-u_1\right). \tag{45}$$

Hence, we have to estimate such terms. Since $f$ is piecewise linear, the second term on the right hand side vanishes. Moreover, from the definition (3) and applying the Cauchy–Schwartz and the inverse inequalities, we have that the first term on the right hand side is bounded as

$$a\left(\frac{f}{\sigma}-u,\frac{f}{\sigma}-u_1\right) = \sum_{K\in\mathcal{T}_h}\frac{\varepsilon}{\sigma}\left(\nabla f,\nabla\left(\frac{f}{\sigma}-u_1\right)\right)_K \leq \sum_{K\in\mathcal{T}_h}\frac{\varepsilon}{\sigma}\|\nabla f\|_{0,K}\left\|\nabla\left(\frac{f}{\sigma}-u_1\right)\right\|_{0,K}$$

$$\leq C\sum_{K\in\mathcal{T}_h}\frac{\varepsilon}{\sigma h_K}\|\nabla f\|_{0,K}\left\|\frac{f}{\sigma}-u_1\right\|_{0,K} \leq C\sum_{K\in\mathcal{T}_h}\left(\frac{4\varepsilon}{\sigma^2 C_K\alpha_K}\|\nabla f\|_{0,K}^2 + \frac{\varepsilon C_K\alpha_K}{4h_K^2}\left\|\frac{f}{\sigma}-u_1\right\|_{0,K}^2\right)$$

$$\leq C\sum_{K\in\mathcal{T}_h}\left(\frac{4\varepsilon}{C_0\sigma^2 C_K\alpha_K}\|\nabla f\|_{0,K}^2 + \frac{C_0\varepsilon}{4h_K^2}\left\|\frac{f}{\sigma}-u_1\right\|_{E,K}^2\right), \tag{46}$$

where $C_0 < 1$. It remains to estimate the third term on right hand side of (45). Clearly, as long as $f$ vanishes on $\partial\Omega$, by the consistency of (31), the third term also vanishes and the result follows. Now, suppose that $f$ is nonzero on $\partial\Omega$. Therefore, again by the consistency of (31), we have that

$$a_e\left(u-u_1,\frac{f}{\sigma}\right)=a_e\left(u-u_1,\frac{f-\overline{f}}{\sigma}\right). \tag{47}$$

By using (47), (3), (15), the consistency of (31), and the Cauchy–Schwartz inequality we obtain that

$$\begin{aligned}
a_e\left(u-u_1,\frac{f}{\sigma}\right) &= \sum_{K\in\Omega_{layer}}\frac{\varepsilon}{\sigma}\left(\nabla\left(\mathcal{I}-\mathcal{M}_K\mathcal{L}\right)(u-u_1),\nabla\left(f-\overline{f}\right)\right)_K+\left(\left(\mathcal{I}-\mathcal{M}_K\mathcal{L}\right)(u-u_1),f-\overline{f}\right)_K\\
&= \sum_{K\in\Omega_{layer}}\frac{\varepsilon}{\sigma}\left[\left(\nabla\left(\mathcal{I}-\mathcal{M}_K\mathcal{L}\right)(u-\frac{f}{\sigma}),\nabla\left(f-\overline{f}\right)\right)_K+\left(\nabla\left(\mathcal{I}-\mathcal{M}_K\mathcal{L}\right)(\frac{f}{\sigma}-u_1),\nabla\left(f-\overline{f}\right)\right)_K\right]\\
&\quad+\left[\left(\left(\mathcal{I}-\mathcal{M}_K\mathcal{L}\right)(u-\frac{f}{\sigma}),f-\overline{f}\right)_K+\left(\left(\mathcal{I}-\mathcal{M}_K\mathcal{L}\right)(\frac{f}{\sigma}-u_1),f-\overline{f}\right)_K\right]\\
&= \sum_{K\in\Omega_{layer}}\frac{\varepsilon}{\sigma}\left[\left(\nabla\left(u-\frac{f}{\sigma}\right),\nabla(f-\overline{f})\right)_K+\left(\nabla\left(\frac{f}{\sigma}-u_h\right),\nabla\left(f-\overline{f}\right)\right)_K\right]\\
&\quad+\left[\left(u-\frac{f}{\sigma},f-\overline{f}\right)_K+\left(\frac{f}{\sigma}-u_h,f-\overline{f}\right)_K\right]\\
&\le C\sum_{K\in\Omega_{layer}}\left[\frac{\varepsilon}{\sigma}\left(\left\|\nabla\left(u-\frac{f}{\sigma}\right)\right\|_{0,K}+\left\|\nabla\left(\frac{f}{\sigma}-u_h\right)\right\|_{0,K}\right)h_K\left\|\nabla(f-\overline{f})\right\|_{\infty,K}\right.\\
&\quad\left.+\left(\left\|u-\frac{f}{\sigma}\right\|_{0,K}+\left\|\frac{f}{\sigma}-u_h\right\|_{0,K}\right)h_K\left\|f-\overline{f}\right\|_{\infty,K}\right]
\end{aligned}$$

$$\begin{aligned}
&\le C\sum_{K\in\Omega_{layer}}\left[\frac{\varepsilon\gamma_1}{\sigma}\left\|\nabla\left(u-\frac{f}{\sigma}\right)\right\|_{0,K}^2+\frac{\varepsilon\gamma_1}{\sigma}\left\|\nabla\left(\frac{f}{\sigma}-u_h\right)\right\|_{0,K}^2+\frac{\varepsilon h_K^2}{\sigma\gamma_1}\left\|\nabla(f-\overline{f})\right\|_{\infty,K}^2\right.\\
&\quad\left.\gamma_2\left\|u-\frac{f}{\sigma}\right\|_{0,K}^2+\gamma_2\left\|\frac{f}{\sigma}-u_h\right\|_{0,K}^2+\frac{h_K^2}{\gamma_2}\left\|f-\overline{f}\right\|_{\infty,K}^2\right]
\end{aligned}$$

where $\gamma_1$ and $\gamma_2$ are positive constants. It turns out from items $(vi)$ and $(vii)$ of Lemma 1 that

$$\begin{aligned}
\left\|\frac{f}{\sigma}-u_h\right\|_{0,K}^2 &= \left\|(\mathcal{I}-\sigma\mathcal{M}_K)\left(\frac{f}{\sigma}-u_1\right)\right\|_{0,K}^2 \le C_\gamma \gamma_{max}^K \left\|\frac{f}{\sigma}-u_1\right\|_{0,K}^2 \\
&\le \frac{C_\gamma}{2}\frac{C_K\varepsilon}{\sigma h_K^2}\left\|\frac{f}{\sigma}-u_1\right\|_{0,K}^2, \qquad (48)\\
\left\|\nabla\left(\frac{f}{\sigma}-u_h\right)\right\|_{0,K}^2 &= \left\|\nabla\left((\mathcal{I}-\sigma\mathcal{M}_K)\left(\frac{f}{\sigma}-u_1\right)\right)\right\|_{0,K}^2 \le \frac{C_\zeta}{h_K^2}\zeta_{max}^K \left\|\frac{f}{\sigma}-u_1\right\|_{0,K}^2 \\
&\le \frac{C_\zeta}{4h_K^2}\left\|\frac{f}{\sigma}-u_1\right\|_{0,K}^2, \qquad (49)
\end{aligned}$$

where we have used that

$$\gamma_{max}^K \le \frac{1}{2}\frac{C_K\varepsilon}{\sigma h_K^2}, \quad \zeta_{max}^K \le \frac{1}{4},$$

for all $K\in\mathcal{T}_h$. The behavior of such eigenvalues is illustrated by the Figures 12 and 13 in the Appendix. Hence, based on (48) and (49), we set $\gamma_1$ and $\gamma_2$ as

$$\gamma_1 = \frac{C_K C_0 \alpha_K \sigma}{2C_\zeta}, \quad \gamma_2 = \frac{C_0\alpha_K\sigma}{4C_\gamma}, \qquad (50)$$

and using (48) and (49) we have that

$$\begin{aligned}
a_e\left(u-u_1,\frac{f}{\sigma}\right) &\le C\sum_{K\in\Omega_{layer}}\left[\frac{C_K\alpha_K\varepsilon}{2C_\zeta}\left\|\nabla\left(u-\frac{f}{\sigma}\right)\right\|_{0,K}^2 + \frac{\varepsilon}{8h_K^2}\|\frac{f}{\sigma}-u_1\|_{E,K}^2\right. \qquad (51)\\
&\left.+\frac{2C_\zeta h_K^2\varepsilon}{\alpha_K\sigma^2}\|\nabla(f-\overline{f})\|_{\infty,K}^2 + \frac{C_K\alpha_K\sigma}{4C_\gamma}\left\|u-\frac{f}{\sigma}\right\|_{0,K}^2 + \frac{\varepsilon}{8h_K^2}\|\frac{f}{\sigma}-u_1\|_{E,K}^2 + \frac{4C_\gamma h_K^2}{\alpha_K\sigma}\|f-\overline{f}\|_{\infty,K}^2\right]\\
&\le C\sum_{K\in\Omega_{layer}}\left[\alpha_K\varepsilon\left\|\nabla\left(u-\frac{f}{\sigma}\right)\right\|_{0,K}^2 + \alpha_K\sigma\left\|u-\frac{f}{\sigma}\right\|_{0,K}^2 + \frac{\varepsilon}{4h_K^2}\|\frac{f}{\sigma}-u_1\|_{E,K}^2\right.\\
&\left.+\frac{h_K^2}{\alpha_K\sigma}\left(\frac{\varepsilon}{\sigma}h_l^{-1}\|f\|_{\infty,\partial K}^2+\|f\|_{\infty,\partial K}^2\right)\right].
\end{aligned}$$

Now, choosing $C_0$ properly, adding (46) and (51), and reordering the terms, and since $\alpha_K \leq 1$ for all $K \in \mathcal{T}_h$, we obtain from (45) that

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \|\frac{f}{\sigma} - u_1\|^2_{E,K} \leq C \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{\sigma^2\alpha} \|\nabla f\|_0^2 + C \sum_{K\in\Omega_{layer}} \Bigg[ \alpha_K h_K^2 \left\| \nabla \left( u - \frac{f}{\sigma} \right) \right\|^2_{0,K} \\ + \frac{\alpha_K \sigma h_K^2}{\varepsilon} \left\| u - \frac{f}{\sigma} \right\|^2_{0,K} + \frac{h_K^4}{\alpha_K \sigma \varepsilon} \left( \frac{\varepsilon}{\sigma} \|\nabla \left( f - \overline{f} \right)\|^2_{\infty,K} + \|f - \overline{f}\|^2_{\infty,K} \right) \Bigg] \\ \leq C \Bigg[ \frac{h^2}{\sigma^2\alpha} \|\nabla f\|_0^2 + h_l^2 \left\| \nabla \left( u - \frac{f}{\sigma} \right) \right\|^2_{0,\Omega_{layer}} + \frac{h_l^2 \sigma}{\varepsilon} \left\| u - \frac{f}{\sigma} \right\|^2_{0,\Omega_{layer}} \\ + \frac{h_l^2}{\alpha\sigma\varepsilon} \left( \frac{\varepsilon}{\sigma} + h_l \right) \|f\|^2_{\infty,\partial\Omega} \Bigg],\end{aligned}$$

the result follows using Lemma 3, and redefining the constants. □

We are ready to present the main convergence result.

**Theorem 6.** *Let $u$ be the solution of (2) and $u_1$ be the solution of (31). There exists $C$ such that*

$$\begin{aligned}\|u - u_1\|_E \leq \frac{C}{\sigma} \Big\{ h_l^{1/2} \left[ h_l^{1/2} \left( h_l^{1/2} \varepsilon^{-1/2} \sigma^{1/2} + 1 \right) \left( 1 + \alpha^{-1/2} \right) + 1 \right] \|f\|_{\infty,\partial\Omega} \\ + h\alpha^{-1/2} \|\nabla f\|_0 + \left( h + \varepsilon^{1/2}\sigma^{-1/2} \right) \left\| \nabla \overline{f} \right\|_0 \Big\}.\end{aligned}$$

*Proof.* The result follows using triangle inequality, Corollary 4, Lemma 5, and redefining the constants. □

*Remark* 4. The convergence result presented in Theorem 6 points out that the error depends on the form of $f$, and we can identify the following behavior:

*i)* supposing that $f$ vanishes on $\partial\Omega$, then

$$\lim_{\varepsilon\to 0} \|u - u_1\|_E \leq C \frac{h}{\sigma} \|\nabla f\|_0, \tag{52}$$

since $\alpha \to 3/4$ when $\varepsilon \to 0$. Moreover, if $f$ is supposed to be constant or linear in $\Omega_0$ and $h_l \simeq \varepsilon^p$ with $p \in (0, 1/2]$, thus we have convergence, i.e.,

$$\lim_{\varepsilon\to 0} \|u - u_1\|_E = 0. \tag{53}$$

*ii)* if $f$ is nonzero on $\partial\Omega$ and $h_l \simeq \varepsilon^p$ with $p \in (1/3, 1/2]$, then

$$\lim_{\varepsilon\to 0} \|u - u_1\|_E \leq C \frac{h}{\sigma} \left( \|\nabla f\|_0 + \left\| \nabla \overline{f} \right\|_0 \right), \tag{54}$$

since $\alpha$ is bounded. As long as $f$ is constant or linear in $\Omega$ we recover the convergence (53).

The convergence results presented above are also valid for the $L^2$ norm. We stress that the assumption $C_1\varepsilon^{1/2} \leq h_l \leq C_2\varepsilon^{1/3}$ used to obtain uniform convergence is not too strong. Indeed, the mesh refinement is concentrated along the boundary. Moreover, we note that if we consider $\varepsilon$ of order $10^{-6}$ for example, what corresponds a strong boundary layer, we just need to chose a first range of element with characteristic length $h_l \in [C_1\, 10^{-3}, C_2\, 10^{-2}]$. This numerical aspect is shown in Section 4. Similar numerical results are obtained using the formulation (27), and that indicates we have equivalent convergence estimates. Such analysis is out of the scope of this work.

Numerical validations point out that uniform convergence is recovered in the interior domain $\Omega^0$ without any boundary refinement as long as $f$ is constant. Such aspect was not analyzed in this work.

3.2. **Case $h \to 0$.** In this subsection we perform a convergence analysis with respect to $h$. In what follows, we consider that the positive constant $C$ is independent of $h$ but might depend on $\varepsilon$ and $\sigma$. First, recall that we denote by $u_g$ the solution of the Galerkin formulation (5). Hence, it is well known (see [9] for instance) that there exists constant $C$ such that

$$\sigma \left\| u - u_g \right\|_0^2 + \varepsilon \left\| \nabla (u - u_g) \right\|_0^2 \leq C h^2 \left\| u \right\|_2^2 . \tag{55}$$

Our goal consists on estimating the Galerkin error in the norm (35). This is done in the following lemma.

**Lemma 7.** Let $u$ be the solution of (2) and $u_g$ be the solution of (5). There exists a constant $C$ such that

$$\sum_{K \in \mathcal{T}_h} \frac{\varepsilon}{h_K^2} \left\| u - u_g \right\|_{E,K}^2 \leq C h^2 \left\| u \right\|_2^2 . \tag{56}$$

*Proof.* From the norm definition (33) we obtain that

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \frac{\varepsilon}{h_K^2} \left\| u - u_g \right\|_{E,K}^2 &= \sum_{K \in \mathcal{T}_h} \left( \frac{\varepsilon C_K}{\sigma h_K^2} \alpha_K \sigma \left\| u - u_g \right\|_{0,K}^2 + \varepsilon \left\| \nabla (u - u_g) \right\|_{0,K}^2 \right) \\ &\leq C \left( \sigma \left\| u - u_g \right\|_0^2 + \varepsilon \left\| \nabla (u - u_g) \right\|_0^2 \right) \end{aligned}$$

since $\varepsilon C_K \alpha_K \sigma^{-1} h_K^{-2}$ is bounded for all $K \in \mathcal{T}_h$, and the result follows using (55). □

**Lemma 8.** Let $u_g$ be the solution of (5) and $u_1$ be the solution of (31). There exist a constant $C$ such that

$$\sum_{K \in \mathcal{T}_h} \frac{\varepsilon}{h_K^2} \left\| u_g - u_1 \right\|_{E,K}^2 \leq C h^2 \left\| u \right\|_2^2 . \tag{57}$$

*Proof.* From Lemma 2 and the consistency of (31) we have that

$$C\sum_{K\in\mathcal{T}_h}\frac{\varepsilon}{h_K^2}\left\|u_g-u_1\right\|_{E,K}^2\le a_e\left(u_g-u_1,u_g-u_1\right)=a_e\left(u_g-u,u_g-u_1\right)$$
$$=a\left(u_g-u,u_g-u_1\right)-a_h\left(\mathcal{M}_K\mathcal{L}\left(u_g-u\right),u_g-u_1\right)$$
$$\le\left|a_h\left(\mathcal{M}_K\mathcal{L}\left(u_g-u\right),u_g-u_1\right)\right|$$

since the Galerkin method is also consistent. Hence, it follows from the Cauchy–Schwartz inequality that

$$C\sum_{K\in\mathcal{T}_h}\frac{\varepsilon}{h_K^2}\left\|u_g-u_1\right\|_{E,K}^2\le\sum_{K\in\mathcal{T}_h}\left(\varepsilon\|\nabla\left(\mathcal{M}_K\mathcal{L}\left(u_g-u\right)\right)\|_{0,K}\|\nabla\left(u_g-u_1\right)\|_{0,K}\right.$$
$$\left.+\sigma\|\mathcal{M}_K\mathcal{L}\left(u_g-u\right)\|_{0,K}\|u_g-u_1\|_{0,K}\right)$$
$$\le\sum_{K\in\mathcal{T}_h}\left(\frac{2\varepsilon}{C}\|\nabla\left(\mathcal{M}_K\mathcal{L}\left(u_g-u\right)\right)\|_{0,K}^2+\frac{2h_K^2\sigma^2}{\varepsilon CC_K\alpha_K}\|\mathcal{M}_K\mathcal{L}\left(u_g-u\right)\|_{0,K}^2+\frac{C\varepsilon}{2h_K^2}\|u_g-u_1\|_{E,K}^2\right).$$

Then using Lemma 1, items $(iv)$, $(v)$, we obtain that

$$\sum_{K\in\mathcal{T}_h}\frac{\varepsilon}{h_K^2}\|u_g-u_1\|_{E,K}^2\le C\sum_{K\in\mathcal{T}_h}\left(\frac{\varepsilon}{\sigma^2h_K^2}\xi_{max}^K+\frac{h_K^2\theta_{max}^K}{\varepsilon\alpha_K}\right)\|\mathcal{L}\left(u_g-u\right)\|_{0,K}^2$$
$$\le C\sum_{K\in\mathcal{T}_h}\frac{h_K^2}{\varepsilon}\left(\left(\frac{\varepsilon}{\sigma h_K^2}\right)^2\xi_{max}^K+\frac{\theta_{max}^K}{\alpha_K}\right)\left(\|\sigma\left(u_g-u\right)\|_{0,K}^2+\|\varepsilon\Delta u\|_{0,K}^2\right)$$
$$\le C\sum_{K\in\mathcal{T}_h}\frac{h_K^2}{\varepsilon}\left(\|\sigma\left(u_g-u\right)\|_{0,K}^2+\|\varepsilon\Delta u\|_{0,K}^2\right)\le Ch^2\left(\frac{\sigma}{\varepsilon}h^2+\varepsilon\right)\|u\|_2^2,$$

where we have used that $\theta_{max}^K\alpha_K^{-1}$ and $(\varepsilon\sigma^{-1}h_K^{-2})^2\xi_{max}^K$ are bounded, and (55). Such behavior is illustrated by Figures 12 and 14 in the Appendix. The result follows by redefining the constants. □

**Theorem 9.** *Let $u$ be the solution of (2) and $u_1$ be the solution of (31). There exist $C$ such that*

$$\left(\sum_{K\in\mathcal{T}_h}\frac{\varepsilon}{h_K^2}\left\|u-u_1\right\|_{E,K}^2\right)^{1/2}\le Ch\|u\|_2. \tag{58}$$

*Proof.* Using triangle inequality, and from Lemmas 7 and 8 we have that

$$\sum_{K\in\mathcal{T}_h}\frac{\varepsilon}{h_K^2}\left\|u-u_1\right\|_{E,K}^2\le\sum_{K\in\mathcal{T}_h}\frac{\varepsilon}{h_K^2}\left\|u-u_g\right\|_{E,K}^2+\sum_{K\in\mathcal{T}_h}\frac{\varepsilon}{h_K^2}\left\|u_g-u_1\right\|_{E,K}^2\le Ch^2\left\|u\right\|_2^2.$$

□

*Remark* 5. The convergence result (58) is equivalent to the standard Galerkin error in the energy norm (55). The asymptotic behavior of the norm's coefficient is presented in (36).

## 4. Numerical Results

### 4.1. **Source problem.**

Let us first consider the unit source problem ($f = 1/2$) defined on the unit square, and subject to the boundary conditions described in Figure 3. We use the unstructured mesh shown in Figure 4.

For a fixed $\sigma = 1$ and small $\varepsilon$, boundary layers appear close to the domain boundary. Figures 5, 6 show the solutions of the Galerkin and the multiscale methods, for $\varepsilon = 10^{-6}$. As predicted, the present method perform better than the Galerkin method. For $\varepsilon = 1$, all methods have comparable performance, see Figure 7.

Next, we take $f$ piecewise linear, $f(x, y) = x$ for $0 \leq x \leq 0.5$ and $f(x, y) = 1-x$ otherwise. Again, the multiscale method perform better than the Galerkin method. We remark that the solution obtained from the enriched formulation (26) is more diffusive than the one obtained from the modified enriched formulation (27) as shown in Figure 10 and Figure 11.

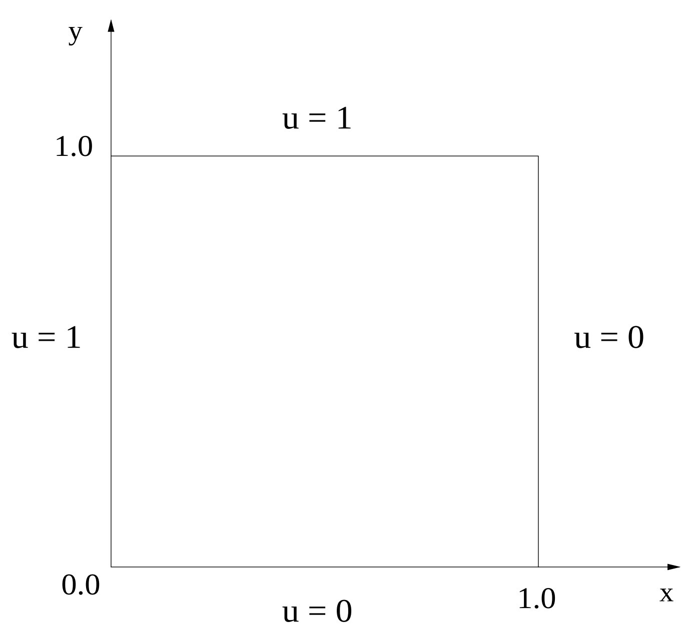


FIGURE 3. Problem statement.

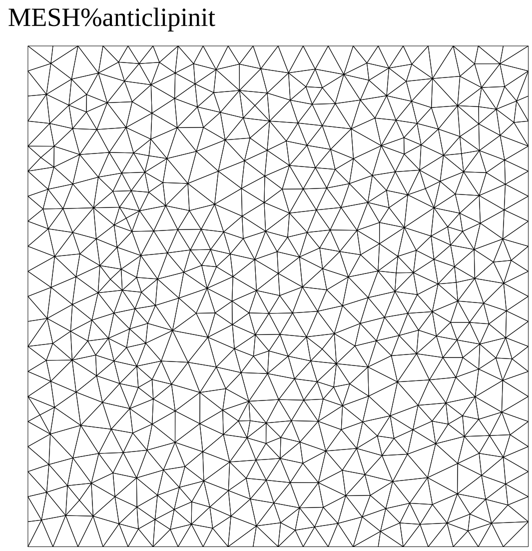


FIGURE 4. The unstructured mesh.

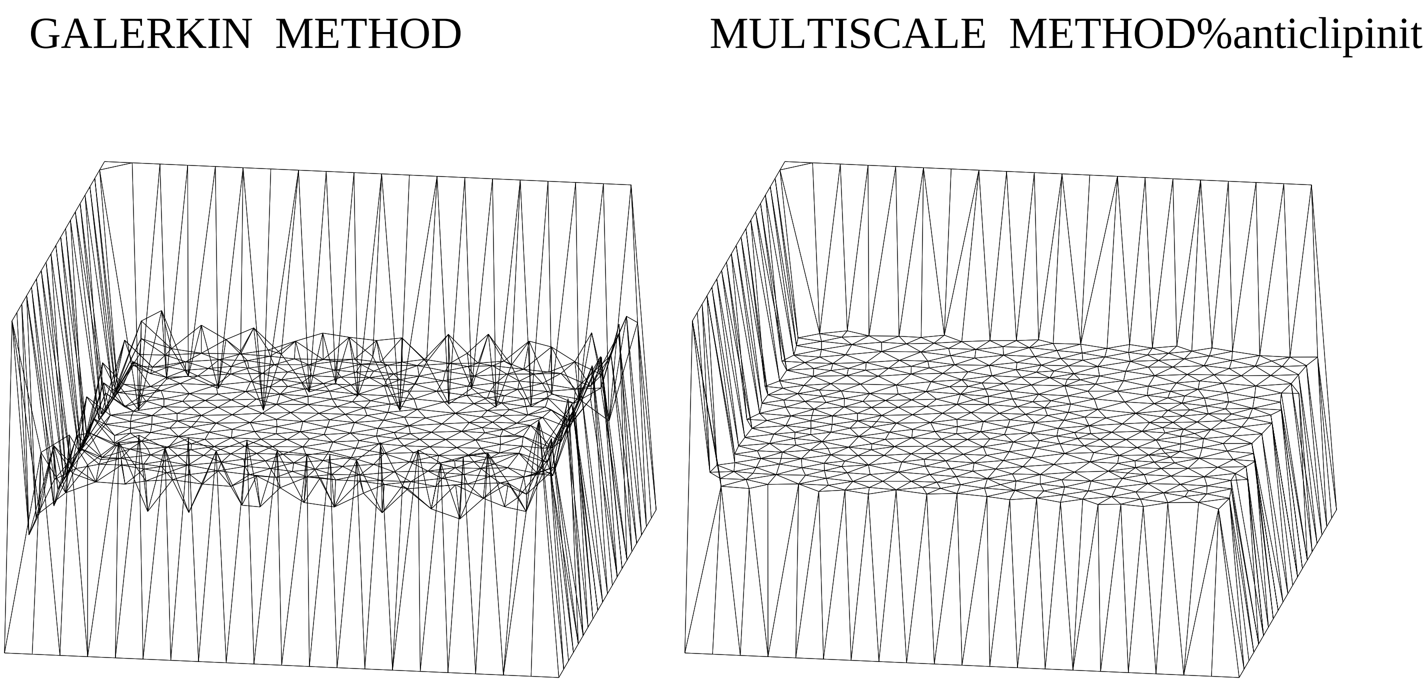


FIGURE 5. Comparison between Galerkin and the multiscale methods for $\varepsilon = 10^{-6}$.

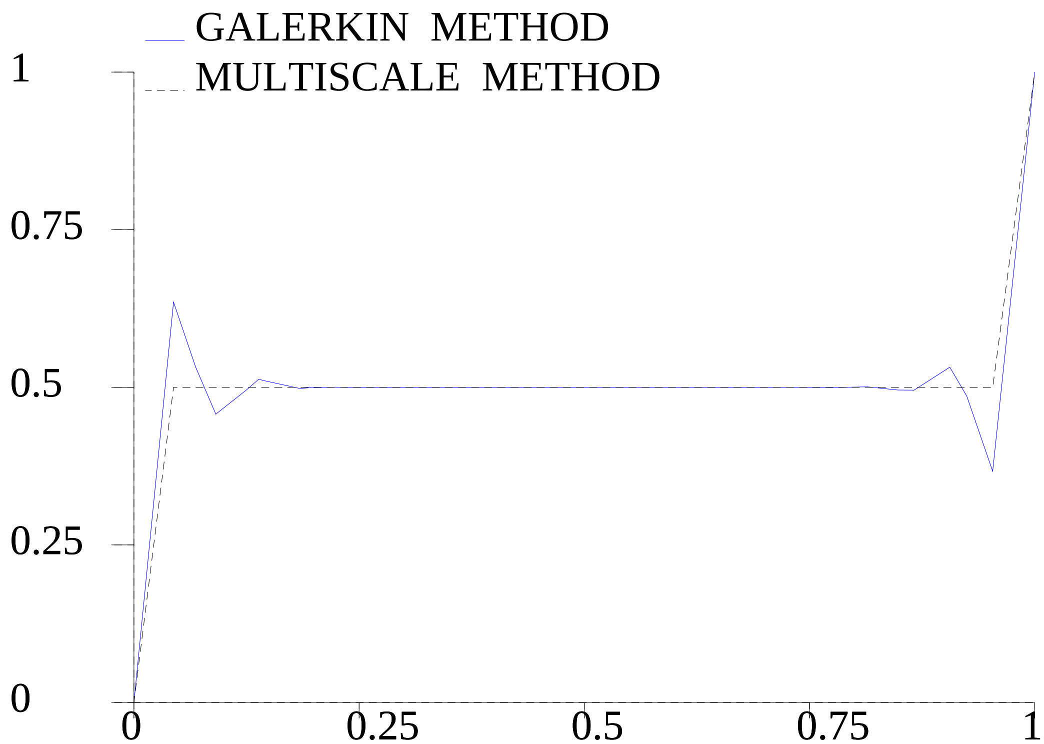


FIGURE 6. Profile of solutions at $x = 0.5$ ($\varepsilon = 10^{-6}$).

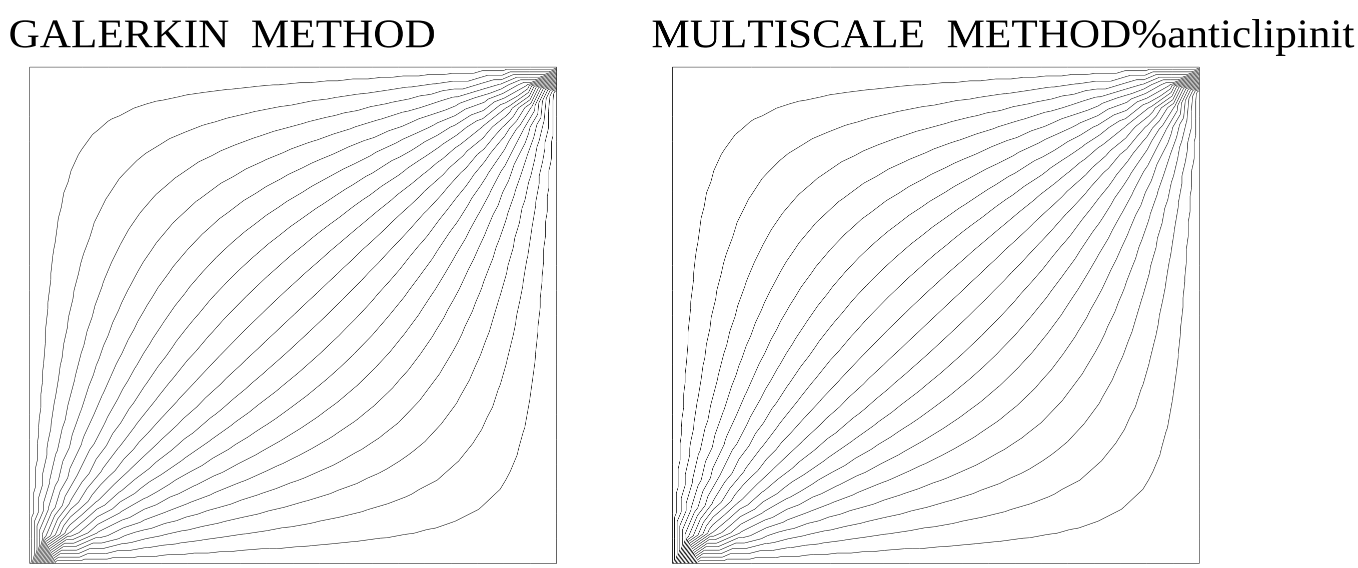


FIGURE 7. Isovalues of solutions obtained with Galerkin and the multiscale methods ($\varepsilon = 1$).

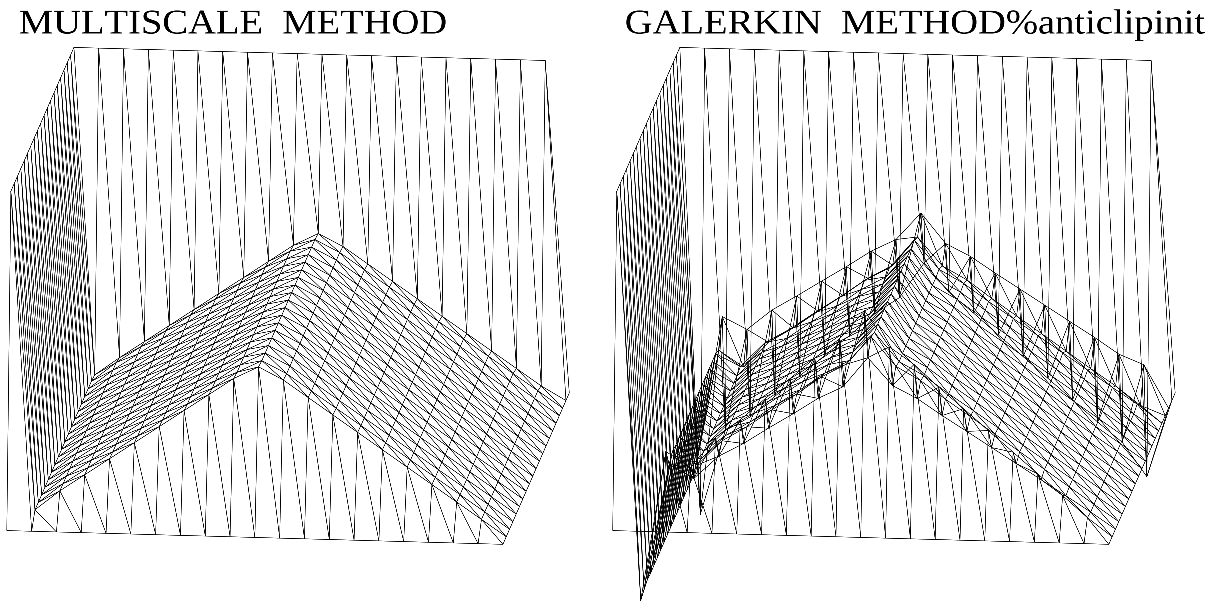


FIGURE 8. Solutions for piecewise linear $f$ with modified multiscale, multiscale and Galerkin methods ($\varepsilon = 10^{-6}$).

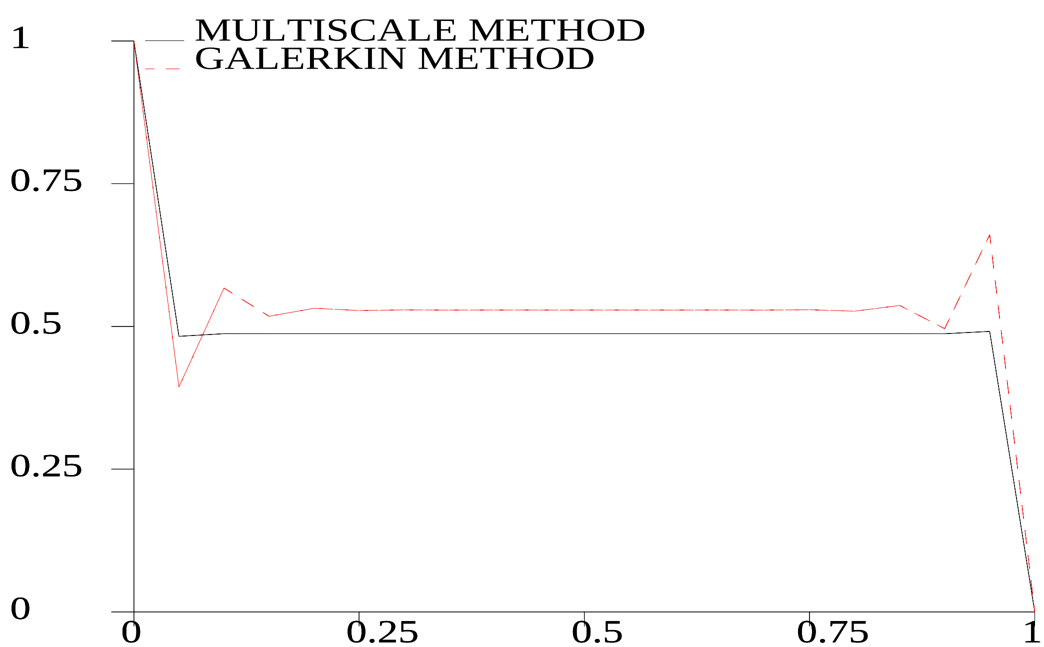


FIGURE 9. Profile of solutions at $x = 0.5$ ($\varepsilon = 10^{-6}$).

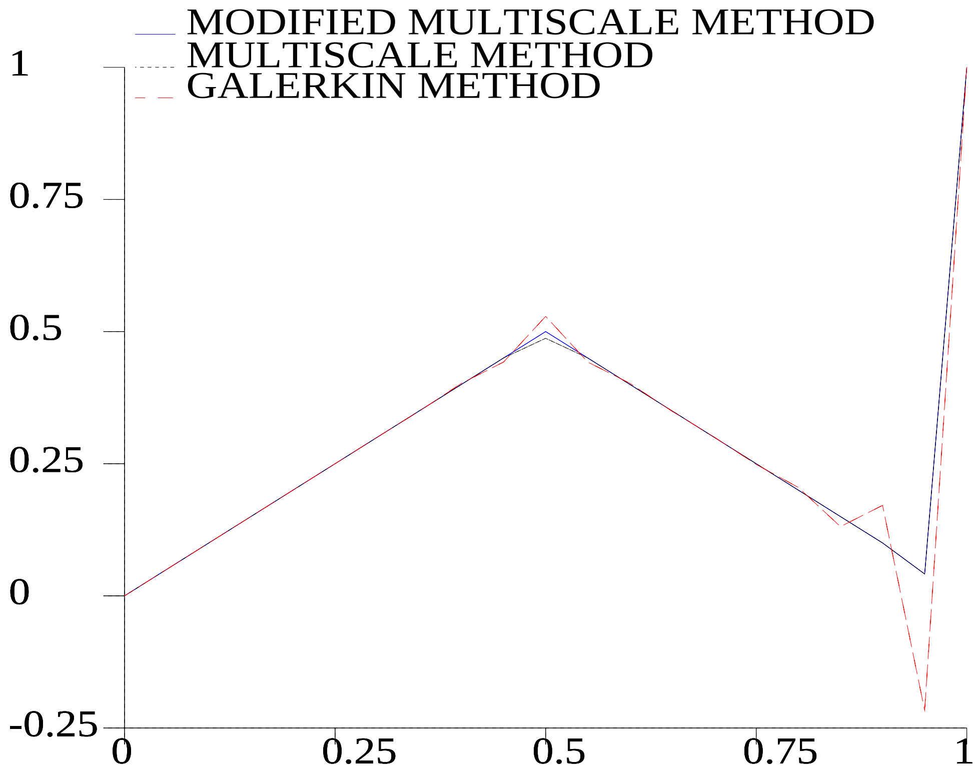


FIGURE 10. Profile of solutions at $y = 0.5$ ($\varepsilon = 10^{-6}$).

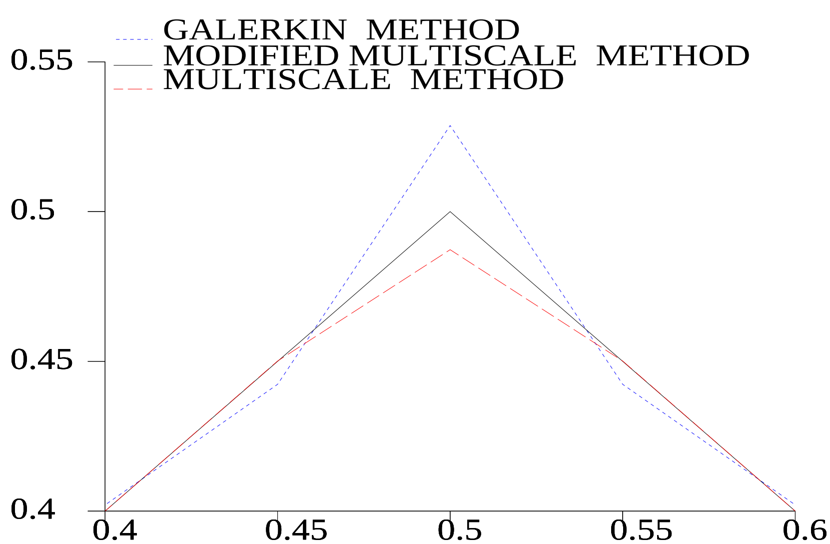


FIGURE 11. Zoom of profile of solutions with Galerkin, modified multiscale and multiscale methods at $y = 0.5$ ($\varepsilon = 10^{-6}$).

## 5. Appendix

We present in this section the expression of the eigenvalues introduced in Lemma 1, and we show graphically the behavior of some coefficients and eigenvalues. To simplify the formulas, we introduce $\beta_K$ defined by

$$\beta_K = \sqrt{\frac{\sigma h_K^2}{C_K \varepsilon}}.$$

The expression of eigenvalues are given by

$$\lambda_{min}^K = \frac{1}{\beta_K^2}\left(1 + \frac{3}{\beta_K \sinh\beta_K} - \frac{3\cosh\beta_K}{\beta_K \sinh\beta_K} + \frac{\beta_K}{2\sinh\beta_K}\right),$$
$$\lambda_{max}^K = \frac{1}{\beta_K^2}\left(1 - \frac{\beta_K}{\sinh\beta_K}\right),$$
$$-\rho_{min}^K = -3/2\left(1 + \frac{2}{\beta_K \sinh\beta_K} - \frac{2\cosh\beta_K}{\beta_K \sinh\beta_K}\right),$$
$$\rho_{max}^K = \frac{3}{\beta_K}\left(\frac{\cosh\beta_K}{\sinh\beta_K} - \frac{1}{\sinh\beta_K}\right),$$
$$\gamma_{max}^K = \frac{1}{4\beta_K^2 \sinh(\beta_K)^2}\left(\cosh(\beta_K)^2 - 8\cosh(\beta_K) - \beta_K^2 + 4\beta\sinh(\beta_K) + 7\right),$$
$$\zeta_{max}^K = \frac{1}{8\sinh(\beta_K)^2}\left(2\cosh(\beta_K)^2 - 2\cosh(\beta_K)\sinh(\beta_K) - 1\right)$$
$$\Big(6\cosh(\beta_K)^4 + 6\cosh(\beta_K)^3\sinh(\beta_K) - 9\cosh(\beta_K)^2 - 6\cosh(\beta_K)\sinh(\beta_K) + 3$$
$$+6\,{\beta_K}^2\cosh(\beta_K)^2 + 6{\beta_K}^2\sinh(\beta_K)\cosh(\beta_K) - 3{\beta_K}^2$$
$$-\Big(\big(\cosh(\beta_K)^4 - 2\cosh(\beta_K)^2 + 34{\beta_K}^2\cosh(\beta_K)^2 + 1 - 34\,{\beta_K}^2 + {\beta_K}^4\big)$$
$$\big(1 + 8\cosh(\beta_K)^4 - 8\cosh(\beta_K)^2 + 8\cosh(\beta_K)^3\sinh(\beta_K) - 4\cosh(\beta_K)\sinh(\beta_K)\big)\Big)^{1/2}\Big),$$
$$\theta_{max}^K = \frac{3}{\beta_K \sinh\beta_K}\left(1 + \frac{-21\,(\cosh\beta_K)^2 - 24\cosh\beta_K - 5\,{\beta_K}^2 + 45 + 2\,{\beta_K}^2(\cosh\beta_K)^2}{36\sinh\beta_K\beta_K}\right),$$
$$\xi_{max}^K = \frac{1}{8\,(\beta_K\sinh\beta_K)^2}\left({\beta_K}^4 F(\beta_K) + {\beta_K}^2 G(\beta_K) + \beta_K H(\beta_K)\right),$$

where the functions $F$, $G$ and $H$ are given by an intricate nonlinear combination of $\sinh\beta_K$ and $\cosh\beta_K$. Instead of presenting such expressions here, we simply plot $\frac{\xi_{max}^K}{\beta_K^4}$ with respect to $\beta_K$.

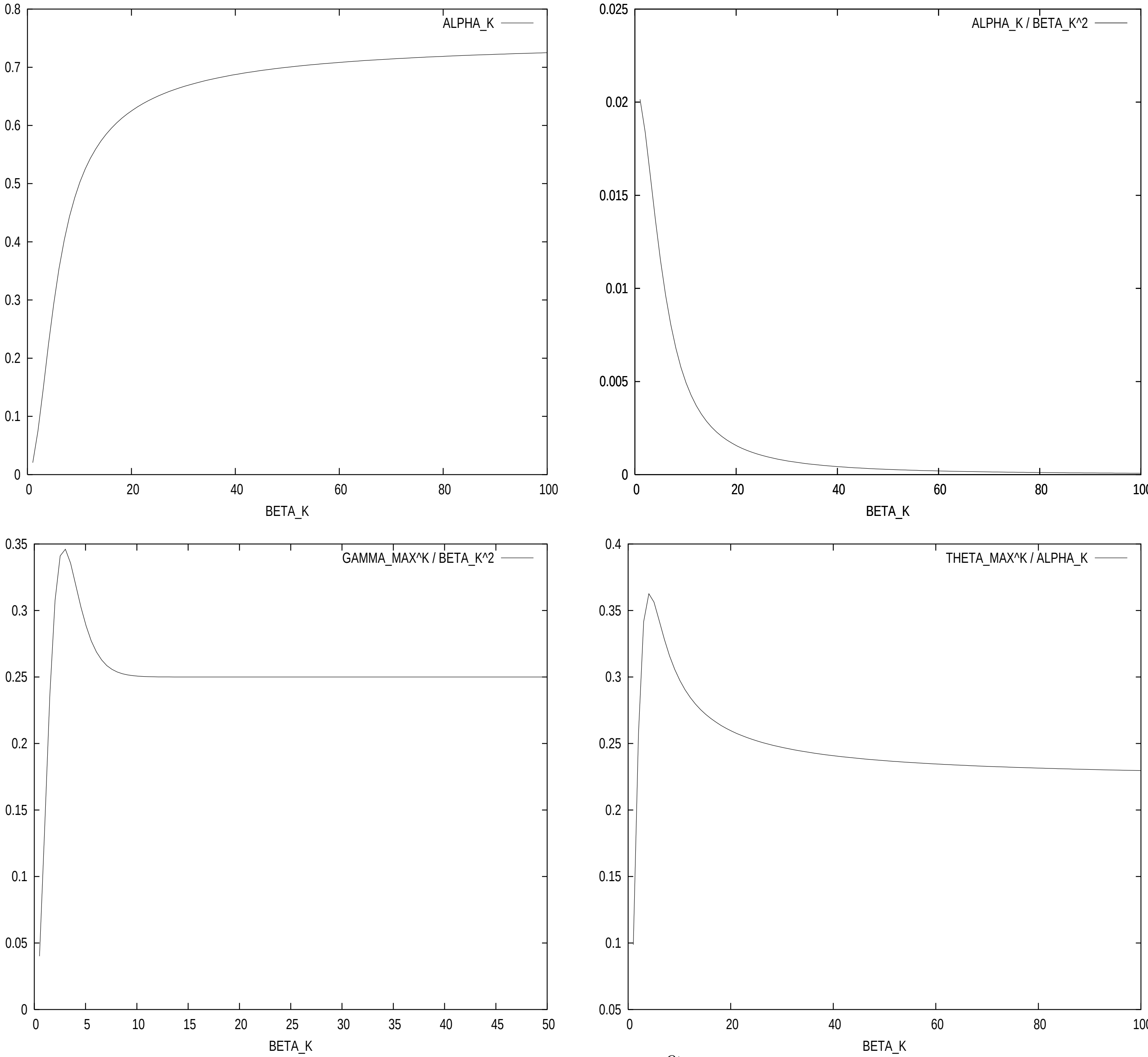


FIGURE 12. Behavior of parameters $\alpha_K$ and $\dfrac{\alpha_K}{\beta_K^2}$ in terms of $\beta_K$. We note that $\gamma_{max}^K \beta_K^2$ and the quotient $\dfrac{\theta_{max}^K}{\alpha_K}$ are bounded.

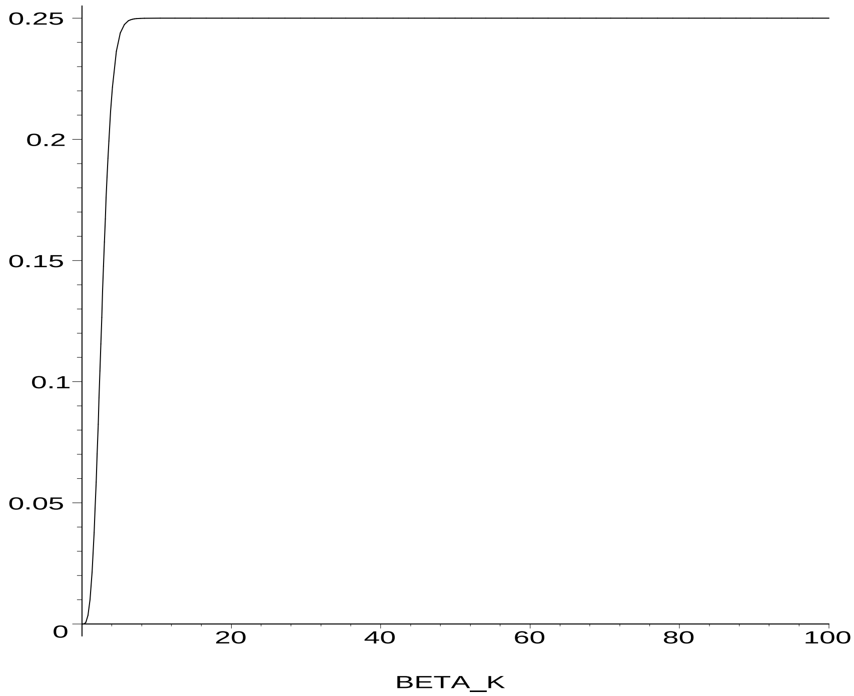


FIGURE 13. The eigenvalue $\zeta^K_{max}$ is bounded.

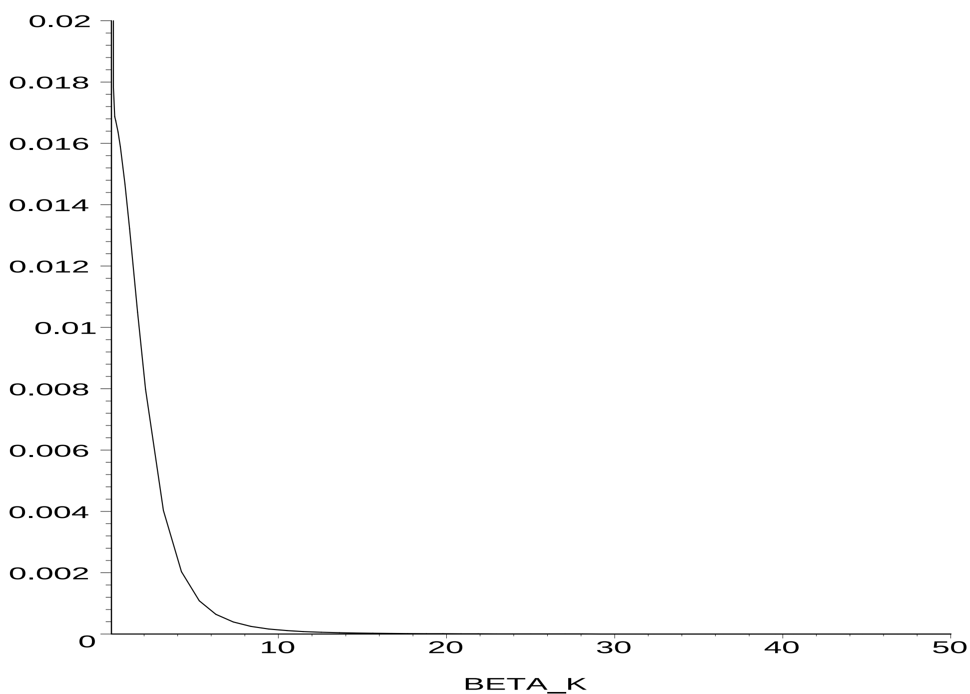


FIGURE 14. The quantity $\dfrac{\xi^K_{max}}{\beta^4_K}$ is bounded.

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