ANALYSIS OF NUMERICAL ERRORS IN LARGE EDDY SIMULATION

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ABSTRACT. We consider the question of "numerical errors" in large eddy simulation. It is often claimed that straightforward discretization and solution using centered methods of models for large eddy motion can simulate the motion of turbulent flows with complexity independent of the Reynolds number and depending only on the resolution "δ" of the eddies sought. This report considers precisely this question analytically: is it possible to prove error estimates for discretizations of *actually used* large eddy models whose error constants depend only on δ but not Re? We consider the most common, simplest and most mathematically tractable model and the most mathematically clear discretization. In two cases, we prove such an error estimate and carefully detail why our argument fails in the most general case. Our analysis aims to assume as little time regularity on the true solution as possible.

1. Introduction.

The laminar or turbulent flow of an incompressible fluid is modeled by solutions (u,p) of the incompressible Navier-Stokes equations:

$$
\begin{aligned}
&u_t + u\cdot\nabla u + \nabla p - Re^{-1}\Delta u = f \quad \text{and} \quad \nabla\cdot u = 0, \quad \text{in} \quad \Omega\times(0,T] \\
&u(x,0) = u_0(x), \quad x\in\Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega\times(0,T], \quad \text{and} \quad \int_\Omega p\,dx = 0.
\end{aligned}
$$

Here $\Omega\subset\mathbb{R}^d (d=2,3)$, is a bounded, simply connected domain with polygonal Γ, $u:\Omega\times(0,T]\to\mathbb{R}^d$ is the fluid velocity, $p:\Omega\to\mathbb{R}$ is the fluid pressure, $f(x,t)$ is the (known) body force, $u_0(x)$ the initial flow field and Re the Reynolds number.

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Unfortunately, when Re is large the resulting turbulent flow is typically so complex that, so called, direct numerical simulation of (u, p) is not practically feasible.

Weak solutions to (1.1) are unique for $d = 3$ if, for example,

$$\int_0^T \|\nabla u\|^4_{L^2(\Omega)} dt < \infty.$$

(There numerous generalizations of this basic result, [Ga99], [L69].) In fact, one conjecture of Leray is that "turbulence" in nature is associated with a breakdown of uniqueness of weak solutions to (1.1). With this in mind, solutions u to (1.1) with $\|\nabla u\|_{L^2(\Omega)} \in L^4(0, T)$ are frequently described as "laminar". Thus, the regularity which can be reasonably assumed is of critical importance.

There are numerous approaches to the simulation of turbulent flows in practical settings. One of the most promising current approaches is large eddy simulation (LES) in which approximations to local spacial averages of u are calculated. A spacial length scale δ is selected and the velocity scales of $O(\delta)$ and larger are approximated directly while the effects of those smaller than $O(\delta)$ on the $O(\delta)$ and larger eddies are modelled. In computational turbulence studies using LES it is often reported that *the resulting computational complexity is independent of the Reynolds number* (but dependent on the resolution sought, δ). There has been little or no analytical support for this observation however. The goal of this report is to begin numerical analysis in support of this claim.

To be more specific, a smooth, nonnegative function $g(x)$ with $g(0) = 1$ and $\int_{\mathbb{R}^d} g\ dx = 1$ is selected and the mollifer $g_\delta(x)$ is defined in the usual way:

$$g_\delta(x) = \delta^{-d} g(x/\delta).$$

(One common example is a Gaussian, $g(x) = (6/\pi)^{d/2}\ \exp\,(-6x_j x_j))$. The spacial averaging/filtering operation is now defined by convolution

$$\overline{u}(x, t) = g_\delta * u(x, t),\ \overline{p} = g_\delta * p,\ \overline{f} = g_\delta * f,\ \text{etc.}$$

In LES, approximations to $(\overline{u}, \overline{p})$ are sought rather than to (u, p). The usual procedure is to first filter the Navier-Stokes equations:

$$\begin{aligned}
&\overline{u}_t + \nabla \cdot (\overline{u}\ \overline{u}) + \nabla \overline{p} - Re^{-1} \Delta \overline{u} = \overline{f} + \nabla \cdot \mathbb{T},\ \text{in } \Omega \\
&\nabla \cdot \overline{u} = 0,\ \text{in } \Omega,\quad \text{where the "Reynolds' stress tensor" } \mathbb{T} \text{ is :} \\
&\mathbb{T} = \mathbb{T}(\overline{u}, u) = \overline{u}\ \overline{u} - \overline{u\ u}.
\end{aligned}$$

Closure is addressed by a modelling step in which $\mathbb{T}$ is written in terms of $\overline{u}$. The resulting (closed) space filtered Navier-Stokes equations are solved numerically. In this procedure, there are three essential issues:
1. The "modelling error" committed in approximating $\mathbb{T}$.
2. The "numerical error" in solving the resulting system.
3. Correct boundary conditions for the flow averages.

In this report, we study the numerical error analytically. Since there are many models in LES (see, e.g., [IL99], [GL00], [HMJ99], [FP99], [BFR80], [L99], [S98],[P92] for examples) and few analytical studies, we take herein the simplest model commonly in use presented, for example, in Ferziger and Peric [FP99; Section 9.3].

To describe the model, let $\mathbb{D}(u)$ be the deformation tensor associated with the indicated velocity field by:

$$\mathbb{D}(u) = \frac{1}{2}(\nabla u + \nabla u^t) = \frac{1}{2}(u_{i,x_j} + u_{j,x_i}).$$

The Reynolds stresses are thought of as a turbulent diffusion process based upon the Boussinesq assumption or eddy viscosity hypothesis that "turbulent fluctuations are dissipative in the mean", [IL99], [F93], [MP93], [P92]. We will accordingly consider a model of the form

$$\nabla \cdot \mathbb{T} \sim \nabla \cdot (\nu_{\mathrm{TURB}}(\overline{u}, \delta) \;\; \mathbb{D}(\overline{u})),$$

where $\nu_{\mathrm{TURB}} \doteq \nu_{\mathrm{TURB}}(\overline{u}, \delta)$ is the, so-called, turbulent viscosity or eddy viscosity. This turbulent viscosity's determination can be very complex, involving even solutions of accompanying systems of nonlinear, partial differential equations. In the simplest case, the turbulent viscosity depends on the mean flow $\overline{u}$ through the magnitude of the deformation of $\overline{u}$, $\nu_{\mathrm{TURB}} = \nu_{\mathrm{TURB}}(\mathbb{D}(\overline{u}))$, with a functional dependence. Under the Boussinesq assumption, $\nabla \cdot \mathbb{T}$ should act like a physical viscosity. Following the reasoning of Ladyzhenskaya [L70], thermodynamic considerations imply that the Taylor series of $\nu_{\mathrm{TURB}}(\mathbb{D})$ should be dominated by odd degree terms. The simplest case is of linear dependence upon $|\mathbb{D}|$

$$\nu_{\mathrm{TURB}}(|\mathbb{D}|) = a_0(\delta) + a_1(\delta)|\mathbb{D}(\overline{u})|, \tag{1.1}$$

where $|\mathbb{D}|$ denotes the Frobenius norm of $\mathbb{D}$. For specificity and for accord with the most commonly used Smagorinsky [S63] model, we take $a_0(\delta) \geq 0$ and $a_1(\delta) = C_s\ \delta^2$. Other scalings are possible, [L96], though less tested, as are many other subgridscale models [IL99], [S98]. Here C_s is typically either chosen to be around 0.1 or taken to be a function $C_s = C_s(x,t)$ and extrapolated as in the "dynamic subgridscale model" of Germano, [GPMC91].

With the model (1.1) the resulting system of equations for the approximations (w, q) to $(\overline{u}, \overline{p})$ is:

$$\begin{cases} w_t + \nabla \cdot (w\ w) + \nabla q - Re^{-1}\Delta w \\ \quad - \nabla \cdot (a_0(\delta) + C_s\delta^2|\mathbb{D}(w)|)\mathbb{D}(w)) = \overline{f}, \text{ in } \Omega \times (0,T], \\ \nabla \cdot w = 0, \text{ in } \Omega \times (0,T], \\ w(x,0) = w_0(x), \text{ in } \Omega, \text{ and } \displaystyle\int_\Omega q\ dx = 0. \end{cases} \tag{1.2}$$

Boundary conditions must be supplied for the large eddies. It is physically clear that large eddies do not adhere to solid walls. (For example, tornadoes and hurricanes move while touching the earth and lose energy as they move.) Therefore, in [GL00], [S00], (see also [P92] for the use of similar boundary conditions in a conventional turbulence model), it was proposed that the large eddies w should satisfy a no-penetration condition and a slip with friction condition on $\partial\Omega$:

$$\begin{cases} w \cdot \hat{n} = 0, \text{ on } \Gamma = \partial\Omega, \text{ and} \\ w \cdot \hat{\tau}_l = 0 \text{ on } \Gamma_0, \text{ meas } (\Gamma_0) > 0, \\ w \cdot \hat{\tau}_l + \beta^{-1} \overrightarrow{t} \cdot \hat{\tau}_l = 0, \text{ on } \Gamma_1, l = 1, d-1, \end{cases} \tag{1.3}$$

where $\vec{t}$ is the Cauchy stress vector on Γ, for background information see Serrin [S59], $\beta = \beta(\delta, Re)$ is the friction coefficient (calculated explicitly in [S00]), $\hat{n}$ the outward unit normal and $\hat{\tau}_j$ an orthonormal system of tangent vector's on Γ. The friction coefficient β can be calculated once a specific filter is chosen, [S00]. It has the property ([S00]) that no slip conditions are recovered as $\delta \to 0$:

$$\beta^{-1}(Re,\ \delta) \to 0 \text{ as } \delta \to 0.$$

A Dirichlet boundary condition $w = w_{\text{inflow}}$ on Γ_0, is appropriate if Γ_0 is an inflow boundary upon which $\overline{u}$ can be calculated from the known, inflow velocity field.

The Cauchy stress vector $\vec{t}$ includes the action of both the viscous stresses and Reynolds stresses and is given by:

$$\vec{t}(w) := \hat{n} \cdot [-q\mathbb{I} + 2Re^{-1}\mathbb{D}(w) + a_0(\delta)\mathbb{D}(w) + C_s\delta^2|\mathbb{D}(w)|\mathbb{D}(w)]. \tag{1.4}$$

Standard properties of convolution operators imply that the flow averages $(\overline{u}, \overline{p})$ are $C^\infty(\Omega)$ in space, have bounded kinetic energy ($\int_\Omega |\overline{u}|^2 dx \le \int_\Omega |u|^2 dx \le C(\Omega, f, u_0)$), have no solution scales smaller than $O(\delta)$ and converge to u as $\delta \to 0$, [IJL00]. On the other hand, it is *not* obvious, nor has it been proven yet, that solutions (w, q) to the large eddy model approximating $(\overline{u}, \overline{p})$ share *any* of these properties! Nevertheless, the spacial regularity of solutions (w, q) we shall consider to be a *modelling* issue (beyond the scope of this report studying *numerical errors* in LES). The *time regularity* of solutions (w, q) is still an important consideration. For example, we shall show that solutions of this model satisfy

$$\int_0^T \|\nabla w\|_{L^3}^3 dt < \infty,$$

uniformly in Re. One goal is to keep the assumed time regularity as close to $L^3(0, T)$ as possible and below $L^4(0, T)$. The fundamental error analysis of Heywood and Rannacher [HR82] for the Navier-Stokes equations is based, in part, on a laminar-type assumption $\nabla u \in L^\infty(0, T; L^2(\Omega))$. Weakening this to an assumption of the form $\nabla u \in L^p(0, T; L^2(\Omega))$ for $p < 4$ (as we seek to do herein) is nontrivial.

2. Preliminaries.

This section sets the notation used in the report, describes the function spaces employed and collects several useful inequalities. The notation used is standard for the most part. The $L^p(\Omega)$ norms, for $p \neq 2$, are denoted explicitly as $||f||_{L^p}$. Sobolev spaces $W^{k,p}(\Omega)$ are defined in the usual way, [A75]. The associated norm is denoted $||\cdot||_{k,p}$. If the domain in question is *not* Ω (e.g., $\Omega \times (0, T)$) then it will be explicitly indicated. If $p = 2$ these norms will be written $||\cdot||_k$ for the $W^{k,2}(\Omega)$ norm and $||\cdot||_{k,\Gamma}$ for the $W^{k,p}(\Gamma)$ norm and $||\cdot||$ and $||\cdot||_\Gamma$, respectively, for the $L^2(\Omega)$ and $L^2(\Gamma)$ norms. Suppose the polygonal boundary $\Gamma = \partial\Omega$ is composed of faces $\Gamma_0, \Gamma_1, \cdots, \Gamma_J$.

The natural spaces associated with the boundary conditions (1.3) are:

$$X := \{v : \Omega \subset \mathbb{R}^d : v_j \in W^{1,2}(\Omega), v = 0 \text{ on } \Gamma_0 \text{ and } v \cdot \hat{n} = 0 \text{ on } \Gamma_j, j = 1, \cdots, J\}$$

$$Q := L_0^2(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q\ dx = 0\}.$$

The boundary condition in X is defined to hold in the sense of the trace theorem on each Γ_j and $\hat{n}$ is the outward unit normal to $\Gamma = \partial\Omega$. The $L^2(\Omega)$ and $L^2(\Gamma)$ inner products are denoted $(\cdot,\cdot)$ and $(\cdot,\cdot)_\Gamma$ respectively.

If $v \in X, \mathbb{D}(v)$ denotes the usual deformation tensor, defined in the introduction. The unit vector $\hat{\tau}_j$ denotes an orthonormal system of tangent vectors on Γ. Whenever $\hat{\tau}_j$ occurs, it will be understood that the term is to be summed over the two tangent vectors if $d = 3$. For example:

$$||v \cdot \hat{\tau}_j||^2_{\Gamma_j} \text{ if } d = 3 \text{ means } \ (||v \cdot \hat{\tau}_{j,1}||^2_{\Gamma_j} + ||v \cdot \hat{\tau}_{j,2}||^2_{\Gamma_j}).$$

The following dual norms are defined in an equivalent but *slightly* nonstandard way: for $\dfrac{1}{q} + \dfrac{1}{q'} = 1, 1 < q, q' < \infty$,

$$||f||_* := \sup_{v \in X} \frac{(f,v)}{||\mathbb{D}(v)||},$$

$$||f||_{W^{-1,q'}(\Omega\times(0,t))} := \sup_{v \in L^q(0;t;X)} \frac{\int_0^t (f,v)dt'}{(\int_0^t ||\mathbb{D}(v)||^q_{L^q}\, dt')^{1/q}}.$$

Lemma 2.1. *[Inf-Sup Condition]. The velocity-pressure spaces (X,Q) satisfy the inf-sup condition:*

$$\inf_{\lambda \in Q} \sup_{v \in X} \frac{(\lambda, \nabla \cdot v)}{||\lambda|| \left[||\mathbb{D}(v)||^2 + \sum_{j=1}^J ||v \cdot \hat{\tau}_j||^2_{1/2,\Gamma_j} \right]^{1/2}} \ > \ 0, \tag{2.1}$$

where the $1/2, \Gamma$ norm is defined here piecewise on each face Γ_j.

Proof. The trace theorem [G92] and Korn's inequality together show that (2.1) is implied by the usual inf-sup condition

$$\inf_{\lambda \in Q} \sup_{v \in X \cap H^1_0(\Omega)^d} \frac{(\lambda, \nabla \cdot v)}{||\lambda||\ ||\nabla v||} > 0. \quad \square$$

Lemma 2.1 implies that the space of discretely divergence free functions V:

$$V := \{v \in X : (\lambda, \nabla \cdot v) = 0, \text{ for all } \lambda \in Q\},$$

is a well defined, nontrivial, closed subspace of X.

Remark. Since Γ is *not* C^1 discontinuities in τ_j and $\hat{n}$ have forced modifications in the norms to piecewise definition. For example, $v \cdot \hat{\tau}_j \notin H^{1/2}(\Gamma)$ for $v \in H^1(\Omega)$ but $v \cdot \hat{\tau}_j \in H^{1/2}(\Gamma_j), j = 0, \cdots, J$.

The conforming finite element method for this problem begins by selecting finite element spaces $X^h \subset X$ and $Q^h \subset Q$, where "h" denotes as usual a representative meshwidth for (X^h, Q^h), satisfying the usual approximation theoretic conditions required of finite element spaces. The condition that $X^h \subset X$ imposes the restriction that $v^h \cdot \hat{n}_{|\Gamma_j} = 0$, for all $v^h \in X^h$. For intricate boundaries, this could possibly be onerous so it is interesting to consider imposing $v^h \cdot \hat{n}_{|\Gamma} = 0$ with penalty or Lagrange multiplier methods, following, e.g., the work in [Li99]. Nevertheless, there is already considerable computational experience with imposing this condition in finite element methods, see, e.g., [GS98], [ESG82], so we shall not focus on the interesting detail of the treatment of corners. Without these additional regularizations in the numerical method, it is useful in the analysis to assume that (X^h, Q^h) satisfies the discrete analogue of (2.1):

$$\inf_{\lambda^h \in Q^h} \sup_{v^h \in X^h} \frac{(\lambda^h, \nabla \cdot v^h)}{||\lambda^h|| \left[||\mathbb{D}(v^h)||^2 + \sum_{j=1}^J ||v^h \cdot \hat{\tau}_j||^2_{1/2,\Gamma_j}\right]^{1/2}} \geq C > 0, \tag{2.2}$$

where $C > 0$ is independent of h. The next lemma shows, in essence, that if the computational mesh follows the boundary and the velocity space restricted to no-slip boundary conditions and the pressure space satisfy the usual inf-sup condition, (2.2) holds.

Lemma 2.2. *[Discrete Inf-Sup Condition]. If (X^h, Q^h) satisfies*

$$\inf_{\lambda_h \in Q^h} \sup_{v_h \in X^h \cap H_0^1(\Omega)^d} \frac{(\lambda^h, \nabla \cdot v^h)}{\|\lambda^h\| \; \|\nabla v^h\|} \geq C_1 > 0,$$

then (2.2) holds.

Proof. By the Poincaré-Friedrichs inequality and the trace theorem [G92], for any $\lambda_h \neq 0$, $v^h (\neq 0) \in X^h$

$$\frac{(\lambda^h, \nabla \cdot v^h)}{||\lambda^h|| \left[||\mathbb{D}(v^h)||^2 + \sum_{j=1}^J ||v^h \cdot \hat{\tau}_j||^2_{1/2,\Gamma_j}\right]^{1/2}} \geq C \frac{(\lambda^h, \nabla \cdot v^h)}{\|\lambda^h\| \; \|v^h\|_1} \geq C \frac{(\lambda^h, \nabla \cdot v^h)}{\|\lambda^h\| \; \|\nabla v^h\|}. \qquad \square$$

Thus, (2.2) will be assumed throughout this report. Under (2.2), the space of discretely divergence free functions

$$V^h := \{v^h \in X^h : (\lambda^h, \nabla \cdot v^h) = 0, \text{ for all } \lambda^h \in Q^h\}$$

is a nontrivial closed subspace of X^h, [GR86], [G89].

We shall frequently use Young's inequality in the form:

$$ab \leq \frac{\epsilon}{q} a^q + \frac{\epsilon^{-q'/q}}{q'} b^{q'}, \; 1 \leq q, q' \leq \infty, \; \frac{1}{q} + \frac{1}{q'} = 1.$$

The generalization of Hölder's inequality:

$$\int_\Omega |u| \; |v| \; |w| dx \leq ||u||_{L^p} ||v||_{L^q} ||w||_{L^r}, \; \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1, \; 1 \leq p, q, r \leq \infty$$

is also useful. We shall frequently use the Sobolev embedding theorem, often, but not always, in the form that in 3 dimensions $W^{1,3}(\Omega) \hookrightarrow L^p(\Omega)$ for $1 \leq p < \infty$.

The nonlinear form in the subgridscale term: for $v, w \in W^{1,3}(\Omega)$

$$(|\mathbb{D}(w)| \mathbb{D}(w), \mathbb{D}(v)) \tag{2.3}$$

is of p-Laplacian type (with $p = 3$). Thus, it is strongly monotone and locally Lipschitz continuous in the sense made precise in the following well-known lemma, see, e.g., [L96], [DG91].

Lemma 2.3. *[Strong Monotonicity and Local Lipschitz-Continuity].*

There are constants $\underline{C}$ *and* $\overline{C}$ *such that for all* $u_1, u_2, v \in (W^{1,3}(\Omega))^d$ *and* $d = 2$ *or* 3, *with* $r = \max\{||\mathbb{D}(u_1)||_{L^3}, ||\mathbb{D}(u_2)||_{L^3}\}$,

$$(|\mathbb{D}(u_1)|\mathbb{D}(u_1) - |\mathbb{D}(u_2)|\mathbb{D}(u_2), \mathbb{D}(u_1 - u_2)) \geq \underline{C}||\mathbb{D}(u_1 - u_2)||_{L^3}^3, \text{ and}$$
$$(|\mathbb{D}(u_1)|\mathbb{D}(u_1) - |\mathbb{D}(u_2)|\mathbb{D}(u_2), \mathbb{D}(v)) \leq \overline{C}r||\mathbb{D}(u_1 - u_2)||_{L^3}||\mathbb{D}(v)||_{L^3}. \quad \square$$

Korn's Inequalities.

Korn's inequalities relate L^p norms of the deformation tensor $\mathbb{D}(v)$ to those same norms of the gradient for $1 < p < \infty$, see Galdi [Ga99], Gobert [G62, G71], Temam [T83] or Fichera [F72], and fail if $p = 1$.

Theorem 2.1. *[Korn's Inequalities]. There is a* $C > 0$ *such that for* $1 < p < \infty$

$$||v||_{W^{1,p}}^p \leq C(\Omega)[||v||_{L^p}^p + ||\mathbb{D}(v)||_{L^p}^p],$$

for all $v \in (W^{1,p}(\Omega))^d$.

Further, if $\gamma(v)$ *is a semi-norm on* $L^p(\Omega)$ *which is a norm on the constants, then*

$$||\nabla v||_{L^p} \leq C(\Omega)[\gamma(v) + ||\mathbb{D}(v)||_{L^p}],$$

holds for $1 < p < \infty$ *and for all* $v \in (W^{1,p}(\Omega))^d$.

As a consequence of Korn's inequality it follows that, taking $\gamma(v) = ||v||_{L^p(\Gamma_0)}$, if $\text{meas}(\Gamma_0) > 0$ then

$$||\nabla v||_{L^p}^p \leq \|v\|_{1,p}^p \leq C_K\|\mathbb{D}(v)\|_{L^p}^p,$$

for all $v \in \{v \in W^{1,p}(\Omega)^d : v_{|\Gamma_0} = 0\}$.

One version of the Poincaré-Morrey inequality is helpful for domains Ω *which do not possess an axis of symmetry.* For such domains, it is known that

$$||v||^2 \leq C(\Omega)\{||\mathbb{D}(v)||^2 + ||v \cdot \hat{n}||_\Gamma^2\}, \text{ for all } v \in (H^1(\Omega))^d.$$

At a critical point in Section 4, we shall use the Gagliardo-Nirenberg inequality in $W^{1,p}(\Omega) \cap X$ to reduce the time regularity required for w. This inequality [A75], [N59], [Ga99], [D93] states that provided Γ satisfies a weak regularity condition (holding in particular for polygonal domains) and $\text{meas}(\Gamma_0) > 0$ for all $v \in W^{1,p}(\Omega) \cap X, 1 \leq q, s \leq \infty$:

$$||v||_{L^q} \leq C||\nabla v||_{L^p}^a||v||_{L^s}^{1-a}, \text{ for all } v \in W^{1,p}(\Omega) \cap X, \tag{2.4}$$

where, for $\Omega \subset \mathbb{R}^3$, (improvable if $\Omega = \mathbb{R}^2$), $p \geq 3, q \geq s,\ 0 \leq a < 1$ and

$$a = \left(\frac{1}{s} - \frac{1}{q}\right)\left(\frac{1}{3} - \frac{1}{p} + \frac{1}{s}\right)^{-1}$$

In particular, note that taking $q = 6, p = 3$ and $s = 2$ gives

$$||v||_{L^6(\Omega)} \leq C||\nabla v||_{L^3(\Omega)}^{2/3}||v||^{1/3}.$$

The following combination of this and Korn's inequality will be useful in Section 4.

Lemma 2.4. *Let $meas(\Gamma_0) > 0$ and $\Omega \subset \mathbb{R}^3$ (or $\mathbb{R}^2$). Then,*

$$||v||_{L^6} \leq C||v||^{1/3}||\mathbb{D}(v)||^{2/3}_{L^3}, \ C = C(\Omega).$$

Proof. This follows immediately from (2.4) and Korn's inequality. $\square$

3. Finite Element Formulation.

This section develops the finite element method for the LES model. The stability of the model is also studied. In particular, we show w and $w^h \in L\infty(0,T;L^2(\Omega)) \cap L^3(0,T;H^1(\Omega))$ uniformly in $\mathbb{R}^3$. Lastly, the error in an equilibrium projection is considered.

The variational formulation is derived in the usual way by multiplication of (1.2) by $(v,q) \in (X,Q)$ and applying the divergence theorem. The boundary integral terms require careful treatment (following, e.g., [L99]) on account of the slip with friction condition on Γ. Let $\alpha \geq 0$ be a constant. The formulation which results is to find $w : [0,T] \to X, q : (0,T] \to Q$ satisfying:

$$(3.1) \qquad \begin{cases} (w_t, v) + \beta(\delta, Re) \sum\limits_{j=1}^{J} (w \cdot \hat{\tau}_j, v \cdot \hat{\tau}_j)_{\Gamma_j} - (q, \nabla \cdot v) + \alpha(\nabla \cdot w, \nabla \cdot v) + \\ ((2Re^{-1} + a_0(\delta) + C_s\delta^2|\mathbb{D}(w)|)\mathbb{D}(w), \mathbb{D}(v)) - (w \cdot \nabla v, w) = (\overline{f}, v), \\ \qquad\qquad \text{for all } v \in X, \\ (\lambda, \nabla \cdot w) = 0, \ \text{for all } \lambda \in Q, \end{cases}$$

and $w(x,0) = \overline{u}_0(x) \in X$. For compactness, define the nonlinear and trilinear form:

$$\begin{aligned} a(u,w,v) &:= \alpha(\nabla \cdot w, \nabla \cdot v) + \sum_{j=1}^{J} \beta(w \cdot \hat{\tau}_j, v \cdot \hat{\tau}_j)_{\Gamma_j} + \\ &\quad + ((2Re^{-1} + a_0(\delta) + C_s\delta^2|\mathbb{D}(u)|)\mathbb{D}(w), \mathbb{D}(v)), \\ b(u,w,v) &:= \frac{1}{2}(u \cdot \nabla w, v) - \frac{1}{2}(u \cdot \nabla v, w). \end{aligned}$$

It is a simple index calculation to check that for $u, v, w \in X$ (since such functions have zero normal components on Γ) $-(w \cdot \nabla v, w) = b(w,w,v)$. Thus, the variational formulation can be rewritten as: find $(w,q) : [0,T] \to (X,Q)$ satisfying: $w(x,0) = \overline{u}_0(x)$ and

(3.2)

$$(w_t, v) + a(w,w,v) + b(w,w,v) + (\lambda, \nabla \cdot w) - (q, \nabla \cdot v) = (\overline{f}, v), \text{ for all } (v,\lambda) \in (X,Q).$$

Using the strong monotonicity properties of the Ladyzhenskaya-Smagorinsky subgridscale model detailed in Lemma 2.3 (see also [L96] and [DG91]), it is easy to prove that the LES model (1.2), (1.3) satisfies the analogue of Leray's inequality for the Navier-Stokes equations.

Lemma 3.1. *[Leray's inequality for the LES model]. A weak solution of (3.2) satisfies:*

$$\frac{1}{2}||w(t)||^2 + \int_0^t [\sum_{j=1}^{J} \beta||w \cdot \hat{\tau}_j||^2_{\Gamma_j} + (2Re^{-1} + a_0(\delta))||\mathbb{D}(w)||^2 + \underline{C}C_s\delta^2||\mathbb{D}(w)||^3_{L^3}]dt'$$
$$\leq \frac{1}{2}||w(0)||^2 + \int_0^t (\overline{f}, w)dt'.$$

Proof. The weak form is precisely (3.2) integrated in time from $t' = 0$ to $t' = t$. Set $v = w, \lambda = q$ in the time integrated form of (3.2), and use Lemma 2.3. $\square$

Remarks.

1. Because of the slip with friction boundary conditions (1.3), it is important to choose the formulation of the viscous terms, as in (3.1), (3.2), involving the deformation tensor.
2. Leray's inequality immediately implies stability in various norms (which we will develop) and is the key, first step in proving existence of weak solutions to (1.2), (1.3). This last question is fully investigated (under different boundary conditions) in remarkable papers by Ladyzhenskaya [L67], Parés [P92] and Du and Gunzburger [DG91].

The continuous-in-time finite element method for (1.1), (1.2) uses the variational formulation (3.2) as follows. First, velocity-pressure finite element spaces $X^h \subset X \cap W^{1,3}(\Omega)^d$, $Q^h \subset Q$ (2.2) are selected. Next, the least squares parameter $\alpha \geq 0$ is selected.
The finite element approximations to (w, q) are maps $(w^h, q^h) : [0, T] \to (X^h, Q^h)$ satisfying:

$$(3.3)\quad (w_t^h, v^h) + a(w^h; w^h, v^h) + b(w^h, w^h, v^h) + (\lambda^h, \nabla \cdot w^h) - (q^h, \nabla \cdot v^h) = (\overline{f}, v^h)$$

for all $(v^h, \lambda^h) \in (X^h, Q^h)$ where $w^h(0) \in X^h$ is an approximation to $w(0) = \overline{u}_0$.

It is straightforward to verify that Leray's inequality holds for w^h as well as w.

Lemma 3.2. *[Leray's inequality for w^h]. Any solution of (3.3) satisfies:*

$$\frac{1}{2}||w^h(t)||^2 + \int_0^t [\beta \sum_{j=1}^{J} ||w^h \cdot \hat{\tau}_j||^2_{\Gamma_j} + (2Re^{-1} + a_0(\delta))||\mathbb{D}(w^h)||^2 + \alpha||\nabla \cdot w^h||^2 +$$
$$\underline{C}\, C_s\delta^2||\mathbb{D}(w^h)||^3_{L^3}]dt' \leq \frac{1}{2}||w^h(0)||^2 +$$
$$+ \int_0^t (\overline{f}, w^h)dt'. \quad \square$$

Using various inequalities in the RHS, stability bounds for w^h follow from Lemma 3.2.

Proposition 3.1. *[Stability of w^h]. The solution w^h of (3.3) satisfies:*

(3.4; a)

$$\frac{1}{2}||w^h(t)||^2 + \int_0^t [\sum_{j=1}^J \beta||w^h\cdot\hat{\tau}_j||^2_{\Gamma_j} + (Re^{-1}+a_0(\delta))||\mathbb{D}(w^h)||^2 + \alpha||\nabla\cdot w^h||^2 +$$

$$\underline{C}\ C_s\delta^2||\mathbb{D}(w^h)||^3_{L^3}]dt' \leq \frac{1}{2}||w^h(0)||^2 + \frac{Re}{4}\int_0^t ||\overline{f}||^2_* dt'.$$

(3.4; b)

$$\frac{1}{2}||w^h(t)||^2 + \int_0^t [\sum_{j=1}^J \beta||w^h\cdot\hat{\tau}_j||^2_{\Gamma_j} + (Re^{-1}+a_0(\delta))||\mathbb{D}(w^h)||^2 + \alpha||\nabla\cdot w^h||^2 +$$

$$\frac{2}{3}\underline{C}\ C_s\delta^2||\mathbb{D}(w^h)||^3_{L^3}]dt' \leq \frac{1}{2}||w^h(0)||^2 + \frac{2}{3}(\underline{C}\ C_s)^{-1/2}\delta^{-1}||\overline{f}||^{3/2}_{W^{-1,3/2}(\Omega\times(0,t))}$$

(3.4; c)

$$||w^h(t)||^2 + 2\int_0^t e^{t-t'}[\sum_{j=1}^J \beta||w^h\cdot\hat{\tau}_j||^2_{\Gamma_j} + (Re^{-1}+a_0(\delta))||\mathbb{D}(w^h)||^2 + \alpha||\nabla\cdot w^h||^2 +$$

$$\underline{C}\ C_s\delta^2||\mathbb{D}(w^h)||^3_{L^3}]dt' \leq e^t||w^h(0)||^2 + \int_0^t e^{t-t'}||\overline{f}||^2 dt'.$$

Proof. Inequality. (3.4;a) follows by applying Young's inequality to Lemma 3.2. The bound (3.4;b) follows from the definition of the dual norm and $ab \leq \frac{\epsilon}{3}a^3 + \frac{2}{3}\epsilon^{-1/2}b^{3/2}$ applied in the same manner.

For (3.4;c), set $v^h = w^h$ and $\lambda^h = q^h$ in (3.3), use Lemma 2.3 and apply the Young's inequality on the RHS. This gives:

$$\frac{d}{dt}||w^h||^2 - ||w^h||^2 + 2[\sum_{j=1}^J \beta||w^h\cdot\hat{\tau}_j||^2_{\Gamma_j} + (2Re^{-1}+a_0(\delta))||\mathbb{D}(w^h)||^2 + \alpha||\nabla\cdot w^h||^2 +$$

$$+\underline{C}\ C_s\delta^2||\mathbb{D}(w^h)||^3_{L^3}] \leq ||\overline{f}||^2.$$

Inequality (3.4;c) now follows by using an integrating factor. □

In the analysis of the error in the approximation of the time dependent problem, it is useful to have a clear description of the error in the Stokes projection under slip with friction boundary conditions, [L99], [Li99]. It is also necessary that any dependence on Re, δ and β be made explicit.

Under the discrete inf-sup condition, the Stokes projection $\Pi : (X, Q) \to (X^h, Q^h)$ is defined as follows. Let $\Pi(w, q) = (\tilde{w}, \tilde{q})$, where $(\tilde{w}, \tilde{q})$ satisfies:

$$\alpha(\nabla\cdot(w-\tilde{w}), \nabla\cdot v^h) + (2Re^{-1}+a_0(\delta))(\mathbb{D}(w-\tilde{w}), \mathbb{D}(v^h)) + \sum_{j=1}^J \beta((w-\tilde{w})\cdot\hat{\tau}_j, v^h\cdot\hat{\tau}_j)_{\Gamma_j} +$$

$$+(q-\tilde{q}, \nabla\cdot v^h) = 0, \text{ for all } v^h \in X^h,$$

$$(\nabla\cdot(w-\tilde{w}), \lambda^h) = 0, \text{ for all } \lambda^h \in Q^h.$$

This is equivalent to the following formulation provided $w \in V$ and $v^h \in V^h$. Given (w, q), find $\tilde{w} \in V^h$ satisfying:

$$\alpha(\nabla \cdot (w - \tilde{w}), \nabla \cdot v^h) + (2Re^{-1} + a_0(\delta))(\mathbb{D}(w - \tilde{w}), \mathbb{D}(v^h)) + \sum_{j=1}^{J} \beta((w - \tilde{w}) \cdot \hat{\tau}_j, v^h \cdot \hat{\tau}_j)_{\Gamma_j} +$$

$$+ (q - \lambda^h, \nabla \cdot v^h) = 0, \text{ for all } v^h \in V^h \text{ and } \lambda^h \in Q^h.$$

Under the discrete inf-sup condition, it is well-known that $(\tilde{w}, \tilde{q})$ is a quasi-optimal approximation of (w, q). The dependence of the stability and error constants upon Re and $\beta = \beta(Re, \delta)$ is important to the error analysis. That dependence is described in the next lemma and proposition.

Lemma 3.3. *[Stability of the projection $\tilde{w}$]. Let $w \in V$ be given. Then, $\tilde{w}$ satisfies:*

$$\alpha||\nabla \cdot \tilde{w}||^2 + (2Re^{-1} + a_0(\delta))||\mathbb{D}(\tilde{w})||^2 + \sum_{j=1}^{J} \beta||\tilde{w} \cdot \hat{\tau}_j||^2_{\Gamma_j} \leq$$

$$\leq \alpha^{-1}||q||^2 + (2Re^{-1} + a_0(\delta))||\mathbb{D}(w)||^2 + \sum_{j=1}^{J} \beta||w \cdot \hat{\tau}_j||^2_{\Gamma_j}, \ \textit{if } \alpha > 0, \ \textit{and}$$

$$\alpha||\nabla \cdot \tilde{w}||^2 + \frac{1}{2}(2Re^{-1} + a_0(\delta))||\mathbb{D}(\tilde{w})||^2 + \sum_{j=1}^{J} \beta||\tilde{w} \cdot \hat{\tau}_j||^2_{\Gamma_j}$$

$$\leq C[(2Re^{-1} + a_0(\delta))^{-1}||q||^2 + (2Re^{-1} + a_0(\delta))||\mathbb{D}(w)||^2 + \sum_{j=1}^{J} \beta||w \cdot \hat{\tau}_j||^2_{\Gamma_j}], \ \textit{if } \alpha = 0.$$

Proof. Set $v^h = \tilde{w} \in V^h$ in the second formulation of the Stokes projection. This gives immediately:

$$\alpha||\nabla \cdot \tilde{w}||^2 + (2Re^{-1} + a_0(\delta))||\mathbb{D}(\tilde{w})||^2 + \sum_{j=1}^{J} \beta||\tilde{w} \cdot \hat{\tau}_j||^2_{\Gamma_j} =$$

$$= (2Re^{-1} + a_0(\delta))(\mathbb{D}(w), \mathbb{D}(\tilde{w})) + \sum_{j=1}^{J} \beta(w \cdot \hat{\tau}_j, \tilde{w} \cdot \hat{\tau}_j)_{\Gamma_j} + (q - \lambda^h, \nabla \cdot \tilde{w}) \leq$$

$$\leq \frac{1}{2}(2Re^{-1} + a_0(\delta))[||\mathbb{D}(w)||^2 + ||\mathbb{D}(\tilde{w})||^2] + \sum_{j=1}^{J} \frac{\beta}{2}[||w \cdot \hat{\tau}_j||^2_{\Gamma_j} +$$

$$+ ||\tilde{w} \cdot \hat{\tau}_j||^2_{\Gamma_j}] + \frac{\alpha}{2}||\nabla \cdot \tilde{w}||^2 + \frac{1}{2\alpha}||q||^2,$$

from which the first result follows. If $\alpha = 0$ the term $(q, \nabla \cdot \tilde{w})$ is bounded by noting that $\nabla \cdot \tilde{w} = \text{trace } (\mathbb{D}(\tilde{w}))$ so that

$$(q, \nabla \cdot \tilde{w}) \leq ||q|| \ ||\mathbb{D}(\tilde{w})|| \leq$$

$$\leq \frac{1}{4}(2Re^{-1} + a_0(\delta))||\mathbb{D}(\tilde{w})||^2 + (2Re^{-1} + a_0(\delta))^{-1}||q||^2. \qquad \square$$

Proposition 3.2. *Suppose the discrete inf-sup condition (2.2) holds. Then,* $(\tilde{w}, \tilde{q})$ *exists uniquely in* (X^h, Q^h) *and satisfies:*

$$\alpha||\nabla\cdot(w-\tilde{w})||^2 + (2Re^{-1}+a_0(\delta))||\mathbb{D}(w-\tilde{w})||^2 + \sum_{j=1}^{J}\beta||(w-\tilde{w})\cdot\hat{\tau}_j||^2_{\Gamma_j} \le$$
$$\le C \inf_{v^h\in X^h,\lambda^h\in Q^h} \{(2Re^{-1}+a_0(\delta))||\mathbb{D}(w-v^h)||^2 +$$
$$+\sum_{j=1}^{J}\beta||(w-v^h)\cdot\hat{\tau}_j||^2_{\Gamma_j} + \min\{\alpha^{-1},(Re^{-1}+a_0)^{-1}\}||q-\lambda^h||^2\}.$$

Proof. The proof follows standard arguments, carefully tracking the dependence of the constants upon Re and β. $\square$

Note that the use of least squares penalization of the divergence allows an error estimate for the Stokes projection whose constants are essentially independent of the Reynolds number in a suitably weighted norm.

4. The Convergence Theorem.

Let us first note that for standard piecewise polynomial finite element spaces it is known that, e.g., the L^2-projection of a function in $L^p, p \ge 2$, is in L^p itself and the L^2-projection operator is stable in $L^p, p\ge 2$, [CT87].

Let $e = w - w^h$ and let $\tilde{w}$ denote an approximation of w in $V^h \cap W^{1,3}(\Omega)$, for example, the L^2-projection. We shall explicitly assume that the $L^3(\Omega)$ norm of $(\tilde{w})$ is bounded. By the above remark, this assumption is reasonable for piecewise polynomial finite element spaces. The error is decomposed as $e = (w-\tilde{w}) - (w^h - \tilde{w}) = \eta - \phi^h$, where $\eta = w - \tilde{w}$ and $\phi^h = w^h - \tilde{w} \in V^h$. An error equation is obtained by subtracting (3.2) from (3.3) and using the fact that $w \in V$. This gives, for any $v^h \in V^h \cap W^{1,3}(\Omega)$ and $\lambda^h \in Q^h$:

$$\begin{aligned}(4.1)\qquad &(e_t, v^h) + a(w;w,v^h) - a(w^h;w^h,v^h)\\ &+b(w,w,v^h) - b(w^h,w^h,v^h) - (q-\lambda^h, \nabla\cdot v^h) = 0.\end{aligned}$$

This is rewritten, adding and subtracting terms and setting $v^h = \phi^h$, as follows:

$$\begin{aligned}(4.2)\qquad &(\phi^h_t,\phi^h) + a(w^h;w^h,\phi^h) - a(\tilde{w};\tilde{w},\phi^h) =\\ &= b(w,w,\phi^h) - b(w^h,w^h,\phi^h) + (\eta_t,\phi^h)\\ &+ a(w;w,\phi^h) - a(\tilde{w};\tilde{w},\phi^h) - (q-\lambda^h,\nabla\cdot\phi^h).\end{aligned}$$

The monotonicity lemma (Lemma 2.3) implies that with $r := \max\{||\mathbb{D}(w)||_{L^3}, ||\mathbb{D}(\tilde{w})||_{L^3}\}$

$$a(w^h;w^h,\phi^h) - a(\tilde{w};\tilde{w},\phi^h) \ge \underline{C}C_s\delta^2||\mathbb{D}(\phi^h)||^3_{L^3} + \alpha||\nabla\cdot\phi^h||^2$$
$$+ (2Re^{-1}+a_0(\delta))||\mathbb{D}(\phi^h)||^2 + \sum_{j=1}^{J}\beta||\phi^h\cdot\hat{\tau}_j||^2_{\Gamma_j},$$

and

$$a(w;w,\phi^h) - a(\tilde{w},\tilde{w},\phi^h) \le (2Re^{-1}+a_0(\delta))||\mathbb{D}(\phi^h)||\ ||\mathbb{D}(\eta)|| + \sum_{j=1}^{J}\beta||\phi^h\cdot\hat{\tau}_j||_{\Gamma_j}\ ||\eta\cdot\hat{\tau}_j||_{\Gamma_j} +$$
$$+ C_s\overline{C}\delta^2 r||\mathbb{D}(\eta)||_{L^3}||\mathbb{D}(\phi^h)||_{L^3}.$$

Remark. If $\tilde{w}$ is taken to be the Stokes projection of (w, q) into V^h then the term "$Re^{-1}||\mathbb{D}(\phi^h)||\ ||\mathbb{D}(\eta)||$" on this last RHS does not occur.

Inserting these two bounds in (4.2) and using the Cauchy-Schwarz and Young's inequalities gives:

$$\frac{1}{2}\frac{d}{dt}||\phi^h||^2 + \underline{C}C_s\delta^2||\mathbb{D}(\phi^h)||^3_{L^3} + \alpha||\nabla\cdot\phi^h||^2 + (2Re^{-1} + a_0(\delta))||\mathbb{D}(\phi^h)||^2 +$$
$$+\sum_{j=1}^{J}\beta||\phi^h\cdot\hat{\tau}_j||^2_{\Gamma_j} \leq$$
$$\leq |b(w,w,\phi^h) - b(w^h,w^h,\phi^h)| + \frac{1}{2}||\eta_t||^2 + \frac{1}{2}||\phi^h||^2 + \frac{1}{2}(2Re^{-1} + a_0(\delta))||\mathbb{D}(\phi^h)||^2$$
$$+\frac{1}{2}(2Re^{-1} + a_0(\delta))||\mathbb{D}(\eta)||^2 + \sum_{j=1}^{J}\left(\frac{\beta}{2}||\phi^h\cdot\hat{\tau}_j||^2_{\Gamma_j} + \frac{\beta}{2}||\eta\cdot\hat{\tau}_j||^2_{\Gamma_j}\right) + \frac{\epsilon_1}{3}\delta^2||\mathbb{D}(\phi^h)||^3_{L^3} +$$
$$+\frac{2}{3}\epsilon_1^{-1/2}\delta^2\overline{C}^{3/2}C_s^{3/2}r^{3/2}||\mathbb{D}(\eta)||^{3/2}_{L^3} + \frac{\alpha}{2}||\nabla\cdot\phi^h||^2 + \frac{1}{2\alpha}||q-\lambda^h||^2.$$

Picking $\epsilon_1 = \underline{C}C_s$ and collecting terms gives:

(4.3)
$$\frac{1}{2}\frac{d}{dt}||\phi^h||^2 + \frac{2}{3}\underline{C}C_s\delta^2||\mathbb{D}(\phi^h)||^3_{L^3} + \frac{\alpha}{2}||\nabla\cdot\phi^h||^2 +$$
$$\frac{1}{2}(2Re^{-1} + a_0(\delta))||\mathbb{D}(\phi^h)||^2 + \sum_{j=1}^{J}\frac{1}{2}\beta||\phi^h\cdot\hat{\tau}_j||^2_{\Gamma_j} \leq |b(w,w,\phi^h) - b(w^h,w^h,\phi^h)| +$$
$$\frac{1}{2}||\eta_t||^2 + \frac{1}{2}(2Re^{-1} + a_0(\delta))||\mathbb{D}(\eta)||^2 + \sum_{j=1}^{J}\frac{1}{2}\beta||\eta\cdot\hat{\tau}_j||^2_{\Gamma_j} +$$
$$\frac{2}{3}\underline{C}^{-1/2}C_s\overline{C}^{3/2}r^{3/2}\delta^2||\mathbb{D}(\eta)||^{3/2}_{L^3} + \frac{1}{2}\alpha^{-1}||q-\lambda^h||^2$$
$$+\frac{1}{2}||\phi^h||^2.$$

This is the basic differential inequality for the error. Three cases will be considered, revolving around the treatment of the first term on the RHS of (4.3).

Remark. If $\alpha = 0$ a completely analogous estimate holds with the pressure term modified to be either nonuniform in Re (e.g., "$C(2Re^{-1} + a_0(\delta))^{-1}||q-\lambda^h||^2_{-1}$") or nonuniform in δ.

Consider the convection terms:

$$b(w,w,\phi^h) - b(w^h,w^h,\phi^h) = b(w,\eta-\phi^h,\phi^h) + b(\eta-\phi^h,w^h,\phi^h). \tag{4.4}$$

The terms containing "η" shall be bounded first. Consider $b(w,\eta,\phi^h)$ and $b(\eta,w^h,\phi^h)$. Using the inequalities in Section 2 appropriately gives:

$$|b(w,\eta,\phi^h)| = \left|\frac{1}{2}[(w\cdot\nabla\eta,\phi^h) - (w\cdot\nabla\phi^h,\eta)]\right|$$
$$\leq \frac{1}{2}\left[||\phi^h||\ ||\nabla\eta||_{L^{s'}}||w||_{L^s} + ||\nabla\phi^h||_{L^3}||w||_{L^q}||\eta||_{L^{q'}}\right]$$

where $\frac{1}{2}+\frac{1}{s'}+\frac{1}{s}=1$ and $\frac{1}{3}+\frac{1}{q}+\frac{1}{q'}=1$. Picking $s'=3, s=6, q=2$ and $q'=6$ gives:

$$
\begin{aligned}
|b(w,\eta,\phi^h)| &\le \frac{1}{2}[||\phi^h||\ ||\nabla\eta||_{L^3}||w||_{L^6}+||\nabla\phi^h||_{L^3}||w||\ ||\eta||_{L^6}] \\
(4.5;a) \qquad &\le \frac{1}{4}||\phi^h||^2+\frac{1}{4}||w||^2_{L^6}||\nabla\eta||^2_{L^3} \\
&+\frac{1}{2}\frac{\epsilon_3}{3}||\nabla\phi^h||^3_{L^3}+\frac{2}{3}\cdot\frac{1}{2}\epsilon_3^{-1/2}||w||^{3/2}||\eta||^{3/2}_{L^6}.
\end{aligned}
$$

The term $b(\eta,w^h,\phi^h)$ is bounded similarly as follows:
(4.5; b)

$$
\begin{aligned}
&|b(\eta,w^h,\phi^h)| = |\frac{1}{2}[(\eta\cdot\nabla w^h,\phi^h)-(\eta\cdot\nabla\phi^h,w^h)]| \le \frac{1}{2}||\nabla w^h||_{L^3}||\eta||_{L^6}||\phi^h|| \\
&+\frac{1}{2}||\nabla\phi^h||_{L^3}||\eta||_{L^6}||w^h|| \le \frac{1}{4}||\nabla w^h||^2_{L^3}||\phi^h||^2+\frac{1}{4}||\eta||^2_{L^6}+ \\
&+\frac{1}{2}\frac{\epsilon_3}{3}||\nabla\phi^h||^3_{L^3}+\frac{2}{3}\cdot\frac{1}{2}\epsilon_3^{-1/2}||\eta||^{3/2}_{L^6}||w^h||^{3/2}.
\end{aligned}
$$

Korn's inequality and the stability bound of Section 3 immediately imply that $\mathbb{D}(w^h)\in L^3(0,T;L^3)$ uniformly in Re so that $||\nabla w^h||^2_{L^3}\in L^1(0,T)$, uniformly in Re. Thus, this bound suffices for a later application of Gronwall's inequality.

The first term containing only $\phi^h, b(w,\phi^h,\phi^h)$, is zero due to skew symmetry. Thus, there only remains the term $b(\phi^h,w^h,\phi^h)$. Estimating the term $b(\phi^h,w^h,\phi^h)$ is the essential, core difficulty in obtaining an error bound which is uniform in Re. There are only a few natural ways to bound this using Hölder's inequality and the Sobolev embedding theorem. There are two cases in which the analysis is successful (i) $a_0\neq 0$ and ∇w less regular in time, (ii) $a_0=0$ and ∇w very regular, $\nabla w\in L^2(0,T;L^\infty(\Omega))$. There is one important case (iii) in which the analysis fails: $a_0=0$ and ∇w having absolutely minimal time regularity.

To highlight subsequent analysis and, hopefully, spur further study, we shall first present this general case (iii) and explain the failure of the analysis.

If we assume only that $\nabla w\in L^3(\Omega)$, there is no need to add and subtract terms since á priori bounds on $||\nabla w^h||_{L^3}$ have been proven which are uniform in Re. Thus, we can use Hölder's inequality to write:

$$
\begin{aligned}
|b(\phi^h,w^h,\phi^h)| &= \left|\frac{1}{2}(\phi^h\cdot\nabla w^h,\phi^h)-\frac{1}{2}(\phi^h\cdot\nabla\phi^h,w^h)\right| \le \\
&\le \frac{1}{2}||\nabla w^h||_{L^3}||\phi^h||_{L^s}||\phi^h||_{L^{s'}}+\frac{1}{2}||\nabla\phi^h||_{L^3}||\phi^h||\ ||w^h||_{L^6},
\end{aligned}
$$

where $\frac{1}{3}+\frac{1}{s'}+\frac{1}{s}=1$ and $1\le s',s\le\infty$. Thus, picking $s'=2,\ s=6$ and using the embedding $W^{1,3}(\Omega)\to L^6(\Omega)$ gives:
(4.5; c)

$$
\begin{aligned}
&|b(\phi^h,w^h,\phi^h)| \le \frac{C(\Omega)}{2}||\nabla w^h||_{L^3}||\phi^h||\ ||\phi^h||_{1,3}+\frac{C(\Omega)}{2}||\nabla\phi^h||_{L^3}||\phi^h||\ ||w^h||_{1,3} \le \\
&+\frac{\epsilon}{6}||\phi^h||^3_{1,3}+C\frac{1}{3}\epsilon^{-1/2}||\nabla w^h||^{3/2}_{L^3}||\phi^h||^{3/2}+\frac{\epsilon}{6}||\nabla\phi^h||^3_{L^3}+C\frac{1}{3}\epsilon^{-1/2}||w^h||^{3/2}_{1,3}||\phi^h||^{3/2}.
\end{aligned}
$$

Remark. Using Lemma 2.4 instead of the embedding of $W^{1,3} \to L^6$ changes the critical exponent on $||\phi^h||$ "3/2" to 12/7 but not the final conclusion.

Combining (4.5;a,b,c) gives an initial bound on the convection term's difference:

$$
\begin{aligned}
&|b(w,w,\phi^h) - b(w^h,w^h,\phi^h)| \le ||\phi^h||^2\{\frac{1}{4} + \frac{1}{4}||\nabla w^h||^2_{L^3}\}+ \\
&+ ||\phi^h||^{3/2}\{\frac{1}{3}\epsilon^{-1/2}||\nabla w^h||^{3/2}_{L^3} + \frac{1}{3}\epsilon^{-1/2}||w^h||^{3/2}_{1,3}\}+ \\
&+ \frac{\epsilon}{2}||\nabla\phi^h||^3_{L^3} + ||\nabla\eta||^2_{L^3}\frac{1}{4}||w||^2_{L^6}+ \\
&||\eta||^{3/2}_{L^6}\{\frac{1}{3}\epsilon^{-1/2}||w||^{3/2} + \frac{1}{3}\epsilon^{-1/2}||w^h||^{3/2}\} + \frac{1}{4}||\eta||^2_{L^6} \\
&+ \frac{\epsilon}{6}||\phi^h||^3_{1,3}.
\end{aligned}
\tag{4.6}
$$

Thus, for all $a_0(\delta)=0$,

$$\frac{1}{2}\,\frac{d}{dt}||\phi^h||^2+\underline{C}C_s\delta^2||\mathbb{D}(\phi^h)||^3_{L^3}+\alpha||\nabla\cdot\phi^h||^2$$
$$+\,2Re^{-1}||\mathbb{D}(\phi^h)||^2+\sum_{j=1}^{J}\beta||\phi^h\cdot\hat{\tau}_j||^2_{\Gamma_j}\le$$
$$\le Re^{-1}||\mathbb{D}(\phi^h)||^2+Re^{-1}||\mathbb{D}(\eta)||^2+\sum_{j=1}^{J}\frac{\beta}{2}||\phi^h\cdot\hat{\tau}_j||^2_{\Gamma_j}+$$
$$+\sum_{j=1}^{J}\frac{\beta}{2}||\eta\cdot\hat{\tau}_j||^2_{\Gamma_j}+\frac{\epsilon_2}{3}\delta^2||\mathbb{D}(\phi^h)||^3_{L^3}$$
$$+\,\frac{2}{3}\epsilon_2^{-1/2}\delta^2\overline{C}^{3/2}r^{3/2}||\mathbb{D}(\eta)||^{3/2}_{L^3}+$$
$$+\left\{\frac{1}{4}+\frac{1}{4}||\nabla w^h||^2_{L^3}\right\}||\phi^h||^2+$$
$$+\,||\phi^h||^{3/2}\left\{\frac{1}{3}\epsilon^{-1/2}||\nabla w^h||^{3/2}_{L^3}+\frac{1}{3}\epsilon^{-1/2}||w^h||^{3/2}_{W^{1,3}}\right\}$$
$$+\,\frac{\epsilon}{2}||\nabla\phi^h||^3_{L^3}$$
$$+\,||\nabla\eta||^2_{L^3}\left\{\frac{1}{4}||w||^2_{L^6}\right\}+||\eta||^{3/2}_{L^6}\left\{\frac{1}{3}\epsilon^{-1/2}||w||^{3/2}+\frac{1}{3}\epsilon^{-1/2}||w^h||^{3/2}\right\}$$
$$+\,\frac{1}{4}||\eta||^2_{L^6}+\frac{1}{2}||\eta_t||^2+\frac{1}{2}||\phi^h||^2$$
$$+\,\frac{\alpha}{2}||\nabla\cdot\phi^h||^2+\frac{1}{2\alpha}||q-\lambda^h||^2+\frac{\epsilon}{6}||\phi^h||^3_{1,3}.$$

Since $X^h\subset X$ and meas $(\Gamma_0)>0$, applying Korn's inequality and collecting terms gives:

$$\frac{1}{2}\frac{d}{dt}||\phi^h||^2+(\underline{C}C_s\delta^2-\frac{1}{3}\epsilon_2\delta^2-\frac{2}{3}\epsilon C_K)||\mathbb{D}(\phi^h)||^3_{L^3}+$$
$$+\,Re^{-1}||\mathbb{D}(\phi^h)||+\frac{\alpha}{2}||\nabla\cdot\phi^h||^2+$$
$$+\sum_{j=1}^{J}\frac{1}{2}\beta||\phi^h\cdot\hat{\tau}_j||^2_{\Gamma_j}\le\left\{Re^{-1}||\mathbb{D}(\eta)||^2+\right.$$
$$+\sum_{j=1}^{J}\frac{1}{2}\beta||\eta\cdot\hat{\tau}_j||^2_{\Gamma_j}+\frac{2}{3}\epsilon_2^{-1/2}\delta^2\overline{C}^{3/2}r^{3/2}||\mathbb{D}(\eta)||^{3/2}_{L^3}+$$
$$+\,\frac{1}{4}||\eta||^2_{L^6}+\frac{1}{2}||\eta_t||^2+\frac{1}{4}||w||^2_{L^6}||\nabla\eta||^2_{L^3}+$$
$$\left.+\,(\frac{1}{3}\epsilon^{-1/2}||w||^{3/2}+\frac{1}{3}\epsilon^{-1/2}||w^h||^{3/2})||\eta||^{3/2}_{L^6}+\frac{1}{2\alpha}||q-\lambda^h||^2\right\}+$$
$$+\,\{\frac{3}{4}+\frac{1}{4}||\nabla w^h||^2_{L^3}\}||\phi^h||^2+\{\frac{1}{3}\epsilon^{-1/2}||\nabla w^h||^{3/2}_{L^3}+\frac{1}{3}\epsilon^{-1/2}||w^h||^{3/2}_{W^{1,3}}\}||\phi^h||^{3/2}.$$

Thus, pick $\epsilon_2=\underline{C}C_s$ (hence independent of δ) and ϵ so that $\frac{2}{3}\epsilon C_K=\frac{1}{3}\underline{C}C_s\delta^2$, i.e.,

$\epsilon = \frac{1}{2}\underline{C}C_s C_K^{-1}\delta^2$ $(= 0(\delta^2))$. This gives

$$\frac{1}{2}\frac{d}{dt}||\phi^h||^2 + \underline{C}C_s\delta^2||\mathbb{D}(\phi^h)||^3_{L^3} + Re^{-1}||\mathbb{D}(\phi^h)||^2 + \frac{\alpha}{2}||\nabla\cdot\phi^h||^2 + \sum_{j=1}^{J}\frac{1}{2}\beta||\phi^h\cdot\hat{\tau}_j||_{\Gamma_j} \leq$$

$$+\Bigg\{Re^{-1}||\mathbb{D}(\eta)||^2 + \sum_{j=1}^{J}\frac{1}{2}\beta||\eta\cdot\hat{\tau}_j||^2_{\Gamma_j} + \frac{2}{3}\underline{C}C_s^{-1/2}\overline{C}^{3/2}\delta^2 r^{3/2}||\mathbb{D}(\eta)||^{3/2}_{L^3} + \frac{1}{4}||\eta||^2_{L^6} + \frac{1}{2}||\eta_t||^2 +$$

$$\frac{1}{4}||w||^2_{L^6}||\nabla\eta||^2_{L^3} + \Bigg(\frac{1}{3}(\frac{2}{3})^{-1/2}\underline{C}^{-1/2}C_s C_K^{1/2}\delta^{-1}||w||^{3/2} +$$

$$+\frac{1}{3}(\frac{2}{3})^{-1/2}C^{-1/2}C_s C_K^{1/2}\delta^{-1}||w^h||^{3/2}\Bigg)\cdot||\eta||^{3/2}_{L^6} + \frac{1}{2\alpha}||q-\lambda^h||^2\Bigg\} +$$

$$+\{\frac{3}{4} + \frac{1}{4}||\nabla w^h||^2_{L^3}\}||\phi^h||^2 + \left(\frac{2}{3}\right)^{1/2}\underline{C}^{-1/2}C_s C_K^{1/2}\delta^{-1}||w^h||^{3/2}_{W^{1,3}}\}\cdot||\phi^h||^{3/2}.$$

Consider the three terms bracketed on this R.H.S. The first is approximation theoretic; the second is an L^1 function multiplying $||\phi^h(t)||^2$; the third is an L^1 function multiplying $\frac{1}{2}||\phi^h(t)||^{3/2}$. Let $y(t) := ||\phi^h(t)||^2$. This inequality may then be written as:

$$\frac{d}{dt}y(t) + \text{ (nonnegative terms) } \leq C(t)h^\gamma + a(t)y(t) + b(t)\delta^{-1}y^{3/4}(t),$$

where $a(t), b(t) \in L^1(0,T)$.

The final step would normally be to apply Gronwall's inequality to deduce $y(t) = \frac{1}{2}||\phi^h(t)||^2$ to be bounded by its initial values and approximation theoretic terms. Unfortunately, the term $y^{3/4}$ is not Lipschitz, so the argument fails at this last step.

Tracing the inequalities backward, the problem term arises from the steps used to bound $b(\phi^h, w^h, \phi^h)$ to obtain Re independence. The error analysis in the successful cases (i) and (ii) centers therefore on alternate bounds for this term. We shall first consider case (i).

Remark. If the estimate in (4.5;c) is improved as noted in the last remark the term $y(t)^{3/4}$ is changed to $y(t)^{6/7}$ but the final conclusion still holds.

Theorem 4.1. *(i) Assume $\alpha > 0$ and either $a_0(\delta) > 0$ for $\delta > 0$. Let*

$$a^h(t) := 1 + ||\mathbb{D}(w^h)||^2_{L^3} + a_0||\mathbb{D}(w^h)||^2_{L^3} + a_0^{-1/2}\alpha^{-3/2}||\mathbb{D}(w^h)||^3_{L^3}.$$

Then, there is a $C_1 = C_1(\delta)$, independent of Re and h, such that

$$||a^h(t)||_{L^1(0,T)} \leq C_1(\delta).$$

Further, there is a $C_2 = C_2(\delta)$, independent of Re and h, such that

$$\delta^{-1}(||w^h||^{3/2}_{L^\infty(0,T;L^2(\Omega))} + ||w||^{3/2}_{L^\infty(0,T;L^2(\Omega))}) \leq C_2(\delta).$$

(ii) Given $v^h \in V^h$, let $\rho = \rho(w, v^h) = \max\{\|\mathbb{D}(w)\|_{L^3}, \|\mathbb{D}(v^h)\|_{L^3}\}$. Then, the error $w - w^h$ satisfies:

$$
\begin{aligned}
&||w - w^h||^2_{L^\infty(0,T;L^2)} + \delta^2||\mathbb{D}(w - w^h)||^3_{L^3(0,T;L^3)} \\
&+ \alpha||\nabla \cdot w^h||^2_{L^2(0,T;L^2)} + (2Re^{-1} + Ca_0(\delta))||\mathbb{D}(w - w^h)||^2_{L^2(0,T;L^2)} \leq \\
&\leq ||(w - w^h)(0)||^2 + C \inf_{v^h \in V^h \cap W^{1,3}(\Omega)^d, \lambda^h \in Q^h} \Big\{ ||w - v^h||_{L^\infty(0,T;L^2)} + \\
&+ \delta^2||\mathbb{D}(w - w^h)||^3_{L^3(0,T;L^3)} + \\
&+ \exp(C_1(\delta)T)\Big[||\eta_t||^2_{L^2(0,T;L^2)} + (2Re^{-1} + Ca_0(\delta))||\mathbb{D}(w - v^h)||^2_{L^2(0,T;L^2)} + \\
&+ \sum_{j=1}^{J} \beta||(w - v^h) \cdot \hat{\tau}_j||^2_{L^2(0,T;L^2(\Gamma_j))} + \rho^{3/2}\delta^2||\mathbb{D}(w - v^h)||^{3/2}_{L^{3/2}(0,T;L^3)} \\
&+ ||w - v^h||^2_{L^2(0,T;L^6)} + \alpha^{-1}||q - \lambda^h||^2_{L^2(0,T;L^2)} + \\
&+ C_2(\delta)||w - v^h||^{3/2}_{L^{3/2}(0,T;L^6)} + \\
&+ \int_0^T ||w||^2_{L^6}||\mathbb{D}(w - v^h)||^2_{L^3} dt' \Big] \Big\}.
\end{aligned}
$$

(iii) In the last term of the error estimate above: $||w||^2_{L^6}||\mathbb{D}(w)||^2_{L^3} \in L^1(0, T)$, uniformly in h and Re, provided either $w \in L^{12/5}(0, T; L^6(\Omega))$ or $\mathbb{D}(w) \in L^{18/5+\epsilon}(0, T; L^3(\Omega))$, for some $\epsilon > 0$.

Proof. Consider the case when $a = a_0(\delta) > 0$ for $\delta > 0$. This analysis follows closely the previous discussion except for the treatment of the $b(\phi^h, w^h, \phi^h)$ term and the final application of Gronwall's inequality.

Consider therefore $b(\phi^h, w^h, \phi^h)$. Integration by parts and using the fact that $\phi^h \cdot \hat{n} = 0$ on Γ gives:

$$
\begin{aligned}
b(\phi^h, w^h, \phi^h) &:= \frac{1}{2}(\phi^h \cdot \nabla w^h, \phi^h) - \frac{1}{2}(\phi^h \cdot \nabla \phi^h, w^h) = \\
&= (\phi^h \cdot \nabla w^h, \phi^h) + \frac{1}{2}(\nabla \cdot \phi^h, \phi^h \cdot w^h) \\
&\leq ||\nabla w^h||_{L^3}||\phi^h||^2_{L^3} + \frac{1}{2}|(\nabla \cdot \phi^h, \phi^h \cdot w^h)|.
\end{aligned}
$$

Using the embedding $L^3 \hookrightarrow H^{1/2}$ in $d = 3$ and 2 and Young's inequality gives

$$
\leq \frac{\epsilon}{2}||\nabla \phi^h||^2 + \frac{1}{2\epsilon}C||\nabla w^h||^2_{L^3}||\phi^h||^2 + \frac{1}{2}|(\nabla \cdot \phi^h, \phi^h \cdot w^h)|. \tag{4.7}
$$

Consider now the last term on the above RHS. By Hölder's inequality:

$$
|(\nabla \cdot \phi^h, \phi^h \cdot w^h)| \leq ||\nabla \cdot \phi^h||\ ||\phi^h||_{L^{r'}}||w^h||_{L^s},
$$

where $\dfrac{1}{r'} + \dfrac{1}{s} = \dfrac{1}{2}$. Thus,

$$
|(\nabla \cdot \phi^h, w^h \cdot \phi^h)| \leq \frac{\alpha}{4}||\nabla \cdot \phi^h||^2 + \alpha^{-1}||\phi^h||^2_{L^{r'}}||w^h||^2_{L^s}. \tag{4.8}
$$

The Sobolev embedding theorem implies that for any $s, 1 \leq s < \infty$ in 3 or 2 dimensions $L^s(\Omega) \hookrightarrow W^{1,3}(\Omega)$. Thus,

$$||w^h||^2_{L^s} \leq C(s,\Omega)\ ||w^h||^2_{W^{1,3}(\Omega)} \leq C(s,\Omega)||\mathbb{D}(w^h)||^2_{L^3}.$$

This implies that for any $r' > 2$

$$|(\nabla \cdot \phi^h, w^h \cdot \phi^h)| \leq \frac{\alpha}{4}||\nabla \cdot \phi^h||^2 + C(r',\Omega)\alpha^{-1}||\phi^h||^2_{L^{r'}}||\mathbb{D}(w^h)||^2_{L^3}.$$

Consider the last term on the above RHS. The Sobolev embedding theorem also implies

$$||\phi^h||_{L^{r'}} \leq C(r',\Omega)||\phi^h||_{H^t(\Omega)}, \text{ for } t > \frac{3}{2} - \frac{3}{r'}.$$

(The final result is not improved by applying here instead the Gagliardo-Nirenberg inequality.) As $r' \to 2, t \to 0$ in this inequality. Thus, picking $r' = r'(t) > 2$ close enough to 2 implies, using an embedding inequality and Korn's inequality,

$$\begin{aligned} &||\phi^h||^2_{L^{r'}}||\mathbb{D}(w^h)||^2_{L^3} \leq C(t,\Omega)||\phi^h||^2_t||\mathbb{D}(w^h)||^2_{L^3} \leq \\ &\leq C(t,\Omega)||\phi^h||^{2(1-t)}\ ||\mathbb{D}(\phi^h)||^{2t}||\mathbb{D}(w^h)||^2_{L^3}, \end{aligned}$$

for any $t > 0$. Thus, for these values of r' and s

$$\frac{1}{\alpha}||\phi^h||^2_{L^r}||w^h||^2_{L^s} \leq \frac{C}{\alpha}||\mathbb{D}(\phi^h)||^{2t}||\phi^h||^{2(1-t)}||\mathbb{D}(w^h)||^2_{L^3}$$

for any $t > 0$. For conjugate exponents $q = 3$ and $q' = \dfrac{3}{2}$ we then have:

$$\begin{aligned} \frac{1}{2\alpha}||\phi^h||^2_{L^{r'}}||w^h||^2_{L^s} \leq \frac{\epsilon}{3}||\mathbb{D}(\phi^h)||^{6t} + \\ + \epsilon^{-1/2}\alpha^{-3/2}C||\phi^h||^3(1-t)||\mathbb{D}(w^h)||^3_{L^3}. \end{aligned}$$

Picking $t = \dfrac{1}{3} > 0$ gives for these values of r' and s:

$$\begin{aligned} \frac{1}{\alpha}||\phi^h||^2_{L^{r'}}||w^h||^2_{L^s} \leq \frac{\epsilon}{3}||\mathbb{D}(\phi^h)||^2 + \\ + C(r',s,t,\Omega)\epsilon^{-1/2}\alpha^{-3/2}||\phi^h||^2||\mathbb{D}(w^h)||^3_{L^3}. \end{aligned}$$

Using this bound, (4.7) and (4.8) gives finally:

$$\begin{aligned} b(\phi^h, w^h, \phi^h) \leq \frac{\epsilon_1}{2}||\nabla\phi^h||^2 + \frac{C}{2\epsilon_1}||\nabla w^h||^2_{L^3}||\phi^h||^2 + \\ + \frac{\alpha}{8}||\nabla\cdot\phi^h||^2 + \frac{\epsilon_2}{6}||\mathbb{D}(\phi^h)||^2 + C\epsilon_2^{-1/2}\alpha^{-3/2}||\mathbb{D}(w^h)||^3_{L^3}||\phi^h||^2. \end{aligned}$$

Remark. It appears on first consideration that this last term $(\nabla \cdot \phi^h, w^h \cdot \phi^h)$ can be agreeably bounded more directly and easily by:

$$\begin{aligned} |(\nabla\cdot\phi^h, w^h\cdot\phi^h)| &\leq C\|\nabla\cdot\phi^h\|\ \|\nabla w^h\|\ \|\phi^h\|^{1/2}\ \|\nabla\phi^h\|^{1/2} \\ &\leq C||\nabla\phi^h||^{3/2}||\phi^h||^{1/2}||\nabla w^h|| \leq \epsilon||\nabla\phi^h||^2 + C(\epsilon)||\nabla w^h||^4||\phi^h||^2. \end{aligned}$$

This bound, while certainly true, is not sufficient because of the condition that inevitably arises from using it that w^h or $w \in L^4(0,T;H^1(\Omega))$. The extra work in the bound we use reduces the time regularity requirements arising from this term to $w^h \in L^3(0,T;W^{1,3}(\Omega))$ (which is bounded uniformly in Re by problem data in Section 3).

Substituting this bound for $b(\phi^h, w^h, \phi^h)$ in the derivation of the upper estimate (4.6) for the difference of the convection terms gives:

(4.9)

$$
\begin{aligned}
|b(w,w,\phi^h) - b(w^h,w^h,\phi^h)| &\leq ||\phi^h||^2\left\{\frac{1}{4} + \frac{1}{4}C||\mathbb{D}(w^h)||_{L^3}^2 + \epsilon_1^{-1}C||\mathbb{D}(w^h)||_{L^3}^2 + \right. \\
&\left. C\epsilon_2^{-1/2}\alpha^{-3/2}||\mathbb{D}(w^h)||_{L^3}^3\right\} + ||\mathbb{D}(\phi^h)||_{L^3}^3\{C\epsilon_3\} \\
&+ ||\nabla\cdot\phi^h||^2\ \frac{\alpha}{8} + ||\mathbb{D}(\phi^h)||^2\left\{\frac{\epsilon_2}{3} + C\ \frac{\epsilon_1}{2}\right\} + \\
&||\nabla\eta||_{L^3}^2\left\{\frac{1}{4}||w||_{L^6}^2\right\} + ||\eta||_{L^6}^{3/2}\left\{\frac{1}{3}\epsilon_3^{-1/3}(||w||^{3/2} + ||w^h||^{3/2})\right\} \\
&+ \frac{1}{4}||\eta||_{L^6}^2,
\end{aligned}
$$

where C will be a generic constant independent of h, Re, δ.

To proceed further, (4.9) is inserted in the RHS of (4.3). This yields the differential inequality

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\phi^h||^2 + \left[\frac{2}{3}\underline{C}C_s\delta^2 - C\epsilon_3\right]||\mathbb{D}(\phi^h)||_{L^3}^3 + \\
&\frac{3}{8}\alpha||\nabla\cdot\phi^h||^2 + \left[\frac{1}{2}(2Re^{-1} + a_0(\delta)) - \frac{\epsilon_2}{3} - C\frac{\epsilon_1}{2}\right]||\mathbb{D}(\phi^h)||^2 + \\
&+ \sum_{j=1}^{J}\frac{1}{2}\beta||\phi^h\cdot\hat{\tau}_j||_{\Gamma_j}^2 \leq \frac{1}{2}||\eta_t||^2 + \frac{1}{2}(2Re^{-1} + a_0(\delta))||\mathbb{D}(\eta)||^2 + \\
&+ \sum_{j=1}^{J}\frac{1}{2}\beta||\eta\cdot\hat{\tau}_j||_{\Gamma_j}^2 + C||w||_{L^6}^2||\mathbb{D}(\eta)||_{L^3}^2 + \\
&\frac{2}{3}\underline{C}^{-1/2}C_s\overline{C}^{3/2}\delta^2 r^{3/2}||\mathbb{D}(\eta)||_{L^3}^{3/2} + \\
&+ \left[\frac{1}{3}\epsilon_3^{-1/2}(||w||^{3/2} + ||w^h||^{3/2})\right]||\eta||_{L^6}^{3/2} + \frac{1}{4}||\eta||_{L^6}^2 + \frac{1}{2}\alpha^{-1}||q - \lambda^h||^2 + \\
&+ \left[\frac{3}{4} + \frac{1}{4}C||\mathbb{D}(w^h)||_{L^3}^2 + \epsilon_1^{-1}C||\mathbb{D}(w^h)||_{L^3}^2 + \right. \\
&\left. + C\epsilon_2^{-1/2}\alpha^{-3/2}||\mathbb{D}(w^h)||_{L^3}^3\right]||\phi^h||^2.
\end{aligned}
\tag{4.10}
$$

In the first bracketed coefficient pick $\epsilon_3 = O(\delta^2)$ so that

$$\left[\frac{2}{3}\underline{C}C_s\delta^2 - C\epsilon_3\right] = C\ \delta^2.$$

This makes $\frac{1}{3}\epsilon_3^{-1/2} = C\ \delta^{-1}$. Recalling that this analysis is for the case $a_0 = a_0(\delta) > 0$, pick ϵ_1 and ϵ_2 so that

$$\frac{1}{2}a_0 - \frac{\epsilon_2}{3} - C\ \frac{\epsilon_1}{2} = C\ a_0,$$

i.e. $\epsilon_{1,2} = C\ a_0$. These choices simplify (4.10) to:

(4.11)

$$\begin{aligned}
&\frac{d}{dt}||\phi^h||^2 + C\delta^2||\mathbb{D}(\phi^h)||^3_{L^3} + \frac{3}{4}\alpha||\nabla\cdot\phi^h||^2 \\
&+ (2Re^{-1} + C\ a_0(\delta))||\mathbb{D}(\phi^h)||^2 + \sum_{j=1}^{J}\beta||\phi^h\cdot\hat{\tau}_j||^2_{\Gamma_j} \leq \\
&+ ||\eta_t||^2 + (2Re^{-1} + a_0(\delta))||\mathbb{D}(\eta)||^2 + \sum_{j=1}^{J}\beta||\eta\cdot\hat{\tau}_j||^2_{\Gamma_j} + \\
&+ C||w||^2_{L^6}||\mathbb{D}(\eta)||^2_{L^3} + C\delta^2 r^{3/2}||\mathbb{D}(\eta)||^{3/2}_{L^3} + \\
&+ C\delta^{-1}(||w^h||^{3/2} + ||w||^{3/2})||\eta||^{3/2}_{L^6} + \frac{1}{2}||\eta||^2_{L^6} + \alpha^{-1}||q-\lambda^h||^2 + \\
&+ C\left[1 + ||\mathbb{D}(w^h)||^2_{L^3} + a_0^{-1}||\mathbb{D}(w^h)||^2_{L^3} + a_0^{-1/2}(\delta)\alpha^{-3/2}||\mathbb{D}(w^h)||^3_{L^3}\right]\cdot||\phi^h||^2.
\end{aligned}$$

Before applying Gronwall's inequality, let us first verify that it will indeed give us an error bound that is uniform in the Reynolds number by considering the coefficients on the RHS of (4.11).

By the stability estimates $||w|| \in L^\infty(0,T)$ and $||w^h|| \in L^\infty(0,T)$, uniformly in Re, so the sixth term on the RHS is no problem. Consider the (critical) bracketed coefficient of the last term on the RHS. We must show this coefficient is in $L^1(0,T)$ uniformly in Re. Indeed, by the stability estimates $||\mathbb{D}(w^h)||_{L^3}$ and $||\mathbb{D}(w)||_{L^3} \in L^3(0,T)$, uniformly in Re. Since $T < \infty, L^3(0,T) \subset L^2(0,T)$, and thus

$$a^h(t) := \left[1 + ||\mathbb{D}(w^h)||^2_{L^3} + a_0||\mathbb{D}(w^h)||^2_{L^3} + a_0^{-1/2}\alpha^{-3/2}||\mathbb{D}(w^h)||^3_{L^3}\right] \in L^1(0,T)$$

uniformly in Re, which is the other half of (i).

The last interesting term on the RHS of (4.10) which must be in $L^1(0,T)$ to apply Gronwall's lemma is $||w||^2_{L^6}||\mathbb{D}(\eta)||^2_{L^3}$. The most straightforward approach is to use the Sobolev embedding theorem to bound $||w||_{L^6} \leq C||\mathbb{D}(w)||_{L^3}$ and Korn's inequalities.

Thus, $||w||^2_{L^6} \in L^{3/2}(0,T)$ uniformly in Re by the stability estimates. By Hölder's inequality, $||w||^2_{L^6}||\mathbb{D}(\eta)||^2_{L^3} \in L^1(0,T)$ provided $||\mathbb{D}(\eta)||_{L^3} \in L^6(0,T)$, which is more than we would like to require. (It would be better to simply assume $||w||_{L^6} \in L^{12/5}(0,T)$.)

To improve this, use the Gagliardo-Nirenberg inequalities in equation (2.4) with $v = w$. Noting that $||w||_{L^2} \in L^\infty(0,T)$ uniformly in Re, we thus have by (2.4):

$$||w||^2_{L^6}||\mathbb{D}(\eta)||^2_{L^3} \leq C(\delta)||\mathbb{D}(w)||^{4/3+2\epsilon}_{L^{3-O(\epsilon)}}||\mathbb{D}(\eta)||^2_{L^3}.$$

Now $||\mathbb{D}(w)||_{L^3} \in L^3(0,T)$, uniformly in Re, by the stability bounds in Section 3. Thus, for $\epsilon > 0$ small enough $||\mathbb{D}(w)||^{4/3+2\epsilon}_{L^{3-O(\epsilon)}} \in L^q(\Omega)$, for any $q < 9/4$. Thus, by Hölder's inequality

$$||w||^2_{L^6}||\mathbb{D}(\eta)||^2_{L^3} \in L^1(0,T) \text{ provided } ||\mathbb{D}(\eta)||_{L^3} \in L^{q'}(0,T)$$

for some $q' > 18/5$. This (implicit) condition that $||\mathbb{D}(w)||_{L^3} \in L^{18/5+\epsilon}(0,T)$ is much less restrictive than assuming it to be in $L^6(0,T)$. This assumption (or the one serving the same purpose ensuring this term to be $L^1(0,T)$) is precisely part (iii) in Theorem 4.1.

Gronwall's lemma now implies

$$
\begin{aligned}
&||\phi^h(t)||^2 + C\delta^2 ||\mathbb{D}(\phi^h)||^3_{L^3(0,T;L^3)} + \frac{3}{4}\alpha ||\nabla \cdot \phi^h||^2_{L^2(0,T;L^2)} + \\
&+ (2Re^{-1} + C\ a_0(\delta))||\mathbb{D}(\phi^h)||^2_{L^2(0,T;L^2)} + \sum_{j=1}^{J} \beta ||\phi^h \cdot \hat{\tau}_j||^2_{L^2(0,T;L^2(\Gamma_j))} \leq \\
&\leq ||\phi^h(0)||^2 + \exp(||a^h(t)||_{L^1(0,T)}t)\Bigg[\int_0^t (||\eta_t||^2 + \\
&+ (2Re^{-1} + a_0(\delta))||\mathbb{D}(\eta)||^2 + \sum_{j=1}^{J} \beta ||\eta \cdot \hat{\tau}_j||^2_{\Gamma_j} + C\delta^2 r^{3/2} ||\mathbb{D}(\eta)||^{3/2}_{L^3} + \\
&+ C||w||^2_{L^6} ||\mathbb{D}(\eta)||^2_{L^3} + \frac{1}{2}||\eta||^2_{L^6} + \alpha^{-1}||q - \lambda^h||^2)dt' + \\
&+ C\delta^{-1}(||w^h||^{3/2}_{L^\infty(0,T;L^2)} + ||w||^{3/2}_{L^\infty(0,T;L^2)}) \int_0^t ||\eta||^{3/2}_{L^6} dt'\Bigg].
\end{aligned}
$$

Since we will pick $\tilde{w}$ so that $||\mathbb{D}(\tilde{w})||_{L^3} \leq C||\mathbb{D}(w)||_{L^3}$, $r \leq C||\mathbb{D}(w)||_{L^3}$. As $w - w^h = \eta - \phi^h$, the triangle inequality completes the proof of Theorem 4.1.

(iii) The case $a_0(\delta) \geq 0$ and smoother w.

We now consider the case of smoother w, i.e.,

$$
w \in L^\infty(0,T;W^{1,\infty}(\Omega)),
$$

allowing for the case $a_0(\delta) \equiv 0$. This case is primarily of interest because many tests involve "academic" flow fields given in closed form (as in Section 5). These are typically smooth and bounded. In this case Theorem 4.2 gives an error estimate with constants independent of Re (but depending on δ and α). It is noteworthy in this estimate that *multiplicative constants* depend on δ but the *rate constant* in the (inevitable) exponential term takes the form

$$
\exp(C_3(w)T), \quad C_3 = C_3(||w||_{L^2(0,T;W^{1,\infty}(\Omega))}),
$$

with no *explicit* dependence on δ. $\Box$

Theorem 4.2. *Suppose $a_0(\delta) \geq 0, \alpha > 0$ and $w \in L^\infty(0,T;W^{1,\infty}(\Omega))$, uniformly in Re. Let $a(t) := 2 + ||\nabla w||^2_{L^\infty} + \frac{1}{4}\alpha^{-1}||w||^2_{L^\infty}$, then there is a $C_3 = C_3(w)$ such that*

$$
||a(t)||_{L^1(0,T)} \leq C_3(w).
$$

Let $C_4 = C_4(\delta)$ be such that

$$
||\mathbb{D}(w^h)||_{L^3(0,T;L^3)} \leq C_4(\delta).
$$

Let $\rho = \rho(w, v^h) = \max\{\|\mathbb{D}(w)\|_{L^3}, \|\mathbb{D}(v^h)\|_{L^3}\}$. Then, the error $w - w^h$ satisfies:

$$
\begin{aligned}
&||w - w^h||^2_{L^\infty(0,T;L^2)} + \delta^2 ||\mathbb{D}(w - w^h)||^3_{L^3(0,T;L^3)} + \\
&+ \alpha ||\nabla \cdot w^h||^2_{L^2(0,T;L^2)} + (2Re^{-1} + a_0(\delta))||\mathbb{D}(w - w^h)||^2_{L^2(0,T;L^2)} \leq \\
&\leq ||(w - w^h)(0)||^2 + C \inf_{\substack{v^h(t) \in V^h \\ \lambda^h(t) \in Q^h}} \Bigg\{ ||w - v^h||_{L^\infty(0,T;L^2)} + \\
&\delta^2 ||\mathbb{D}(w - w^h)||^3_{L^3(0,T;L^3)} + \exp(C_3(w)T) \Bigg[||w_t - v^h_t||^2_{L^\infty(0,T;L^2)} + \\
&+ (2Re^{-1} + a_0(\delta))||\mathbb{D}(w - v^h)||^2_{L^2(0,T;L^2)} + \sum_{j=1}^{J} \beta ||(w - v^h) \cdot \hat{\tau}_j||^2_{L^2(0,T;L^2(\Gamma_j))} + \\
&+ \delta^2 \rho^{3/2} ||\mathbb{D}(w - w^h)||^{3/2}_{L^{3/2}(0,T;L^3)} + \alpha^{-1} ||q - \lambda^h||^2_{L^2(0,T;L^2)} \\
&+ ||\nabla w||^2_{L^\infty(0,T,L^\infty)} ||w - v^h||^2_{L^2(0,T;L^2)} + ||w||^2_{L^\infty(0,T;L^\infty)} ||\nabla \cdot (w - v^h)||_{L^2(0,T;L^2)} \\
&C_4 \delta^2 ||w - v^h||^2_{L^6(0,T;L^6)} + C(\epsilon) C_4 \delta^{4/3-\epsilon} ||w - v^h||^2_{L^{18/5}(0,T;L^3)} \Bigg] \Bigg\}.
\end{aligned}
$$

Proof. In this case, the difference in the nonlinear terms is decomposed a bit differently as:

$$
\begin{aligned}
&|b(w, w, \phi^h) - b(w^h, w^h, \phi^h)| = |b(\eta - \phi^h, w, \phi^h) + b(w^h, \eta - \phi^h, \phi^h)| = \\
&= |b(\eta, w, \phi^h) - b(\phi^h, w, \phi^h) + b(w^h, \eta, \phi^h)|.
\end{aligned}
\tag{4.12}
$$

Consider the individual terms on the RHS of (4.12):

$$
\begin{aligned}
|b(\eta, w, \phi^h)| &= |\frac{1}{2}(\eta \cdot \nabla w, \phi^h) - \frac{1}{2}(\eta \cdot \nabla \phi^h, w)| = \\
&= |(\eta \cdot \nabla w, \phi^h) + \frac{1}{2}(\nabla \cdot \eta, \phi^h \cdot w)| \leq \\
&\leq \frac{1}{2} ||\nabla w||^2_{L^\infty} ||\eta||^2 + \frac{3}{4} ||\phi^h||^2 + \frac{1}{4} ||w||^2_{L^\infty} ||\nabla \cdot \eta||^2 \\
|b(\phi^h, w, \phi^h)| &= |(\phi^h \cdot \nabla w, \phi^h) + \frac{1}{2}(\nabla \cdot \phi^h, w \cdot \phi^h)| \leq \\
&\leq ||\nabla w||_{L^\infty} ||\phi^h||^2 + \frac{\alpha}{4} ||\nabla \cdot \phi^h||^2 + \frac{1}{4\alpha} ||w||^2_{L^\infty} ||\phi^h||^2 \\
|b(w^h, \eta, \phi^h)| &= |(w^h \cdot \nabla \eta, \phi^h) + \frac{1}{2}(\nabla \cdot w^h, \eta \cdot \phi^h)| \leq \\
&\leq ||w^h||_{L^6} ||\nabla \eta||_{L^3} ||\phi^h|| + ||\nabla \cdot w^h||_{L^3} ||\eta||_{L^6} ||\phi^h|| \\
&\leq C(||w^h||^2_{L^6} ||\nabla \eta||^2_{L^3} + ||\mathbb{D}(w^h)||^2_{L^3} ||\eta||^2_{L^6}) + \frac{3}{4} ||\phi^h||^2
\end{aligned}
$$

Combining these three estimates gives:

$$
\begin{aligned}
&|b(w, w, \phi^h) - b(w^h, w^h, \phi^h)| \leq (\frac{1}{2} ||\nabla w||^2_{L^\infty} ||\eta||^2 \\
&+ \frac{1}{4} ||w||^2_{L^\infty} ||\nabla \cdot \eta||^2 + C ||w^h||^2_{L^6} ||\nabla \eta||^2_{L^3} + \\
&+ C ||\mathbb{D}(w^h)||^2_{L^3} ||\eta||^2_{L^6} + \frac{\alpha}{4} ||\nabla \cdot \phi^h||^2 + \\
&+ (\frac{3}{2} + ||\nabla w||^2_{L^\infty} + \frac{1}{4\alpha} ||w||^2_{L^\infty}) ||\phi^h||^2.
\end{aligned}
$$

The term $||w^h||_{L^6}$ is bounded using the Gagliardo-Nirenberg inequality as in the last proof:

$$\begin{aligned}
||w^h||_{L^6}^2 &\leq C||w^h||^{2/3-2\epsilon}||\mathbb{D}(w^h)||_{L^{3-O(\epsilon)}}^{4/3+2\epsilon} \\
&\leq C(\epsilon)||w^h||^{2/3-2\epsilon}||\mathbb{D}(w^h)||_{L^3}^{4/3+2\epsilon} \\
&\leq C(\epsilon,\delta)||\mathbb{D}(w^h)||_{L^3}^{4/3+2\epsilon}.
\end{aligned}$$

This gives:

$$\begin{aligned}
&|b(w,w,\phi^h) - b(w^h,w^h,\phi^h) \leq \frac{1}{2}||\nabla w||_{L^\infty}^2||\eta||^2 + \frac{1}{4}||w||_{L^\infty}^2||\nabla\cdot\eta||^2 + \\
&+ C(\epsilon,\delta)||\mathbb{D}(w^h)||_{L^3}^{4/3-2\epsilon}||\mathbb{D}(\eta)||_{L^3}^2 + C||\mathbb{D}(w^h)||_{L^3}^2||\eta||_{L^6}^2 + \\
&+ \frac{\alpha}{4}||\nabla\cdot\phi^h||^2 + (\frac{3}{2} + ||\nabla w||_{L^\infty}^2 + \frac{1}{4\alpha}||w||_{L^\infty}^2)||\phi^h||^2.
\end{aligned} \tag{4.13}$$

This bound is now inserted in the RHS of (4.3) giving:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\phi^h||^2 + \frac{2}{3}\underline{C}\ C_s\delta^2||\mathbb{D}(\phi^h)||_{L^3}^3 + \frac{\alpha}{4}||\nabla\cdot\phi^h||^2 + \\
&\frac{1}{2}(2Re^{-1} + a_0(\delta))||\mathbb{D}(\phi^h)||^2 + \sum_{j=1}^{J}\frac{1}{2}\beta||\phi^h\cdot\hat{\tau}_j||_{\Gamma_j}^2 \leq \\
&\leq \frac{1}{2}||\eta_t||^2 + \frac{1}{2}(2Re^{-1} + a_0(\delta))||\mathbb{D}(\eta)||^2 + \sum_{j=1}^{J}\frac{1}{2}\beta||\eta\cdot\hat{\tau}_j||_{\Gamma_j}^2 \\
&\frac{2}{3}\underline{C}^{-1/2}\ C_s^{-1/2}\ \overline{C}^{3/2}\delta^2 r^{3/2}||\mathbb{D}(\eta)||_{L^3}^{3/2} + \frac{1}{2}\alpha^{-1}||q-\lambda^h||^2 \\
&+ \frac{1}{2}||\nabla w||_{L^\infty}^2||\eta||^2 + \frac{1}{4}||w||_{L^\infty}^2||\nabla\cdot\eta||^2 + \\
&+ C(\epsilon,\delta)||\mathbb{D}(w^h)||_{L^3}^{4/3-2\epsilon}||\mathbb{D}(\eta)||_{L^3}^2 + \\
&+ C||\mathbb{D}(w^h)||_{L^3}^2||\eta||_{L^6}^2 + (2 + ||\nabla w||_{L^\infty}^2 + \frac{1}{4\alpha}||w||_{L^\infty}^2)||\phi^h||^2.
\end{aligned}$$

To apply Gronwall's inequality we need $\left(2 + ||\nabla w||_{L^\infty}^2 + \frac{1}{4\alpha}||w||_{L^\infty}^2\right) \in L^1(0,T)$, in other words

$$w \in L^2(0,T;W^{1,\infty}(\Omega)).$$

In the final result of Gronwall's Lemma, we must also verify that the resulting terms containing $||\mathbb{D}(w^h)||_{L^3}$ are bounded uniformly in Re. To this end, apply Hölder's inequality:

$$\begin{aligned}
&\int_0^T ||\mathbb{D}(w^h)||_{L^3}^{4/3-2\epsilon}||\mathbb{D}(\eta)||_{L^3}^2\,dt \leq \\
&\leq ||\mathbb{D}(w^h)||_{L^{q(4/3-2\epsilon)}(0,T;L^3)}^{4/3-2\epsilon}||\mathbb{D}(\eta)||_{L^{2q'}(0,T;L^3)}^2,
\end{aligned}$$

where $\dfrac{1}{q}+\dfrac{1}{q'}=1$. From the stability estimates, we clearly must take q, q' so that $q\left(\dfrac{4}{3}-2\epsilon\right)\leq 3$. Accordingly, take $q=\dfrac{9}{4}, q'=\dfrac{9}{5}$. This gives:

$$
\begin{aligned}
&\int_0^T ||\mathbb{D}(w^h)||_{L^3}^{4/3-2\epsilon}||\mathbb{D}(\eta)||_{L^3}^2\,dt \\
&\leq C||\mathbb{D}(w^h)||_{L^3(0,T;L^3)}^{4/3-2\epsilon}||\mathbb{D}(\eta)||^2_{L^{18/5}(0,T;L^3)}.
\end{aligned}
\tag{4.14}
$$

Similarly, for q and q' conjugate exponents: $q=\dfrac{3}{2},\ q'=3$

$$
\begin{aligned}
&\int_0^T ||\mathbb{D}(w^h)||_{L^3}^2||\eta||_{L^6}^2 dt \leq ||\mathbb{D}(w^h)||^2_{L^{2q}(0,T;L^3)}||\eta||^2_{L^{2q'}(0,T;L^6)} \\
&\leq ||\mathbb{D}(w^h)||^2_{L^3(0,T;L^3)}||\eta||^2_{L^6(0,T;L^6)}.
\end{aligned}
$$

The stated error estimate now follows from the above Gronwall's inequality and the triangle inequality. $\Box$

5. A Numerical Example.

To give a numerical illustration several decisions must be made, mainly to work on an 'academic' flow problem with a known exact solution or a more realistic flow problem containing the accompanying uncertainties. Since our aim is to illustrate a convergence theorem we have chosen the former. (To assess a model or study the limitations of an algorithm we would naturally have chosen the latter.) Accordingly, we have selected the vortex decay problem of Chorin [C68], used also by others, e.g., Tafti [T96]. The domain is $\Omega=(0,1)^2$ and choose

$$
\begin{aligned}
w_1 &= -\cos(n\pi x)\sin(n\pi y)\exp(-2n^2\pi^2 t/\tau), \\
w_2 &= \sin(n\pi x)\cos(n\pi y)\exp(-2n^2\pi^2 t/\tau), \\
q &= -\frac{1}{4}(\cos(2n\tau x)+\cos(2n\pi y))\exp(-4n^2\pi^2/\tau).
\end{aligned}
\tag{5.1}
$$

For the relaxation time $\tau = Re$ this is a solution of the Navier-Stokes equation consisting of an array of opposite signed vortices which decay as $t\to\infty$. The right hand side f, initial condition and non-homogeneous Dirichlet boundary conditions are chosen so that this (w_1, w_2, q) is the closed form solution of (1.2).

Since we are studying convergence as $h\to 0$ for δ fixed and Re varying we have accordingly chosen

$$
\begin{aligned}
\text{Relaxation time } \tau &= 1000, \\
\text{Vortex configuration } n &= 4, \\
\text{Final time } T &= 20, \\
\text{Eddy Scale } \delta &= 0.1, \\
\text{Smagorinski constant } C_s &= .0648.
\end{aligned}
$$

It is significant that $\delta = 0.1 \leq \dfrac{1}{4}=\dfrac{1}{n}$ so that the vortices are larger than $O(\delta)$ and hence should be "visible" to the model. Tables 5.2 to 5.4 give the error in L^∞

$(0,T;L^2(\Omega))$ and show uniformity in Re and in $L^2(0,T;H^1(\Omega))$ and show weak dependence on Re (since this term in the error estimate is scaled by Re^{-1}).

The least squares constant α is chosen to be zero. Time discretization is by the fractional step θ-method with the indicated time steps. The tables show a decrease of the error as h decreases until it reaches the error introduced by the time stepping procedure. The spacial discretization is done on a uniform square mesh with the Q_2/P_1^{disc} (or Q_2/P_{-1} as denoted in [GS98]) element with the associated meshwidths indicated. The viscous term is treated not as $(\nabla w^h, \nabla v^h)$ but using the deformation tensor formulation, $(\mathbb{D}(w^h), \mathbb{D}(v^h))$ as analyzed herein. Both the Smagorinsky subgridscale model and the convection term are treated implicitly.

Using the above elements and meshes the calculations involved the numbers of degrees of freedom listed below in Table 5.1.

h	velocity d.o.f.	pressure d.o.f.	all d.o.f.
1/8	578	192	770
1/16	2178	768	2946
1/32	8450	3072	11522
1/64	33282	12288	45570

Table 5.1: degrees of freedom in space

These are certainly not extremely large numbers of degrees of freedom. However, their importance is only relative to the Reynolds numbers chosen $Re = 10^2, 10^3, 10^4$ and 10^5 and the resolution sought $\delta = 0.1$. Again, LES is focused on situations in which the number of degrees of freedom is small relative to Re. Thus, the chosen values of h and Re seem appropriate.

The Tables 5.2 – 5.4 present the $L^\infty(0,T;L^2)$ norms of the errors. Note that the trends are exactly as anticipated by the theory; there is none to minimal degradation in the error as Re increases from 10^2 to 10^5 and the error plateau's as h decreases at a value which seems to be the induced error in the time stepping procedure.

$Re \setminus h$	1/8	1/16	1/32	1/64
10^2	$2.1630e-2$	$2.6934e-3$	$1.2458e-3$	$1.2405e-3$
10^3	$3.0767e-2$	$3.1342e-3$	$1.0457e-3$	$1.0468e-3$
10^4	$3.4721e-2$	$4.5764e-3$	$1.1048e-3$	$1.0456e-3$
10^5	$3.6367e-2$	$4.9437e-3$	$1.1720e-3$	$1.0510e-3$

Table 5.2: $||e||_{L^\infty(0,T;L^2)}, \Delta t = 0.02$. Note the uniformity in Re.

$Re \setminus h$	1/8	1/16	1/32	1/64
10^2	$2.1834e-2$	$2.7110e-3$	$6.5885e-4$	$6.1954e-4$
10^3	$3.0992e-2$	$3.1718e-3$	$5.8792e-4$	$5.2093e-4$
10^4	$3.4921e-2$	$4.6586e-3$	$7.8972e-4$	$5.1248e-4$
10^5	$3.7104e-2$	$5.0251e-3$	$8.4838e-4$	$5.1295e-4$

Table 5.3: $||e||_{L^\infty(0,T;L^2)}, \Delta t = 0.01$. Again, observe the uniformity in Re.

$Re \setminus h$	$1/8$	$1/16$	$1/32$	$1/64$
10^2	$2.1914e-2$	$2.7336e-3$	$4.0862e-4$	$3.1028e-4$
10^3	$3.1108e-2$	$3.2122e-3$	$4.2189e-4$	$2.5970e-4$
10^4	$3.5017e-2$	$4.7230e-3$	$7.1411e-4$	$2.5217e-4$
10^5	$3.7464e-2$	$5.0893e-3$	$7.7364e-4$	$2.5169e-4$

Table 5.4: $||e||_{L^\infty(0,T;L^2)}, \Delta t = 0.005$.

Note the uniformity in Re and the reduction of the minimal error as Δt decreases.

The Tables 5.5 – 5.7 present the errors in $L^2(0,T;H^1)$. These are not predicted to be uniform in Re so some error degradation is expected as Re increases. Very mild degradation is indeed observed. The degradation is mild, possibly because

$$Re^{-1}||\mathbb{D}(e)||^2_{L^2(0,T;L^2)} \quad \text{and} \quad \delta^2||\mathbb{D}(e)||^3_{L^3(0,T;L^3)}$$

are predicted to have the same uniform in Re convergence rates. Thus, the theory forecasts $||\mathbb{D}(e)||_{L^2(0,T;L^2)}$ to increase as Re increases until it reaches the (slower) uniform convergence rate predicted by the $||\mathbb{D}(e)||_{L^3(0,T;L^3)}$ bound.

$Re \setminus h$	$1/8$	$1/16$	$1/32$	$1/64$
10^2	$1.4266e+0$	$3.6241e-1$	$9.3784e-2$	$3.3421e-2$
10^3	$1.8118e+0$	$4.2062e-1$	$1.0184e-1$	$3.7432e-2$
10^4	$2.5391e+0$	$5.2415e-1$	$1.3060e-1$	$6.7210e-2$
10^5	$2.8681e+0$	$5.5900e-1$	$1.4338e-1$	$8.3121e-2$

Table 5.5: $||De||_{L^2(0,T;L^2)}, \Delta t = 0.02$.

$Re \setminus h$	$1/8$	$1/16$	$1/32$	$1/64$
10^2	$1.4286e+0$	$3.6234e-1$	$9.1549e-2$	$2.5804e-2$
10^3	$1.8151e+0$	$4.2099e-1$	$9.9348e-2$	$2.7398e-2$
10^4	$2.5596e+0$	$5.2578e-1$	$1.2376e-1$	$4.0530e-2$
10^5	$2.8941e+0$	$5.6168e-1$	$1.3480e-1$	$4.8723e-2$

Table 5.6: $||De||_{L^2(0,T;L^2)}, \Delta t = 0.01$.

$Re \setminus h$	$1/8$	$1/16$	$1/32$	$1/64$
10^2	$1.4296e+0$	$3.6245e-1$	$9.1018e-2$	$2.3493e-2$
10^3	$1.8168e+0$	$4.2134e-1$	$9.8856e-2$	$2.4278e-2$
10^4	$2.5696e+0$	$5.2703e-1$	$1.2283e-1$	$3.0516e-2$
10^5	$2.9068e+0$	$5.6356e-1$	$1.3386e-1$	$3.4719e-2$

Table 5.7: $||De||_{L^2(0,T;L^2)}, \Delta t = 0.005$.

References

[A75] R.A. Adams, *Sobolev Spaces*, Academic Press, N.Y. (1975).

[ABF84] D. Arnold, F. Brezzi and M. Fortin, *A stable finite element for the Stokes equations*, Cacolo **21** (1984), 337-344.

[B83] J. Bardina, *Improved turbulence models based on large eddy simulation of homogeneous, incompressible turbulent flows*, Ph.D. thesis, Stanford University (1983).

[BDK82] G.A. Baker, V. Dougalis and O. Karakashian, *On a higher order accurate fully discrete Galerkin approximation to the Navier-Stokes equations*, Math. Comp. **39** (1982), 339-375.

[BFR80] J. Bardina, J.H. Ferziger and W. Reynolds, *Improved subgrid models for large eddy simulation*, AIAA (1980), 80-1357.

[BHMS99] F. Brezzi, P. Houston, D. Marini and E. Süli, *Modeling subgrid viscosity for advection-diffusion problems*, preprint (1999).

[C68] A.J. Chorin, *Numerical solution for the Navier-Stokes equations*, Math. Comp. **22** (1968), 745-762.

[CF88] P. Constantin and C. Foias, *Navier-Stokes Equations*, Univ. of Chicago Press (1988).

[CT87] M. Crouzeix and V. Thomée, *The stability in L^p and W_1^p of the L^2-projection onto finite element function spaces*, Math. Comp. **48** (1987), 521-532.

[D93] E. DiBenedetto, *Degenerate Parabolic Equations*, Springer, N.Y. (1993).

[DG91] Q. Du and M. Gunzburger, *Analysis of a Ladyzhenskaya model for incompressible viscous flow*, JMAA **155** (1991), 21-45.

[ESG82] M.S. Engleman, R.L. Sani and P.M. Gresho, *The implementation of normal and/or tangential boundary conditions in finite element codes for incompressible fluid flows*, IJNME **2** (1982), 225-238.

[F95] U. Frisch, *Turbulence, the Legacy of A.N. Kolmogorov*, Cambridge (1995).

[F72] G. Fichera, *Existence theorems in elasticity*, in Handbook de Physik, vol. IV. a/2, Springer, Berlin (1972).

[FP99] J. Ferziger and M. Peric, *Computational Methods for Fluid Dynamics*, 2nd edition, Springer, Berlin (1999).

[G99] J.-L. Guermond, *Stabisation par viscośité de sous-maille pour l'approximation de Galerkin des opérateurs linéaires monotines*, C.R.A.S. **328** (1999), 617-622.

[G98] J.-L. Guermond, *Subgrid stabilization of Galerkin approximations of monotone operators*, in: Proc. European Sci. Foundation Com. on Appl. Math. for Indust. Flow Problems, (AMIF), Spain (1998).

[G96] S. Ghosal, *An analysis of numerical errors in large eddy simulations of turbulence*, J. Comput. Phys. **125** (1996), 187-206.

[G89] M. Gunzburger, *Finite Element Methods for Viscous Incompressible Flows*, A Guide to Theory Practice and Algorithms, Academic Press (1989).

[G71] J. Gobert, *Sur une inégalité de coercivité*, J. Math. Anal. Appl. **36** (1971), 518-528.

[G62] J. Gobert, *Une inequation fondametale de la théorie de l'élasticité*, Bull. Soc. Roy. Sci. Liège, no. 3 and 4 (1962).

[G92] P. Grisvard, *Singularities in Boundary Value Problems*, Masson, Paris (1992).

[Ga99] G.P. Galdi, *Lectures in Mathematical Fluid Dynamics*, Ed: G.P. Galdi, J.G. Heywood and R. Rannacher, Birkhauser-Verlag, (1999) (to be published).

[GIL00] G.P. Galdi, T. Iliescu and W. Layton, *Mathematical analysis for a new large eddy simulation model*, technical report, University of Pittsburgh (2000).

[GL00] G.P. Galdi and W. Layton, *Approximating the larger eddies in fluid motion, II: A model for space filtered flow*, Math. Methods and Models in the Applied Sci. **10** (2000), 343-350.

[GPMC91] M. Germano, U. Piomelli, P. Moin and W. Cabot, *A dynamic subgridscale eddy viscosity model*, Physics Fluids A3 **7** (1991), 1760-1765.

[GR86] V. Girault and P.A. Raviart, *Finite Element Methods for the Navier-Stokes Equations*, Springer, Berlin (1986).

[GS98] P.M. Gresho and R.L. Sani, *Incompressible Flow and the Finite Element Method*, John Wiley and Sons, Chichester (1998).

[HMJ99] T.J. Hughes, L. Mazzei and K.E. Jansen, *Large eddy simulation and the variational multiscale method*, Comput. Visual Sci. **3** (2000), 47-59.

[HR82] J. Heywood and R. Rannacher, *Finite element approximation of the nonstationary Navier-Stokes problem I: Regularity of solutions and second order error estimates for spatial discretization*, SIAM JNA **19** (1982), 275-311.

[IJL00] T. Iliescu, V. John, W. Layton, G. Matthias and L. Tobiska, *An assessment of a classical and a new model for large eddy simulation*, tech. report (2000).

[IL99] T. Iliescu and W. Layton, *Approximating the larger eddies in fluid motion III: The Boussinesq model for turbulent fluctuations*, to appear in: Analele Stinfice ale Universitatii "Al. I. Cuz", Ser. Math. (1999).

[JL98] V. John and W. Layton, *Approximating the larger eddies in fluid motion, I: Direct simulation for the Stokes problem*, submitted (1999) (copy available at http://www.math.pitt.edu/∼wjl).

[KM97] A.G. Kravchenko and P. Moin, *On the effect of numerical errors in large eddy simulations of turbulent flows*, J. Comput. Phys. **131** (1997), 310-322.

[L70] O.A. Ladyzhenkaya, *Modification of the Navier-Stokes Equations for large velocity gradients*, BVP's of Math, Phys. and Related Aspects of Funct. Thy., Consultants Bureau, New York (1970).

[L69] O.A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, 2nd edition, Gordan and Breach (1968).

[L96] W. Layton, *A nonlinear subgridscale model for incompressible viscous flow problems*, SIAM J. Sci. Comput. **17** (1996), 347-357.

[Li99] A. Liakos, *Weak imposition of boundary conditions in the Stokes and the Navier-Stokes Equation*, Ph.D. thesis, Univ. of Pittsburgh (1999).

[MP93] B. Mohammadi and O. Pironneau, *Analysis of the $k-\epsilon$ turbulence model*, John Wiley and Sons, N.Y. (1993).

[N81] J.A. Nitsche, *On Korn's second inequality*, RAIRO Analyse Numerique **15** (1981), 237-248.

[N59] L. Nirenberg, *On elliptic partial differential equations*, Annali della Scuola Norm. Sup. **13** (1959), 115-162.

[P94] C. Parés, *Approximation de la solution des equations d'un modele de turbulence par une methode de Lagrange-Galerkin*, Revista de Matematícas Aplicadas **15** (1994), 63-124.

[P92] C. Parés, *Existence,uniqueness and regularity of solutions of the equations of a turbulence model for incompressible fluids*, Applicable Anal. **43** (1992), 245-296.

[RST96] H.-G. Roos, M. Stynes and L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations*, Springer, Berlin (1996).

[S00] N. Sahin, *New Perspectives on Boundary Conditions for Large Eddy Simulation*, tech. report, Math. Dept. University of Pittsburgh (2000).

[S98] P. Sagaut, *Introduction à la simulation des grandes échelles pour les écoulements de fluide incompressible*, Springer, Berlin (1998).

[S63] J. Smagorinsky, *General circulation experiments with the primitive equations*, Monthly Weath Review **91** (1963), 261-341.

[S59] J. Serrin, *Mathematical Principles of Classical fluid mechanics*, in: Encyclopedia of Physics, VIIII/1 (eds J. Flugge and C. Truesdell), Springer, Berlin (1959), 125-263.

[T96] D. Tafti, *Comparison of some upwind-biased high-order formulations with a second-order central-difference scheme for time integration of the incompressible Navier-Stokes equations*, Comp. and Fluids **25** (1996), 647-665.

[T83] R. Temam, *Problems Matematique en Plasticite*, Gauthier-Villars, Paris (1983).