THE POISSON PROBLEM IN A THREE-DIMENSIONAL PLATE

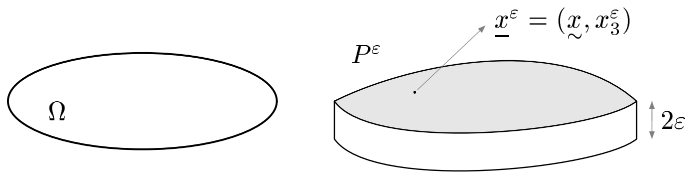
Alexandre L. Madureira

Laboratório Nacional de Computação Científica (LNCC)

Abstract. This note was the basis of a short course presented on June 23, 2000, at University of Pavia, Italy. Here we define and analyse dimension reduction models for the Poisson problem in a thin, three-dimensional plate. After introducing the Poisson problem, and a variational approach for dimension reduction, we present the asymptotic expansion for the exact and model solutions. Then we estimate the modeling error. This work is based on the thesis [3], see the web site http://www.math.psu.edu/dna/education.html#students

1 – Introduction

For a positive number $\varepsilon < 1$, define the three-dimensional plate $P^\varepsilon = \Omega \times (-\varepsilon, \varepsilon)$ and its boundaries $\partial P^\varepsilon_L = \partial\Omega \times (-\varepsilon, \varepsilon)$ and $\partial P^\varepsilon_\pm = \Omega \times \{-\varepsilon, \varepsilon\}$.



Assume that $u^\varepsilon \in H^1(P^\varepsilon)$ satisfies (in the weak sense)

$$\begin{aligned} \Delta u^\varepsilon &= -f^\varepsilon \qquad \text{in } P^\varepsilon, \\ \frac{\partial u^\varepsilon}{\partial n} &= 0 \qquad \text{on } \partial P^\varepsilon_\pm, \\ u^\varepsilon &= 0 \qquad \text{on } \partial P^\varepsilon_L, \end{aligned} \tag{1}$$

where $f^\varepsilon : P^\varepsilon \to \mathbb{R}$. In general, the solution of (1) will depend on ε in a nontrivial way. In fact the above problem is a singularly perturbed one, and as ε goes to zero it "loses" ellipticity. This causes the onset of boundary layers, as we make clear below.

It is possible to characterize the solution of (1) in an alternative way, as the minimizer of the associate energy functional, i.e.,

$$u^\varepsilon = \operatorname*{arg\,min}_{v \in V(P^\varepsilon)} \mathcal{J}(v), \text{ where } \mathcal{J}(v) = \frac{1}{2}\int_{P^\varepsilon} |\,\underline{\nabla}\, v|^2 \, d\underline{x} - \int_{P^\varepsilon} f^\varepsilon v \, d\underline{x},$$

This work has been partially supported by IAN-CNR, Italy

and $V(P^\varepsilon) = \{v \in H^1(P^\varepsilon) : v = 0 \text{ on } \partial P_L^\varepsilon\}$.

Aiming to find a "good" approximation for u^ε, we search for

$$u^\varepsilon(p) = \operatorname*{arg\,min}_{v \in \mathring{H}^1(\Omega;\mathbb{P}_p(-\varepsilon,\varepsilon))} \mathcal{J}(v), \tag{2}$$

where the notation is as follows. For an integer p and a positive real number a, we define $\mathbb{P}_p(-a,a)$ as the space of polynomials of degree p in $(-a,a)$. So $\mathring{H}^1(\Omega;\mathbb{P}_p(-a,a))$ denotes the space of polynomials of degree p with coefficients in $\mathring{H}^1(\Omega)$. The space $\mathring{H}^1(\Omega)$ is the set of functions in the usual Sobolev space $H^1(\Omega)$ with zero trace on $\partial\Omega$. It follows from its definition that $u^\varepsilon(p)$ is the Ritz projection of u^ε into $\mathring{H}^1(\Omega;\mathbb{P}_p(-\varepsilon,\varepsilon))$ and such model is a minimum energy one. Observe that the use of higher polynomial degrees yield higher order models, actually leading to a hierarchy of models that furnish increasingly better solutions.

Rewriting (2) in weak form, it is not hard to check that if $u^\varepsilon(1)(\underset{\sim}{x}) = \omega_0(\underset{\sim}{x}) + \omega_1(\underset{\sim}{x})x_3^\varepsilon$, then

$$\begin{gathered} \Delta_{2D}\,\omega_0 = -\frac{1}{2}f^0, \quad \frac{2\varepsilon^2}{3}\Delta_{2D}\,\omega_1 - 2\omega_1 = -f^1 \qquad \text{in } \Omega, \\ \omega_0 = \omega_1 = 0 \qquad \text{on } \partial\Omega, \end{gathered} \tag{3}$$

where $\Delta_{2D} = \partial_{11} + \partial_{22}$, and

$$f^0(\underset{\sim}{x}^\varepsilon) = \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} f^\varepsilon(\underset{\sim}{x}^\varepsilon, x_3^\varepsilon)\,dx_3^\varepsilon, \quad f^1(\underset{\sim}{x}^\varepsilon) = \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} f^\varepsilon(\underset{\sim}{x}^\varepsilon, x_3^\varepsilon)x_3^\varepsilon\,dx_3^\varepsilon. \tag{4}$$

Note that the equations (3) are independent of each other. We can express in a unique way any function defined on P^ε as a sum of its even and odd parts with respect to x_3^ε. The even part of f^ε appears only in the equation for ω_0, and the odd part of f^ε shows up in the equation for ω_1. Also, the equation determining ω_1 is singularly perturbed, but this is not the case for the equation determining ω_0. If higher order methods were used, we would have two singularly perturbed independent systems of equations, corresponding to the even and odd parts of $u^\varepsilon(p)$. Similar splitting also occurs in plate models for linearized elasticity, where, for an isotropic plate, the equations decouple into two independent problems corresponding to bending and stretching of a plate.

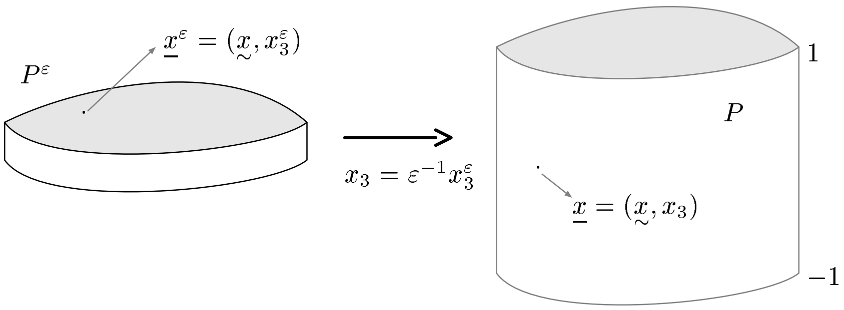
The natural question of how close $u^\varepsilon(p)$ is to u^ε is not easy to answer due to the complex influence of ε in both the original and model solutions. We resolve this, not by comparing the exact and model solutions directly, but rather by first looking at the difference between the solutions and their truncated asymptotic expansions, and then comparing both asymptotic expansions. This is possible because the same projection used to define each model can be used to find the first terms of the asymptotic expansion of the model. This allows us to compare corresponding terms of the expansions. Schematically, this is how it works:

$$\begin{array}{ccc} u^\varepsilon & \longleftrightarrow & \begin{array}{l}\text{Asymptotic}\\ \text{Expansion of } u^\varepsilon\end{array} \\ \updownarrow & & \updownarrow \\ u^\varepsilon(p) & \longleftrightarrow & \begin{array}{l}\text{Asymptotic}\\ \text{Expansion of } u^\varepsilon(p)\end{array} \end{array}$$

Next we develop then the asymptotic expansion for the exact solution, and then do the same for the model solution. Finally we compare them both to find upper bounds for the modeling error.

2 – The asymptotic expansion for the original 3D solution

Our first step to show the influence of ε explicitly is to rewrite (1) in the scaled domain $P = \Omega \times (-1, 1)$. Let $\partial P_L = \partial\Omega \times (-1, 1)$ and $\partial P_\pm = \Omega \times \{-1, 1\}$. Also, $\underline{x} = (\underset{\sim}{x}, x_3)$ is a typical point of P, with $\underset{\sim}{x} = \underset{\sim}{x}^\varepsilon$ and $x_3 = \varepsilon^{-1} x_3^\varepsilon$.



In this new domain we define $u(\varepsilon)(\underline{x}) = u^\varepsilon(\underline{x}^\varepsilon)$, and $f(\underline{x}) = f^\varepsilon(\underline{x}^\varepsilon)$. We conclude from (1) that

$$
\begin{aligned}
(\Delta_{2D} + \varepsilon^{-2}\partial_{33})u(\varepsilon) &= -f \qquad \text{in } P,\\
\frac{\partial u(\varepsilon)}{\partial n} &= 0 \qquad \text{on } \partial P_\pm,\\
u(\varepsilon) &= 0 \qquad \text{on } \partial P_L.
\end{aligned} \tag{5}
$$

We assume that f is independent of ε.

Consider the asymptotic expansion

$$
u(\varepsilon) \sim u^0 + \varepsilon^2 u^2 + \varepsilon^4 u^4 + \cdots . \tag{6}
$$

Formally substituting (6) in (5) and grouping together terms with same power in ε we have

$$
\begin{gathered}
\varepsilon^{-2}\partial_{33}u^0 + [\Delta_{2D}\, u^0 + \partial_{33}u^2] + \varepsilon^2 [\Delta_{2D}\, u^2 + \partial_{33}u^4] + \cdots = -f,\\
\frac{\partial u^0}{\partial n} + \varepsilon^2 \frac{\partial u^2}{\partial n} + \varepsilon^4 \frac{\partial u^4}{\partial n} + \cdots = 0 \text{ on } \partial P_\pm.
\end{gathered}
$$

It is then natural to require that

$$
\partial_{33}u^0 = 0, \tag{7}
$$

$$
\partial_{33}u^2 = -f - \Delta_{2D}\, u^0, \tag{8}
$$

$$
\partial_{33}u^{2k} = -\,\Delta_{2D}\, u^{2k-2}, \qquad \text{for all } k > 1, \tag{9}
$$

along with the boundary conditions

$$
\frac{\partial u^{2k}}{\partial n} = \delta_{k1} g \text{ on } \partial P_\pm, \qquad \text{for all } k \in \mathbb{N}. \tag{10}
$$

Equations (7)–(10) define a sequence of Neumann problems in the interval $x_3 \in (-1,1)$ parametrized by $\underset{\sim}{x} \in \Omega$. If the data for these problems is compatible then the solution can be written as

$$u^{2k}(\underline{x}) = \mathring{u}^{2k}(\underline{x}) + \zeta^{2k}(\underset{\sim}{x}), \qquad \text{for all } k \in \mathbb{N}, \tag{11}$$

where

$$\int_{-1}^{1} \mathring{u}^{2k}(\underset{\sim}{x}, x_3)\, dx_3 = 0, \tag{12}$$

with $\mathring{u}^{2k}$ uniquely determined, but ζ^{2k} an arbitrary function of $\underset{\sim}{x}$ only. From the Dirichlet boundary condition in (5), it would be natural to require that $u^{2k} = 0$ on ∂P_L. This is equivalent to imposing

$$\zeta^{2k} = 0 \text{ on } \partial\Omega, \tag{13}$$

$$\mathring{u}^{2k} = 0 \text{ on } \partial P_L. \tag{14}$$

However, in general, only (13) can be imposed and (14) will not hold. We shall correct this discrepancy latter. Now we show that the functions ζ^{2k}, $\mathring{u}^{2k}$ (and so u^{2k}) are uniquely determined from (7)–(13). In fact, (7) and (10) yields $\mathring{u}^0 = 0$. From the compatibility of (8) and (10) we see that

$$\Delta_{2D}\, \zeta^0(\underset{\sim}{x}) = -\frac{1}{2}\int_{-1}^{1} f(\underset{\sim}{x}, x_3)\, dx_3, \tag{15}$$

which together with (13), determines ζ^0 and then, from (11), u^0. In view of the compatibility condition (15), $\mathring{u}^2$ is fully determined by (8) and (10). Next, the Neumann problem (9), (10) admits a solution for $k > 1$ if and only if $\partial_{11}\zeta^{2k-2} = 0$. But in view of (13), this means $\zeta^{2k-2} = 0$, for $k > 1$, and then $\mathring{u}^{2k}$ is uniquely determined from (9), (10). Note that $u^0 = \zeta^0$ and $u^{2k} = \mathring{u}^{2k}$ for $k \geq 1$.

Observe that u^0 satisfies all the boundary conditions imposed since $\mathring{u}^0 = 0$ and so (14) holds for $k = 0$. In general this is not the case for u^2, u^4, etc, as they do not vanish on the lateral boundary of the domain (although their vertical integrals do). We introduce then, formally, the boundary corrector

$$U \sim \varepsilon^2 U^2 + \varepsilon^3 U^3 + \varepsilon^4 U^4 + \cdots, \tag{16}$$

to correct the values of u^2, u^4, etc. on ∂P_L. We expect also that

$$\begin{aligned} (\Delta_{2D} + \varepsilon^{-2}\partial_{33})U &= 0 \qquad \text{in } P, \\ \frac{\partial U}{\partial n} &= 0 \qquad \text{on } \partial P_{\pm}. \end{aligned} \tag{17}$$

We hope to pose a boundary corrector problem that is independent of ε. In a two-dimensional beam, it is enough to define a *stretched* coordinate in the horizontal direction and pose the corrector problem in the semi-infinite strip $\Sigma = \mathbb{R}^+ \times (-1,1)$. We proceed here analogously, using a new system of (boundary-fitted) horizontal coordinates. In this new system, a point close to the boundary $\partial\Omega$ has as coordinates its distance to $\partial\Omega$ and the arclength along the boundary. We are able then to define a horizontal stretched coordinate, in the normal direction, and pose, after some work, a sequence of problems in, once more, Σ. The geometry of Ω plays an important role, as we see below.

Following the notation of Chen [2], suppose that $\partial\Omega$ is arclength parametrized by $\utilde{z}(\theta) = (X(\theta), Y(\theta))$. Let $\utilde{s} = (X', Y')$, $\utilde{n} = (Y', -X')$ denote the tangent and the outward normal of $\partial\Omega$, and define

$$\Omega_b = \left\{ \utilde{z} - \rho\utilde{n} \,:\, \utilde{z} \in \partial\Omega,\, 0 < \rho < \rho_0 \right\},$$

where ρ_0 is a positive number smaller than the minimum radius of curvature of $\partial\Omega$. With L denoting the arclength of $\partial\Omega$, then

$$\utilde{x}(\rho, \theta) = \utilde{z}(\theta) - \rho\utilde{n}(\theta).$$

is a diffeomorphism between $(0, \rho_0) \times \mathbb{R}/L$ and Ω_b. Extending $\utilde{n}$ and $\utilde{s}$ to Ω_b by

$$\utilde{n}(\rho, \theta) = \utilde{n}(\theta), \qquad \utilde{s}(\rho, \theta) = \utilde{s}(\theta), \tag{18}$$

then, for $\alpha = 1, 2$:

$$\partial_\alpha\theta = \frac{s_\alpha}{J(\theta)}, \qquad \partial_\alpha\rho = -n_\alpha,$$

where $J(\rho, \theta) = 1 - \rho\kappa(\theta)$, and κ is the curvature of $\partial\Omega$. Finally, the change of coordinates yields

$$\partial_\alpha f = \partial_\theta f \partial_\alpha\theta + \partial_\rho f \partial_\alpha\rho, \qquad \text{for } \alpha = 1, 2.$$

The expression for the Laplacian in these new coordinates follows:

$$\begin{aligned} \Delta_{2D} U &= \partial_{\rho\rho} U - \frac{\kappa}{J}\partial_\rho U + \frac{1}{J^2}\partial_{\theta\theta} U + \frac{\rho\kappa'}{J^3}\partial_\theta U \\ &= \partial_{\rho\rho} U + \sum_{j=0}^{\infty} \rho^j \left(a_1^j \partial_\rho U + a_2^j \partial_{\theta\theta} U + a_3^j \partial_\theta U \right), \end{aligned} \tag{19}$$

where we formally replace each coefficient with its respective Taylor expansion, see [1], and

$$a_1^j = -[\kappa(\theta)]^{j+1}, \quad a_2^j = (j+1)[\kappa(\theta)]^j, \quad a_3^j = \frac{j(j+1)}{2}[\kappa(\theta)]^{j-1}\kappa'(\theta).$$

Defining the new variable $\hat\rho = \varepsilon^{-1}\rho$ and using the same name for functions different only up to this change of coordinates, we have from (19) that

$$\Delta_{2D} U = \varepsilon^{-2}\partial_{\hat\rho\hat\rho} U + \sum_{j=0}^{\infty} (\varepsilon\hat\rho)^j \left(a_1^j \varepsilon^{-1} \partial_{\hat\rho} U + a_2^j \partial_{\theta\theta} U + a_3^j \partial_\theta U \right), \tag{20}$$

Aiming to solve (17), we formally use (16) and (20), collect together terms with same order of ε and for $k \geq 2$, pose the following sequence of problems parametrized by θ:

$$
\begin{aligned}
(\partial_{\hat\rho\hat\rho} + \partial_{33})U^k &= F_k \qquad \text{in } \Sigma,\\
\frac{\partial U^k}{\partial n} &= 0 \qquad \text{on } \partial\Sigma_\pm,\\
U^k(0,\theta,x_3) &= u^k(0,\theta,x_3) \qquad \text{for } x_3 \in (-1,1),
\end{aligned} \tag{21}
$$

where

$$
F_k = \sum_{j=0}^{k-2} \hat\rho^j \left(a_1^j \partial_{\hat\rho} U^{k-j-1} + a_2^j \partial_{\theta\theta} U^{k-j-2} + a_3^j \partial_\theta U^{k-j-2} \right),
$$

The next lemma shows that U^k decays to zero exponentially. See [3] for a proof.

Lemma 1. *Let U^k be defined by (21) for any positive positive k. Then*

$$
\int_t^\infty \int_{-1}^1 (U^k)^2 + (\partial_{\hat\rho} U^k)^2 + (\partial_\theta U^k)^2 \, dx_3 \, d\hat\rho \leq c_k(f) e^{-\alpha_k t}.
$$

Combining (6) and the boundary layer expansion we write

$$
u^\varepsilon(\underline{x}^\varepsilon) \sim \zeta^0(\underset{\sim}{x}^\varepsilon) + \sum_{k=1}^\infty \varepsilon^{2k} u^{2k}(\underset{\sim}{x}^\varepsilon, \varepsilon^{-1}x_3^\varepsilon) - \chi(\rho) \sum_{k=2}^\infty \varepsilon^k U^k(\varepsilon^{-1}\rho, \theta, \varepsilon^{-1}x_3^\varepsilon), \tag{22}
$$

where $\chi(\rho)$ is a smooth cutoff function identically one if $0 \leq \rho \leq \rho_0/3$ and identically zero if $\rho \geq 2\rho_0/3$.

We omit many of the details in the results below. Using the convenient hypothesis that f is "smooth", we will bound their (arbitrarily high) Sobolev norms by a general constant $C(f)$. This allows a simplification of the estimates and no major loss of information occurs. We present the convergence estimates of the truncated asymptotic expansion in the $H^1(P^\varepsilon)$ norm without a proof. Let

$$
e_N = u^\varepsilon - \sum_{k=0}^N \varepsilon^{2k} u^{2k}(\underset{\sim}{x}^\varepsilon, \varepsilon^{-1}x_3^\varepsilon) + \chi(\rho) \sum_{k=2}^{2N} \varepsilon^k U^k(\varepsilon^{-1}\rho, \theta, \varepsilon^{-1}x_3^\varepsilon)
$$

Then we have the result below.

Theorem 2. *For any nonnegative integer N, there exists a constant $C(f)$ such that the difference between the truncated asymptotic expansion and the original solution measured in the original domain is bounded as follows:*

$$
\|e_0\|_{H^1(P^\varepsilon)} \leq C(f)\varepsilon^{3/2}, \quad \|e_N\|_{H^1(P^\varepsilon)} \leq C(f)\varepsilon^{2N+1}.
$$

Remark. *If $\int_{-1}^1 f(\underset{\sim}{x}, x_3)\, dx_3 \not\equiv 0$, then*

$$
\|u^\varepsilon\|_{H^1(P^\varepsilon)} \geq C(f)\varepsilon^{1/2},
$$

and then
$$\frac{\|e_0\|_{H^1(P^\varepsilon)}}{\|u^\varepsilon\|_{H^1(P^\varepsilon)}} = O(\varepsilon), \quad \frac{\|e_N\|_{H^1(P^\varepsilon)}}{\|u^\varepsilon\|_{H^1(P^\varepsilon)}} = O(\varepsilon^{2N+1/2}).$$

We compile in the table below the estimates for the error between the original solution and the truncated asymptotic expansion, for $N \geq 1$.

TABLE 1. Convergence rates of the truncated asymptotic expansion

	u^ε	BL	$e_N (N \geq 1)$	Relative Error
$\|\cdot\|_{L^2(P^\varepsilon)}$	$\varepsilon^{1/2}$	ε^3	$\varepsilon^{2N+2}(\varepsilon^{2N+5/2})$	$\varepsilon^{2N+3/2}(\varepsilon^{2N+2})$
$\|\partial_\rho \cdot\|_{L^2(P^\varepsilon)}$	$\varepsilon^{1/2}$	ε^2	$\varepsilon^{2N+1}(\varepsilon^{2N+5/2})$	$\varepsilon^{2N+1/2}(\varepsilon^{2N+2})$
$\|\partial_\theta \cdot\|_{L^2(P^\varepsilon)}$	$\varepsilon^{1/2}$	ε^3	$\varepsilon^{2N+2}(\varepsilon^{2N+5/2})$	$\varepsilon^{2N+3/2}(\varepsilon^{2N+2})$
$\|\partial_{x_3^\varepsilon} \cdot\|_{L^2(P^\varepsilon)}$	$\varepsilon^{3/2}$	ε^2	$\varepsilon^{2N+1}(\varepsilon^{2N+3/2})$	$\varepsilon^{2N-1/2}(\varepsilon^{2N})$
$\|\cdot\|_{H^1(P^\varepsilon)}$	$\varepsilon^{1/2}$	ε^2	$\varepsilon^{2N+1}(\varepsilon^{2N+3/2})$	$\varepsilon^{2N+1/2}(\varepsilon^{2N+1})$

3 – ASYMPTOTIC EXPANSION FOR $u^\varepsilon(p)$

To develop an asymptotic expansion for the solution of the hierarchical models, we reason as before, but use weak equations instead of their strong form. We start by posing a problem for the solution of the minimum energy model in the scaled domain P. If we define $u(p)(\underline{x}) = u^\varepsilon(p)(\underline{x}^\varepsilon)$, then

(23)
$$\int_P \underset{\sim}{\nabla}\, u(p)\, \underset{\sim}{\nabla}\, v + \varepsilon^{-2}\partial_3 u(p)\partial_3 v\, d\underline{x} = \int_P f v\, d\underline{x} \qquad \text{for all } v \in \mathring{H}^1(\Omega; \mathbb{P}_p(-1,1)).$$

Considering the asymptotic expansion

$$u^0(p) + \varepsilon^2 u^2(p) + \varepsilon^4 u^4(p) + \cdots, \tag{24}$$

and formally substituting it for $u(p)$ in (23), we can easily conclude that for all $v \in \mathring{H}^1(\Omega; \mathbb{P}_p(-1,1))$,

$$\begin{gathered} \int_P \partial_3 u^0(p)\partial_3 v\, d\underset{\sim}{x} = 0, \\ \int_P \partial_3 u^2(p)\partial_3 v\, d\underset{\sim}{x} = \int_P \big(f + \Delta_{2D}\, u^0(p)\big) v\, d\underset{\sim}{x}, \\ \int_P \partial_3 u^{2k}(p)\partial_3 v\, d\underset{\sim}{x} = \int_P \Delta_{2D}\, u^{2k-2}(p) v\, d\underset{\sim}{x}, \qquad \text{for } k > 1. \end{gathered} \tag{25}$$

Let $\hat{\mathbb{P}}_p(-1,1)$ be the space of polynomials of degree p in $(-1,1)$ with zero average. Repeating the arguments of the expansion for the exact solution, we set $u^0(p)(\underline{x}) = \zeta^0(\underset{\sim}{x})$ and $u^2(p)(\underset{\sim}{x},\cdot)$ as the Galerkin projection of $u^2(\underset{\sim}{x},\cdot)$ into $\hat{\mathbb{P}}_p(-1,1)$ for almost every $\underset{\sim}{x} \in \Omega$, i.e.,

(26)
$$\int_{-1}^1 \partial_3 u^2(p)(\underset{\sim}{x}, x_3)\partial_3 v(x_3)\, dx_3 = \int_{-1}^1 [f(\underset{\sim}{x}, x_3) + \Delta_{2D}\zeta^0(\underset{\sim}{x})] v(x_3)\, dx_3, \qquad \text{for all } v \in \hat{\mathbb{P}}_p(-1,1).$$

For any integer $k \geq 2$, we define $u^{2k}(p)(\underset{\sim}{x}, \cdot) \in \hat{\mathbb{P}}_p(-1,1)$ by

$$\int_{-1}^{1} \partial_3 u^{2k}(p)(\underset{\sim}{x}, x_3)\partial_3 v(x_3)\, dx_3 = \int_{-1}^{1} \Delta_{2D}\, u^{2k-2}(p)(\underset{\sim}{x}, x_3) v(x_3)\, dx_3, \tag{27}$$

for all $v \in \hat{\mathbb{P}}_p(-1,1)$, and for almost every $\underset{\sim}{x} \in \Omega$.

The ansatz (24) does not satisfy the Dirichlet boundary conditions at ∂P_L and we use then boundary correctors $U^k(p)$. These functions are polynomial in the transverse direction, and are defined in the semi-infinite strip Σ. We need to define the spaces

$$V(\Sigma, p) = \big\{ v \in \mathcal{D}'(\mathbb{R}^+; \mathbb{P}_p(-1,1)) \,:\, \| \underset{\sim}{\nabla} v\|_{L^2(\Sigma)} + \|v(0,\cdot)\|_{L^2(-1,1)} < \infty \big\},$$
$$V_0(\Sigma, p) = \big\{ v \in V(\Sigma, p) \,:\, v(0,\cdot) = 0 \big\}.$$

For any positive integer k, let $U^k(p) \in V(\Sigma, p)$ be the solutions of

$$\begin{gathered} \int_\Sigma \underset{\sim}{\nabla} U^k(p) \cdot \underset{\sim}{\nabla} v \, d\hat{\rho}\, dx_3 = \int_\Sigma F_k(p) v \, d\hat{\rho}\, dx_3 \qquad \text{for all } v \in V_0(\Sigma, p), \\ U^k(p)(0,\theta,x_3) = u^k(p)(0,\theta,x_3) \qquad \text{for all } x_3 \in (-1,1), \\ F_k(p) = \sum_{j=0}^{k-1} \hat{\rho}^j \left(a_1^j \partial_{\hat{\rho}} U^{k-j-1}(p) + a_2^j \partial_{\theta\theta} U^{k-j-2}(p) + a_3^j \partial_\theta U^{k-j-2}(p) \right), \end{gathered} \tag{28}$$

where $u^k = 0$ for k odd and $U^0(p) = U^1(p) = 0$.

A result guaranteeing existence, uniqueness and exponential decay holds for $U^k(p)$.

We have finally that

$$u^\varepsilon(p)(\underline{x}^\varepsilon) \sim \zeta^0(\underset{\sim}{x}^\varepsilon) + \sum_{k=1}^{\infty} \varepsilon^{2k} u^{2k}(p)(\underset{\sim}{x}^\varepsilon, \varepsilon^{-1} x_3^\varepsilon) - \chi(\rho) \sum_{k=2}^{\infty} \varepsilon^k U^k(p)(\varepsilon^{-1}\rho, \theta, \varepsilon^{-1} x_3^\varepsilon),$$

where ζ^0 solves (15).

We present next an estimate, in the $H^1(P^\varepsilon)$ norm, of $u^\varepsilon(p)$ minus its truncated asymptotic expansion. We would like to remark that this result gives a bound that is uniform in p.

Theorem 3. *For any positive integer N, let*

$$e_{2N}(p) = u^\varepsilon(p) - \sum_{k=0}^{N} \varepsilon^{2k} u^{2k}(p) + \chi(\rho) \sum_{k=2}^{2N} \varepsilon^k U^k(p).$$

Then there exists a constant $C(f)$ such that $\|e_{2N}(p)\|_{H^1(P^\varepsilon)} \leq C(f)\varepsilon^{2N+1}$, for all $p \in \mathbb{N}$.

4 – Estimating the error

We start by comparing some terms of the asymptotic expansion of both u^ε and $u^\varepsilon(p)$.

The first three estimates in the lemma below hold since $u^2(p)(\underset{\sim}{x},\cdot)$ is the Galerkin projection of $u^2(\underset{\sim}{x},\cdot)$ into $\hat{\mathbb{P}}_p(-1,1)$ for $\underset{\sim}{x}\in\Omega$. The convergence of the boundary correctors is more involved. It follows from the definition of U^2 and $U^2(p)$. The slow rate of convergence follows from the the singular behaviour of U^2 close to the corners of the plate (it has a singularity of the type $r^\gamma\log r$, for a certain power γ and where r indicates the distance to the corners).

Lemma 4. *For any nonnegative real number s, and for any arbitrarily small $\delta>0$, there exists a constant C such that*

$$\|u^2-u^2(p)\|_{L^2(P^\varepsilon)}\le C\varepsilon^{1/2}p^{-2-s}\|f\|_{L^2(\Omega;H^s(-1,1))},$$
$$\|\underset{\sim}{\nabla}u^2-\underset{\sim}{\nabla}u^2(p)\|_{L^2(P^\varepsilon)}\le C\varepsilon^{1/2}p^{-2-s}\|f\|_{H^1(\Omega;H^s(-1,1))},$$
$$\|\partial_{x_3^\varepsilon}u^2-\partial_{x_3^\varepsilon}u^2(p)\|_{L^2(P^\varepsilon)}\le C\varepsilon^{-1/2}p^{-1-s}\|f\|_{L^2(\Omega;H^s(-1,1))}.$$

Also, for the first boundary corrector,

$$\|\partial_{x_3^\varepsilon}\{\chi[U^2-U^2(p)]\}\|_{L^2(P^\varepsilon)}+\|\partial_\rho\{\chi[U^2-U^2(p)]\}\|_{L^2(P^\varepsilon)}\le C_\delta p^{-6+\delta}\|f\|_{L^2(\partial\Omega;H^5(-1,1))}.$$

Finally, we present the convergence results for the model defined by (2). Let $P_0^\varepsilon=\Omega_0\times(-\varepsilon,\varepsilon)$, where Ω_0 is an open domain such that $\bar{\Omega}_0\subset\Omega$.

Theorem 5. *For any nonnegative real number s, and for any arbitrarily small $\delta>0$, there exist constants C and $C(f)$ such that the error between u^ε and the approximation $u^\varepsilon(p)$ given by the SP(p) model is bounded as*

$$\|u^\varepsilon-u^\varepsilon(p)\|_{L^2(P^\varepsilon)}\le C\varepsilon^{5/2}p^{-2-s}\|f\|_{L^2(\Omega;H^s(-1,1))}+C(f)\varepsilon^3,$$
$$\|\partial_\rho[u^\varepsilon-u^\varepsilon(p)]\|_{L^2(P^\varepsilon)}\le C_\delta\varepsilon^2p^{-6+\delta}\|f\|_{L^2(\partial\Omega;H^5(-1,1))}+C(f)\varepsilon^{5/2},$$
$$\|\partial_\theta[u^\varepsilon-u^\varepsilon(p)]\|_{L^2(P^\varepsilon)}\le C\varepsilon^{5/2}p^{-2-s}\|f\|_{H^1(\Omega;H^s(-1,1))}+C(f)\varepsilon^3,$$
$$\|\underset{\sim}{\nabla}u^\varepsilon-\underset{\sim}{\nabla}u^\varepsilon(p)\|_{L^2(P_0^\varepsilon)}\le C\varepsilon^{5/2}p^{-2-s}\|f\|_{H^1(\Omega;H^s(-1,1))}+C(f)\varepsilon^{9/2},$$
$$\|\partial_{x_3^\varepsilon}u^\varepsilon-\partial_{x_3^\varepsilon}u^\varepsilon(p)\|_{L^2(P^\varepsilon)}\le C\varepsilon^{3/2}p^{-1-s}\|f\|_{L^2(\Omega;H^s(-1,1))}+C(f)\varepsilon^2,$$
$$\|u^\varepsilon-u^\varepsilon(p)\|_{H^1(P^\varepsilon)}\le C\varepsilon^{3/2}p^{-1-s}\|f\|_{L^2(\Omega;H^s(-1,1))}+C(f)\varepsilon^2.$$

Proof. We prove the fifth estimate. Using the triangle inequality, the following holds:

$$\begin{aligned}(29)\qquad &\|\partial_{x_3^\varepsilon}u^\varepsilon-\partial_{x_3^\varepsilon}u^\varepsilon(p)\|_{L^2(P^\varepsilon)}\le\|u^\varepsilon-\zeta^0-\varepsilon^2u^2+\varepsilon^2\chi U^2\|_{H^1(P^\varepsilon)}\\ &\quad+\varepsilon^2\big(\|\partial_{x_3^\varepsilon}u^2-\partial_{x_3^\varepsilon}u^2(p)\|_{L^2(P^\varepsilon)}+|\chi[U^2-U^2(p)]|_{H^1(P^\varepsilon)}\big)\\ &\quad+\|u^\varepsilon(p)-\zeta^0-\varepsilon^2u^2(p)+\varepsilon^2\chi U^2(p)\|_{H^1(P^\varepsilon)}.\end{aligned}$$

From Theorems 2 and 3, we have that

$$\|u^\varepsilon-\zeta^0-\varepsilon^2u^2+\varepsilon^2\chi U^2\|_{H^1(P^\varepsilon)}+\|u^\varepsilon(p)-\zeta^0-\varepsilon^2u^2(p)+\varepsilon^2\chi U^2(p)\|_{H^1(P^\varepsilon)}\le C(f)\varepsilon^3,$$

and from the exponential decay of the boudary correctors,

$$|\chi[U^2 - U^2(p)]|_{H^1(P^\varepsilon)} \leq C(f).$$

Finally we apply Lemma 4 to bound $\|\partial_{x_3^\varepsilon} u^2 - \partial_{x_3^\varepsilon} u^2(p)\|_{L^2(P^\varepsilon)}$, and substituting in (29) we have the result. The other estimates follow from similar arguments. $\square$

We summarize the convergence results in the table below. We present only the leading terms of the errors and in parenthesis we show interior estimates if those are better than the global ones.

We need the following notation. For a nonnegative real number s, let

$$a_s = \|f\|_{L^2(\Omega; H^s(-1,1))}, \quad a_s^1 = \|f\|_{H^1(\Omega; H^s(-1,1))}, \quad a^b = \|f\|_{L^2(\partial\Omega; H^5(-1,1))},$$

TABLE 2. Convergence estimates for the SP(p) models

	$u^\varepsilon - u^\varepsilon(p)$	Relative Error
$\|\cdot\|_{L^2(P^\varepsilon)}$	$\varepsilon^{5/2} p^{-2-s} a_s$	$\varepsilon^2 p^{-2-s} a_s$
$\|\partial_\rho \cdot\|_{L^2(P^\varepsilon)}$	$\varepsilon^2 p^{-6+\delta} a^b$ $(\varepsilon^{5/2} p^{-2-s} a_s^1)$	$\varepsilon^{3/2} p^{-6+\delta} a^b$ $(\varepsilon^2 p^{-2-s} a_s^1)$
$\|\partial_\theta \cdot\|_{L^2(P^\varepsilon)}$	$\varepsilon^{5/2} p^{-2-s} a_s^1$	$\varepsilon^2 p^{-2-s} a_s^1$
$\|\partial_{x_3^\varepsilon} \cdot\|_{L^2(P^\varepsilon)}$	$\varepsilon^{3/2} p^{-1-s} a_s$	$p^{-1-s} a_s$
$\|\cdot\|_{H^1(P^\varepsilon)}$	$\varepsilon^{3/2} p^{-1-s} a_s$	$\varepsilon p^{-1-s} a_s$

SECTION 5 – AN ALTERNATIVE VARIATIONAL APPROACH

We now present the SP$'(p)$ models for (1) and the results related to it. Let $V'(P^\varepsilon) = L^2(P^\varepsilon)$ and $\underline{S}_0'(P^\varepsilon) = \{\underline{\sigma} \in \underline{H}(\operatorname{div}, P^\varepsilon) : \underline{\sigma} \cdot \underline{n} = 0 \text{ on } \partial P_\pm^\varepsilon\}$. Then we have the following principle.

SP$'$: $(u^\varepsilon, \underline{\sigma}^\varepsilon)$ is the unique critical point of

$$L'(v, \underline{\tau}) = \frac{1}{2}\int_{P^\varepsilon} |\underline{\tau}|^2 \, d\underline{x}^\varepsilon + \int_{P^\varepsilon} f^\varepsilon v \, d\underline{x}^\varepsilon + \int_{P^\varepsilon} \operatorname{div} \underline{\tau} v \, d\underline{x}^\varepsilon$$

in $V'(P^\varepsilon) \times \underline{S}_0'(P^\varepsilon)$. Looking for critical points in the spaces $V'(P^\varepsilon, p) = \{v \in V'(P^\varepsilon) : \deg_3 v \leq p\}$ and $\underline{S}_0'(P^\varepsilon, p) = \{\underline{\tau} \in \underline{S}_0'(P^\varepsilon) : \deg_3 \underset{\sim}{\tau} \leq p, \deg_3 \tau_3 \leq p - 1\}$ we derive the SP$_1'(p)$ models. Another option is to choose $\underline{S}_0'(P^\varepsilon, p) = \{\underline{\tau} \in \underline{S}_0'(P^\varepsilon) : \deg_3 \underset{\sim}{\tau} \leq p, \deg_3 \tau_3 \leq p + 1\}$ instead, yielding the SP$_2'(p)$ models. For both SP$_1'(p)$ and SP$_2'(p)$ models, $\operatorname{div} \underline{S}_0'(P^\varepsilon, p) = V(P^\varepsilon, p)$ and $\sigma^\varepsilon(p)$ is the minimizer of the complementary energy

$$J_c(\underline{\tau}) = \frac{1}{2}\int_{P^\varepsilon} |\underline{\tau}|^2 \, d\underset{\sim}{x}$$

over all $\underline{\tau} \in \underline{S}_0'(P^\varepsilon, p)$ such that $\operatorname{div} \underline{\tau} = -\pi_{V'} f^\varepsilon$, where $\pi_{V'} f^\varepsilon$ is the orthogonal L^2 projection on f^ε into $V'(P^\varepsilon, p)$.

We present next some results regarding the SP$_2'(p)$ models, omiting the motivations and the proofs.

For the $\mathrm{SP}_2'(p)$ methods, the asymptotic expansions for $u^\varepsilon(p)$ and $\underset{\sim}{\sigma}^\varepsilon(p)$ are

$$u^\varepsilon(p)(\underline{x}^\varepsilon) \sim \zeta^0(\underset{\sim}{x}^\varepsilon) + \sum_{k=1}^{\infty} \varepsilon^{2k} u^{2k}(p)(\underset{\sim}{x}^\varepsilon, \varepsilon^{-1}x_3^\varepsilon) + \text{boundary correctors},$$

$$\underline{\sigma}(p)(\underline{x}^\varepsilon) \sim \begin{pmatrix} \underset{\sim}{\nabla}\zeta^0 \\ 0 \end{pmatrix} (\underset{\sim}{x}^\varepsilon) + \sum_{k=1}^{\infty} \varepsilon^{2k} \begin{pmatrix} \underset{\sim}{\sigma}^{2k}(p) \\ \varepsilon^{-1}\sigma_3^{2k}(p) \end{pmatrix} (\underset{\sim}{x}^\varepsilon, \varepsilon^{-1}x_3^\varepsilon)$$
$$+ \text{boundary correctors}.$$

Equations (13), (15) define ζ^0. The other terms are determined as below. With $\underset{\sim}{x} \in \Omega$ as a parameter, $u^2(p)(\underline{x}) \in \hat{\mathbb{P}}_p(-1,1)$ and $\sigma_3^2(p)(\underset{\sim}{x},\cdot) \in \mathring{\mathbb{P}}_{p+1}(-1,1)$, where $\mathring{\mathbb{P}}_p(-1,1)$ are the polynomials of degree p that vanish on $\{-1,1\}$. Then

$$\int_{-1}^{1} \sigma_3^2(p)(\underset{\sim}{x},x_3)\tau_3(x_3)\,dx_3 + \int_{-1}^{1} u^2(p)(\underset{\sim}{x},x_3)\partial_3\tau_3(x_3)\,dx_3 = 0$$
$$\text{for all } \tau_3 \in \mathring{\mathbb{P}}_{p+1}(-1,1),$$
$$\int_{-1}^{1} \partial_3\sigma_3^2(p)(\underset{\sim}{x},x_3)v(x_3)\,dx_3 = -\int_{-1}^{1} [f(\underset{\sim}{x},x_3) + \Delta_{2D}\zeta^0(\underset{\sim}{x})]v(x_3)\,dx_3$$
$$\text{for all } v \in \hat{\mathbb{P}}_p(-1,1).$$

Note that $u^2(p)$, $\sigma_3^2(p)$ are mixed method approximations of u^2, $\partial_3 u^2$ (with $\underset{\sim}{x} \in \Omega$ as a parameter).

For all integers $k \geq 2$, let $\sigma_3^{2k}(p)(\underset{\sim}{x},\cdot) \in \mathring{\mathbb{P}}_{p+1}(-1,1)$ and $u^{2k}(p)(\underset{\sim}{x},\cdot) \in \hat{\mathbb{P}}_p(-1,1)$ be such that

$$\int_{-1}^{1} \sigma_3^{2k}(p)(\underset{\sim}{x},x_3)\tau_3(x_3)\,dx_3 + \int_{-1}^{1} u^{2k}(p)(\underset{\sim}{x},x_3)\partial_3\tau_3(x_3)\,dx_3 = 0$$
$$\text{for all } \tau_3 \in \mathring{\mathbb{P}}_{p+1}(-1,1),$$
$$\int_{-1}^{1} \partial_3\sigma_3^{2k}(p)(\underset{\sim}{x},x_3)v(x_3)\,dx_3 = -\int_{-1}^{1} \Delta_{2D}\, u^{2k-2}(p)(\underset{\sim}{x},x_3)v(x_3)\,dx_3$$
$$\text{for all } v \in \hat{\mathbb{P}}_{p+1}(-1,1).$$

Also, $\underset{\sim}{\sigma}^{2k} = \underset{\sim}{\nabla}\, u^{2k}(p)$. We present some details regarding the boundary corrector problem. We expect a pair of correctors $U(p)$, $\underline{\Xi}(p)$ with trace $U_0(p)$ on ∂P_L^ε to satisfy

(30)
$$\int_{P^\varepsilon} \underline{\Xi}(p)\cdot\underline{\tau} + U(p)\operatorname{div}\underline{\tau}\,d\underline{x} = \int_{\partial P_L^\varepsilon} U_0(p)\underline{\tau}\cdot\underline{n}\,d\underset{\sim}{x}\,dx_3 \qquad \text{for all } \underline{\tau} \in \underline{S}_0'(P^\varepsilon,p),$$
$$\int_{P^\varepsilon} \operatorname{div}\underline{\Xi}(p)v\,d\underline{x} = 0 \qquad \text{for all } v \in V'(P^\varepsilon,p).$$

We use (18) to define

$$\Xi_n(p)(\underline{x}^\varepsilon) = \underset{\sim}{\Xi}(p)(\underline{x}^\varepsilon)\cdot\underset{\sim}{n}(\underset{\sim}{x}^\varepsilon), \qquad \Xi_s(p)(\underline{x}^\varepsilon) = \underset{\sim}{\Xi}(p)(\underline{x}^\varepsilon)\cdot\underset{\sim}{s}(\underset{\sim}{x}^\varepsilon),$$
$$\tau_n(p)(\underline{x}^\varepsilon) = \underset{\sim}{\tau}(p)(\underline{x}^\varepsilon)\cdot\underset{\sim}{n}(\underset{\sim}{x}^\varepsilon), \qquad \tau_s(p)(\underline{x}^\varepsilon) = \underset{\sim}{\tau}(p)(\underline{x}^\varepsilon)\cdot\underset{\sim}{s}(\underset{\sim}{x}^\varepsilon),$$

in Ω_b. Then, a long but straightforward computation shows that

$$\operatorname{div} \underset{\sim}{\Xi}(p) = \partial_\rho \Xi_n(p) + \frac{1}{J}\partial_\theta \Xi_s(p) - \frac{\kappa}{J}\Xi_n(p).$$

Hoping that the correctors will decay very quickly, we, in a first step, pose (30) in Ω_b using the boundary fitted coordinates $(\rho, \theta, x_3^\varepsilon)$. Next, we use the "stretched" (in the normal and vertical directions) variables $(\hat\rho, \theta, x_3)$ in order to pose a ε-independent sequence of corrector problems, and define

$$\hat\Xi_n(p)(\hat\rho,\theta,x_3) = \varepsilon \Xi_n(p)(\rho,\theta,x_3^\varepsilon), \qquad \hat\Xi_s(p)(\hat\rho,\theta,x_3) = \Xi_s(p)(\rho,\theta,x_3^\varepsilon),$$
$$\hat\Xi_3(p)(\hat\rho,\theta,x_3) = \varepsilon \Xi_3(p)(\rho,\theta,x_3^\varepsilon).$$

Similar definitions hold for $\hat\tau_n(p)$, $\hat\tau_s(p)$ and $\hat\tau_3(p)$. The motivation for multiplying $\Xi_n(p)$ and $\Xi_3(p)$ by ε is that we expect them to "behave" as ε^{-1}, after all they approximate $\partial_\rho U$ and $\partial_3 U$ in P^ε. All the above described transformations lead to

$$\int_{\hat Q} \big[\varepsilon^{-2}\hat\Xi_n(p)\hat\tau_n + \hat\Xi_s(p)\hat\tau_s + \varepsilon^{-2}\hat\Xi_3(p)\hat\tau_3 + U(p)\big(\varepsilon^{-2}\partial_{\hat\rho}\hat\tau_n + \frac{1}{J}\partial_\theta\hat\tau_s + \varepsilon^{-2}\partial_3\hat\tau_3\big)\big]J$$
$$- \varepsilon^{-1}\kappa U(p)\hat\tau_n \, d\hat Q = \int_0^{2\pi}\int_{-1}^1 U_0(p)(0,\theta,x_3)\hat\tau_n(0,\theta,x_3)\,dx_3\,d\theta,$$
$$\int_{\hat Q} \big[\varepsilon^{-2}\partial_{\hat\rho}\hat\Xi_n(p) + \frac{1}{J}\partial_\theta\hat\Xi_s + \varepsilon^{-2}\partial_3\hat\Xi_3 - \varepsilon^{-1}\frac{\kappa}{J}\hat\Xi_n(p)\big]vJ\,d\hat Q = 0,$$

where $\hat Q = \mathbb{R}^+ \times (0,2\pi) \times (-1,1)$ is a semi-infinite quadrilateral domain with the union of its top and bottom boundaries given by $\partial\hat Q_\pm = \mathbb{R}^+ \times (0,2\pi) \times \{-1,1\}$, and

$$\underline{\tau} \in \big\{\underline{\tau} \in \underline{H}(\operatorname{div},\hat Q) \,:\, \tau_3 = 0 \text{ on } \partial\hat Q_\pm,\, \deg_3 \underset{\sim}{\tau} \le p,\, \deg_3 \tau_3 \le p+1\big\},$$
$$v \in \big\{v \in L^2(\hat Q) \,:\, \deg_3 v \le p\big\}.$$

Replacing $\hat\tau_n$ by $\hat\tau_n/J$, $\hat\tau_3$ by $\hat\tau_3/J$, and v by v/J, formally substituting the Taylor series of the coefficients and

$$U(p)(\underline{x}) \sim \varepsilon^2 U^2(p)(\underline{x}) + \varepsilon^3 U^3(p)(\underline{x}) + \varepsilon^4 U^4(p)(\underline{x}) + \dots,$$
$$\underline{\hat\Xi}(p)(\underline{x}) \sim \varepsilon^2 \underline{\hat\Xi}^2(p)(\underline{x}) + \varepsilon^3 \underline{\hat\Xi}^3(p)(\underline{x}) + \varepsilon^4 \underline{\hat\Xi}^4(p)(\underline{x}) + \dots$$
$$U_0(p) \sim \varepsilon^2 u^2(p) + \varepsilon^3 u^3(p) + \varepsilon^4 u^4(p) + \dots,$$

where $u^k(p) = 0$ for k odd, we arrive at the following sequence of problems, parametrized by $\theta \in \mathbb{R}/L$ and defined in the semi-infinite strip Σ:

$$\int_\Sigma \hat\Xi_n^k(p)\hat\tau_n + \hat\Xi_3^k(p)\hat\tau_3 + U^k(p)\,(\partial_{\hat\rho}\hat\tau_n + \partial_3\hat\tau_3)\; d\hat\rho\,dx_3$$
$$= -\int_{-1}^1 u^k(p)(0,\theta,x_3)\hat\tau_n(0,x_3)\,dx_3 \qquad \text{for all } \underset{\sim}{\tau} \in \underset{\sim}{S}_0'(\Sigma,p),$$
$$\int_\Sigma \big[\partial_{\hat\rho}\hat\Xi_n^k(p) + \partial_3\hat\Xi_3^k(p)\big]v\,d\hat\rho\,dx_3 = \int_\Sigma G_k(p)v\,d\hat\rho\,dx_3 \qquad \text{for all } v \in V'(\Sigma,p),$$
$$\hat\Xi_s^k(p) = \hat\rho\kappa(\theta)\hat\Xi_s^{k-1}(p) + \partial_\theta U^k(p),$$
$$G_k(p) = \sum_{j=0}^{k-2} \hat\rho^j \left(a_1^j \hat\Xi_n^{k-j-1}(p) + a_2^j \partial_{\theta\theta} U^{k-j-2}(p) + a_3^j \partial_\theta U^{k-j-2}(p)\right).$$

Finally,

$$u^\varepsilon(p)(\underline{x}^\varepsilon) \sim \zeta^0(\underset{\sim}{x}^\varepsilon) + \sum_{k\geq 1} \varepsilon^{2k} u^{2k}(p)(\underset{\sim}{x}^\varepsilon, \varepsilon^{-1}x_3^\varepsilon) - \chi(\rho)\sum_{k\geq 2}\varepsilon^k U^k(p)(\varepsilon^{-1}\rho,\theta,\varepsilon^{-1}x_3^\varepsilon),$$

$$\begin{aligned}\underline{\sigma}^\varepsilon(p)(\underline{x}^\varepsilon) \sim \begin{pmatrix}\underset{\sim}{\nabla}\zeta^0\\ 0\end{pmatrix}(\underset{\sim}{x}^\varepsilon) + \sum_{k\geq 1}\varepsilon^{2k}\begin{pmatrix}\underset{\sim}{\sigma}^{2k}(p)\\ \varepsilon^{-1}\sigma_3^{2k}(p)\end{pmatrix}(\underset{\sim}{x}^\varepsilon,\varepsilon^{-1}x_3^\varepsilon)\\ - \chi(\rho)\sum_{k\geq 2}\varepsilon^k\begin{pmatrix}\varepsilon^{-1}\hat{\Xi}_n^k(p)\underset{\sim}{n} + \hat{\Xi}_s^k(p)\underset{\sim}{s}\\ \varepsilon^{-1}\hat{\Xi}_3^k(p)\end{pmatrix}(\varepsilon^{-1}\rho,\theta,\varepsilon^{-1}x_3^\varepsilon).\end{aligned}$$

We present next the various error estimates.

Theorem 6. *For any nonnegative integer N, there exists a constant $C(f)$ such that*

$$\begin{aligned}&\Big\|u^\varepsilon(p) - \zeta^0 - \sum_{k=1}^{N}\varepsilon^{2k}u^{2k}(p) + \chi(\rho)\sum_{k=2}^{2N}\varepsilon^k U^k(p)\Big\|_{L^2(P^\varepsilon)}\\ &+\Big\|\underline{\sigma}^\varepsilon(p) - \begin{pmatrix}\underset{\sim}{\nabla}\zeta^0\\ 0\end{pmatrix} - \sum_{k=1}^{N}\varepsilon^{2k}\begin{pmatrix}\underset{\sim}{\sigma}^{2k}(p)\\ \varepsilon^{-1}\sigma_3^{2k}(p)\end{pmatrix}\\ &\qquad + \chi(\rho)\sum_{k=2}^{2N}\varepsilon^k\begin{pmatrix}\varepsilon^{-1}\hat{\Xi}_n^k(p)\underset{\sim}{n} + \hat{\Xi}_s^k(p)\underset{\sim}{s}\\ \varepsilon^{-1}\hat{\Xi}_3^k(p)\end{pmatrix}\Big\|_{L^2(P^\varepsilon)} \leq C(f)\varepsilon^{2N+1}\end{aligned}$$

The next two lemmas estimate the difference between the first terms of the asymptotic expansion of u^ε and $u^\varepsilon(p)$.

Lemma 7. *For any nonnegative real number s, and for any arbitrarily small $\delta > 0$, there exists a constant C such that*

$$\begin{aligned}\|u^2 - u^2(p)\|_{L^2(P^\varepsilon)} &\leq C\varepsilon^{1/2}p^{-2-s}\|f\|_{L^2(\Omega;H^s(-1,1))},\\ \|\underset{\sim}{\nabla}u^2 - \underset{\sim}{\nabla}u^2(p)\|_{L^2(P^\varepsilon)} &\leq C\varepsilon^{1/2}p^{-2-s}\|f\|_{H^1(\Omega;H^s(-1,1))},\\ \|\sigma_3^2 - \sigma_3^2(p)\|_{L^2(P^\varepsilon)} &\leq C\varepsilon^{1/2}p^{-1-s}\|f\|_{L^2(\Omega;H^s(-1,1))},\\ \|\partial_{x_3^\varepsilon}\sigma_3^2 - \partial_{x_3^\varepsilon}\sigma_3^2(p)\|_{L^2(P^\varepsilon)} &\leq C\varepsilon^{-1/2}p^{-s}\|f\|_{L^2(\Omega;H^s(-1,1))},\\ |\chi[\hat{\Xi}_n^2 - \hat{\Xi}_n^2(p)]|_{L^2(P^\varepsilon)} + |\chi[\hat{\Xi}_3^2 - \hat{\Xi}_3^2(p)]|_{L^2(P^\varepsilon)} &\leq C_\delta\varepsilon p^{-5+\delta}\|f\|_{L^2(\partial\Omega;H^4(-1,1))},\end{aligned}$$

where $\sigma_3^2(\underline{x}) = \partial_{x_3}u^2(\underline{x})$, $\hat{\Xi}_3^2(\underline{x}) = \partial_{x_3}U^2(\underline{x})$.

We end this note by presenting the convergence results for the $\mathrm{SP}'(p)$ model.

Theorem 8. *For any nonnegative real number s, and for any arbitrarily small $\delta > 0$, there exist constants C and $C(f)$ such that the following bounds hold:*

$$\begin{aligned}\|u^\varepsilon - u^\varepsilon(p)\|_{L^2(P^\varepsilon)} &\leq C\varepsilon^{5/2}p^{-2-s}\|f\|_{L^2(\Omega;H^s(-1,1))} + C(f)\varepsilon^3,\\ \|\underset{\sim}{\sigma}^\varepsilon\cdot\underset{\sim}{n} - \underset{\sim}{\sigma}^\varepsilon(p)\cdot\underset{\sim}{n}\|_{L^2(P^\varepsilon)} &\leq C_\delta\varepsilon^2 p^{-5+\delta})\|f\|_{L^2(\partial\Omega;H^4(-1,1))} + C(f)\varepsilon^{5/2},\\ \|\underset{\sim}{\sigma}^\varepsilon\cdot\underset{\sim}{s} - \underset{\sim}{\sigma}^\varepsilon(p)\cdot\underset{\sim}{s}\|_{L^2(P^\varepsilon)} &\leq C\varepsilon^{5/2}p^{-2-s}\|f\|_{H^1(\Omega;H^s(-1,1))} + C(f)\varepsilon^3,\\ \|\underset{\sim}{\sigma}^\varepsilon - \underset{\sim}{\sigma}^\varepsilon(p)\|_{L^2(P_0^\varepsilon)} &\leq C\varepsilon^{5/2}p^{-2-s}\|f\|_{H^1(\Omega;H^s(-1,1))} + C(f)\varepsilon^{9/2},\\ \|\sigma_3^\varepsilon - \sigma_3^\varepsilon(p)\|_{L^2(P^\varepsilon)} &\leq C\varepsilon^{3/2}p^{-1-s}\|f\|_{L^2(\Omega;H^s(-1,1))} + C(f)\varepsilon^2,\end{aligned}$$

where $\underline{\sigma}^\varepsilon(\underline{x}^\varepsilon) = \underline{\nabla}\,u^\varepsilon(\underline{x}^\varepsilon)$.

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Laboratório Nacional de Computação Científica, Petrópolis, Rio de Janeiro, Brazil. CEP: 25651-070, Caixa Postal: 95113. http://www.lncc.br/~alm
E-mail address: alm@lncc.br