Analysis, Numerical Methods, and Applications of Multiscale Problems

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ABSTRACT. Multiscale Problems are omnipresent in real world applications, and present a challenge in terms of numerical approximations. Well-known examples include modeling of plates and shells, composites, flow in porous media, among other examples.

The PDEs that model these problems are characterized by either the presence of a small parameter in the equation (e.g., the viscosity of a turbulent flow), or in the domain itself (as in shell problems). These PDEs are commonly denominated *singular perturbed*.

In this course, I plan to discuss the modeling of some singular perturbed PDEs. Modeling here has two meanings. It can be in the sense of approximating the original PDE by another PDE that's easier to solve, as in plate and shell theory. It can also be in numerical approximation point of view, where the final goal is to develop a numerical scheme that is robust, i.e., that works well for a wide range of parameters.

The techniques involved will be introduced by means of examples. In all cases discussed, we shall derive modeling error estimates by means of asymptotic analysis. The problems I plan to describe are the Reaction–Diffusion Equation, Problem in domain with Rough Boundaries (think of a golf ball), and Plate problems. If time permits, I'll describe new techniques just developed to deal with PDEs with oscillating coefficients.

Duration: 12 hours.

References: There's no single book I'll follow. Some modern papers dealing with the topic will be refereed to whenever necessary.

Pre-requisites: I'll assume basic knowledge of analysis and Finite Element Methods. The main tools will be developed as the course goes.

Web Site: http://math.cudenver.edu/∼alm/courses/asymp.html

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CHAPTER 1

One dimensional Singular Perturbed Problem

In this chapter, we introduce a singular perturbed problem and a numerical difficulty associate with its discretization.

1.1. Advection–Diffusion with constant coefficients

1.1.1. Problem description and a finite element discretization. Consider the following boundary value problem:

$$
\begin{aligned}
-\varepsilon \frac{d^2 u^\epsilon}{dx^2} + \frac{du^\epsilon}{dx} &= 0, \\
u^\epsilon(0) = 1, \quad u^\epsilon(1) &= 0,
\end{aligned}
\tag{1.1.1}
$$

where ε is a positive real number. It is convenient to assume that $\varepsilon \leq 1$. The exact solution is simply

$$u^\epsilon(x) = 1 - \frac{e^{x/\varepsilon} - 1}{e^{1/\varepsilon} - 1}.$$

The function plots for $\varepsilon = 1$, $\varepsilon = 0.1$, and $\varepsilon = 0.01$ follow in figures 1, 2, and 3. It is clear that when ε approaches zero, there is the onset of a boundary layer close to $x = 1$. This is also highlighted by the following fact:

$$\lim_{\varepsilon \to 0} \lim_{\substack{x \to 1 \\ x < 1}} u^\epsilon(x) \neq \lim_{\substack{x \to 1 \\ x < 1}} \lim_{\varepsilon \to 0} u^\epsilon(x).$$

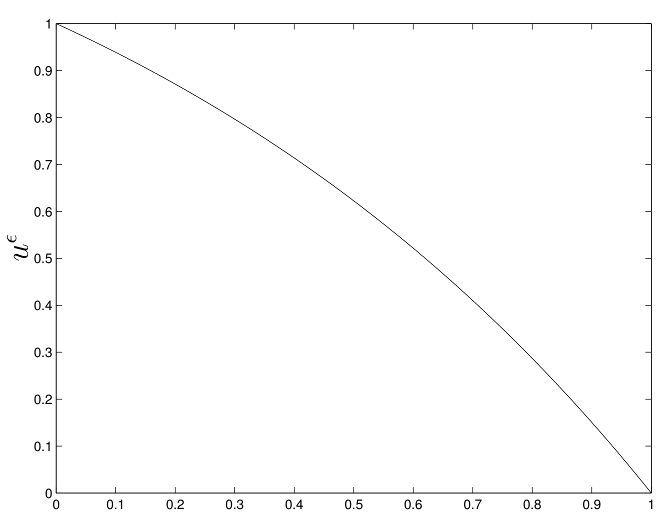


FIGURE 1. Exact solution for $\varepsilon = 1$

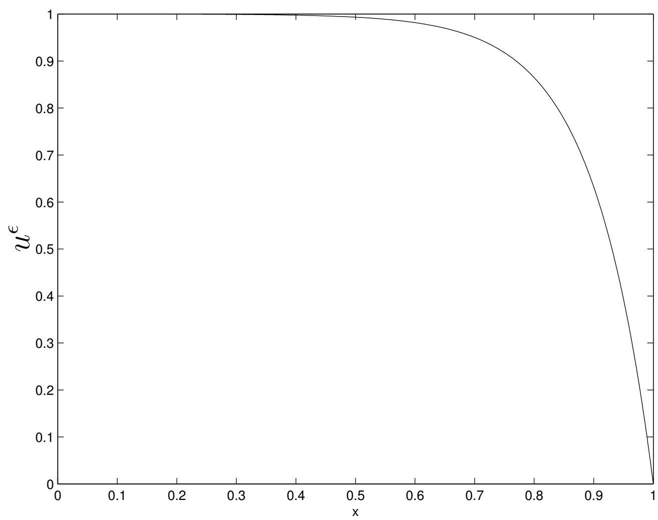


FIGURE 2. Exact solution for $\varepsilon = 0.1$

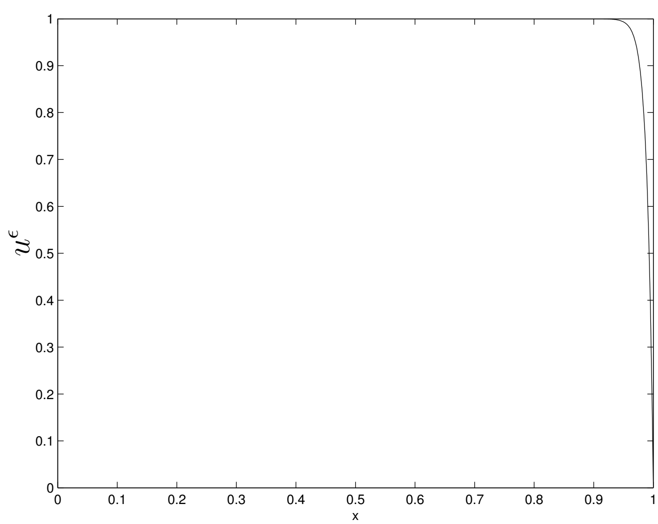


FIGURE 3. Exact solution for $\varepsilon = 0.01$

Let's proceed with a straightforward Galerkin discretization of (1.1.1) using finite element method. We first rewrite (1.1.1) in a weak form, i.e, the exact solution

$$u^\epsilon \in V = \{v \in H^1(0,1) \,:\, v(0) = 1 \text{ and } v(1) = 0\},$$

satisfies

$$a(u^\epsilon, v) := \varepsilon \int_0^1 \frac{du^\epsilon}{dx}\frac{dv}{dx}\,dx + \int_0^1 \frac{du^\epsilon}{dx} v\,dx = 0 \quad \text{for all } v \in H_0^1(0,1). \tag{1.1.2}$$

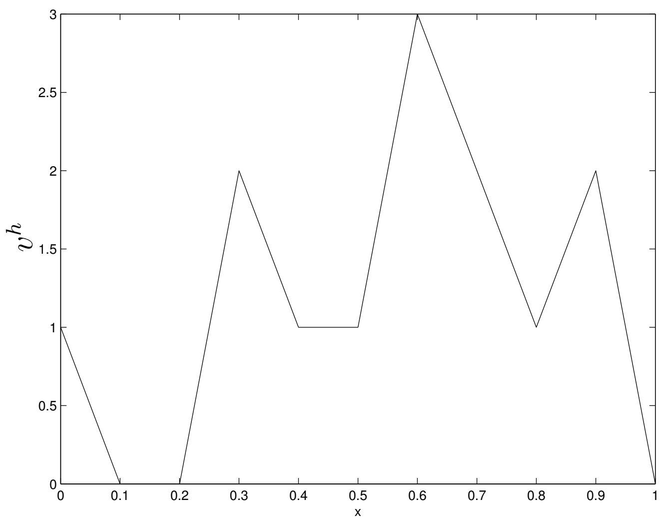


FIGURE 4. Typical piecewise linear function

REMARK. For the domain $(0,1)$, the Sobolev space $H^k(0,1) \subset L^2(0,1)$ is the set of functions with derivatives up to kth order in $L^2(0,1)$. We endow this space with the semi-norm and norm

$$|v|_{H^k(0,1)} = \left(\int_0^1 \left(\frac{\partial^k v}{\partial x^k}\right)^2 dx\right)^{1/2}, \qquad \|v\|_{H^k(0,1)} = \left(\sum_{i=0}^{k} |v|^2_{H^i(0,1)}\right)^{1/2}.$$

We also need $H^1_0(0,1)$, the space of functions in $H^1(0,1)$ vanishing at the boundary $\{0,1\}$.

We introduce now a discretization of the domain $(0,1)$ into finite elements by defining the nodal points $0 = x_0 < x_1 < \cdots < x_{N+1} = 1$, where $x_j = j/(N+1)$. The mesh parameter $h = 1/(N+1)$. Next, we define the finite dimensional $V^h \subset V$, where

$$V^h = \left\{v^h \in V \,:\, v^h \text{ is linear in } (x_{j-1}, x_j) \text{ for } j = 1, \ldots, N+1\right\}.$$

We say that V^h is a space of piecewise linear functions. A typical function of V^h is depicted in figure 4. We finally define

$$V^h_0 = \left\{v^h \in H^1_0(0,1) \,:\, v^h \text{ is piecewise linear}\right\}.$$

The finite element approximation to u^ϵ is $u^h \in V^h$ such that

$$a(u^h, v) = 0 \quad \text{for all } v \in V^h_0. \tag{1.1.3}$$

REMARK. Note that u^h depends on ε, although this is not explicitly indicated in the notation

For a uniform mesh, as described above, with $h = 1/16$ the numerical solution for $\varepsilon = 1$ is as in figure 5. On the other hand, with the same mesh, the numerical solution for $\varepsilon = 0.01$ is as in figure 6. For a more refined mesh, $h = 1/32$, the numerical solution is less oscillatory, but still unsatisfactory, as in figure 7. Eventually, reducing even further the mesh size, the Galerkin approximation will look fine in the "picture norm."

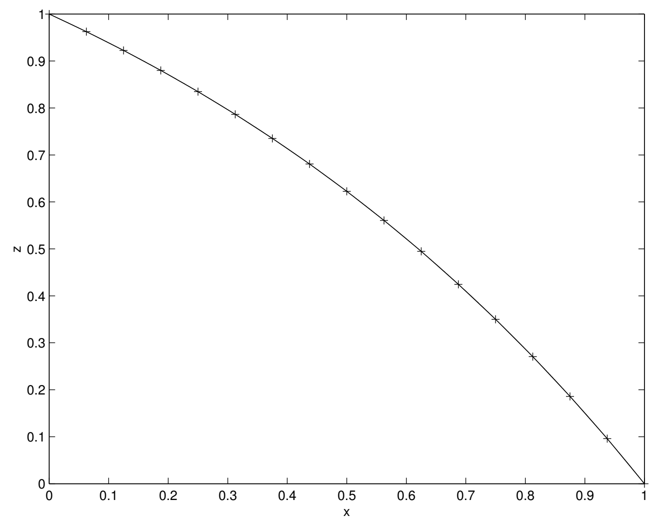


FIGURE 5. Galerkin approximation for $\varepsilon = 1$ and $h = 1/16$

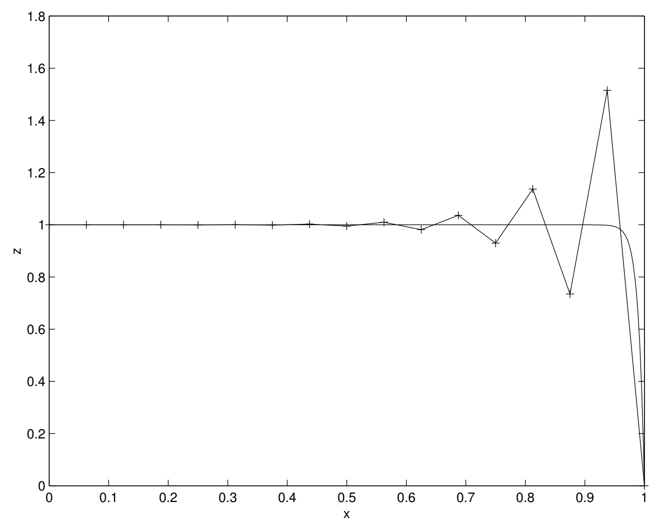


FIGURE 6. Galerkin approximation for $\varepsilon = 0.01$ and $h = 1/16$

1.1.2. So, what goes wrong? To better understand, or, at least, have a feeling of what goes wrong, we develop an error analysis for this problem. We use the convenient convention that the constants that appear in our estimates are independent of the parameters ε and h, unless explicitly indicated. These constants are generally denoted by C.

We first investigate the "continuity" of the bilinear form $a(\cdot, \cdot)$. In fact, it follows from its definition that

$$a(u, v) \leq C\|u\|_{H^1(0,1)}\|v\|_{H^1(0,1)} \quad \text{for all } u, v \in H_0^1(0, 1). \tag{1.1.4}$$

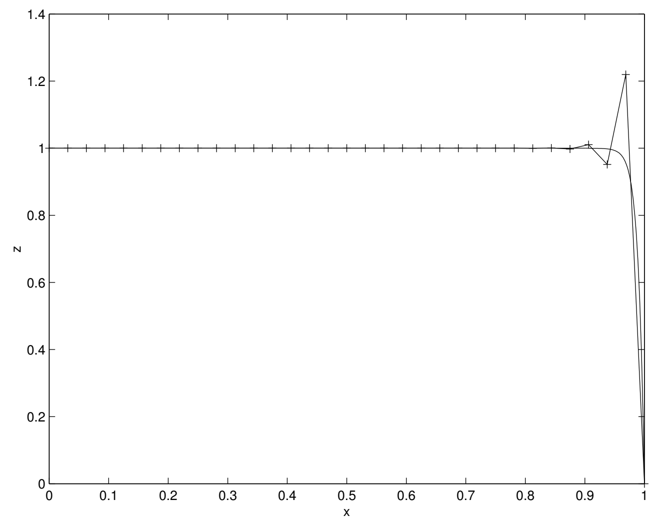


FIGURE 7. Galerkin approximation for $\varepsilon = 0.01$ and $h = 1/32$

The problem starts when we try to derive the *coercivity estimate*:

$$a(v,v) = \varepsilon \int_0^1 \left(\frac{dv}{dx}\right)^2 dx + \int_0^1 \frac{dv}{dx} v \, dx = \varepsilon \int_0^1 \left(\frac{dv}{dx}\right)^2 dx \geq C\varepsilon \|v\|^2_{H^1(0,1)} \quad \text{for all } v \in H^1_0(0,1), \tag{1.1.5}$$

since integration by parts yields $\int_0^1 (dv/dx)v \, dx = 0$, for $v \in H^1_0(0,1)$. We also used Poincaré's inequality at the last step.

We are ready to derive error estimates. Using (1.1.5), and then (1.1.4), we gather that:

$$\begin{aligned} \|u^\epsilon - u^h\|^2_{H^1(0,1)} &\leq C\varepsilon^{-1} a(u^\epsilon - u^h, u^\epsilon - u^h) = C\varepsilon^{-1} a(u^\epsilon - u^h, u^\epsilon - v^h) \\ &\leq C\varepsilon^{-1} \|u^\epsilon - u^h\|_{H^1(0,1)} \|u^\epsilon - v^h\|_{H^1(0,1)} \quad \text{for all } v^h \in V^h. \end{aligned} \tag{1.1.6}$$

Using standard interpolation estimates, we have that $I^h u^\epsilon$, the interpolator of u^ϵ, satisfies

$$\|u^\epsilon - I^h u^\epsilon\|_{H^1(0,1)} \leq h|u^\epsilon|_{H^2(0,1)}.$$

Making $v^h = I^h u^\epsilon$ in (1.1.6), we conclude that

$$\|u^\epsilon - u^h\|_{H^1(0,1)} \leq C\varepsilon^{-1} h |u^\epsilon|_{H^2(0,1)}. \tag{1.1.7}$$

We stop now to try interpret the error estimate we just obtained. First of all, *there is convergence in* h. Indeed, for a fixed ε, the error goes to zero as the mesh size goes to zero.

The problem is that the convergence in h is not uniform in ε. Hence, for ε small, unless the mesh size is very small, the H^1 norm error estimate becomes large. The estimate is even worse than one can think at first glance, since $|u^\epsilon|_{H^2(0,1)} = O(\varepsilon^{-3/2})$. This makes (1.1.7) and the traditional Galerkin method almost useless.

Another way to look at this problem is by first noticing that we would like to have

$$\lim_{\varepsilon \to 0} u^h = \lim_{\varepsilon \to 0} u^\epsilon = 1.$$

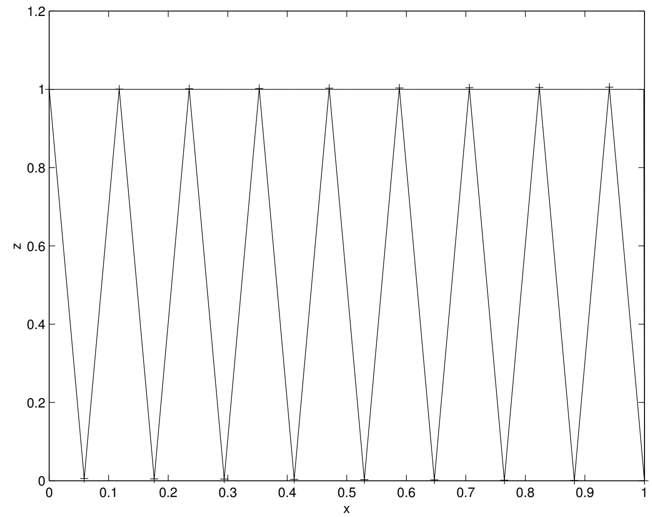


FIGURE 8. Numerical and exact solutions for $\varepsilon = 10^{-5}$ and $N = 16$

After all, it would be just perfect to have a method that converges (with ε) to the correct solution *for a fixed mesh*. This is not happening. Indeed, looking at the matricial problem coming from (1.1.3), it is matter of computation to show that [**26**]

$$-\frac{\varepsilon}{h^2}(u_{j+1} - 2u_j + u_{j-1}) + \frac{1}{2h}(u_{j+1} - u_{j-1}) = 0, \quad u_0 = 1, \ u_{N+1} = 0, \tag{1.1.8}$$

where $u_j = u^h(x_j)$. Assume N even. As ε goes to zero, it "follows that $u_{j+1} = u_{j-1}$." This, and the boundary conditions originate the oscillatory behavior of the approximate solution. See figure 8.

REMARK. Note that although we used a finite element scheme to derive (1.1.8), this scheme is also a finite difference scheme which uses a central difference approximation for the convective term du/dx. The more naive finite difference approximation

$$-\frac{\varepsilon}{h^2}(u_{j+1} - 2u_j + u_{j-1}) + \frac{1}{h}(u_j - u_{j-1}) = 0, \quad u_0 = 1, \ u_{N+1} = 0,$$

yields in fact a better result. See figure (9). In fact, for this scheme, $u_j = u_{j-1}$, as ε goes to zero. Since $u_0 = 1$, it holds that $u_j = 1$ in the $\varepsilon \to 0$ limit:

$$\lim_{\varepsilon \to 0} u^h(x_j) = \lim_{\varepsilon \to 0} u^\epsilon(x_j) = 1, \quad \text{for } j = 1, \ldots, N.$$

The behavior we described above is typical in singular perturbed PDEs, where the onset of boundary layers is a common phenomenon. But this is not all that can happen. For instance, in plate models, in particular for the Reissner–Mindlin equation, as the plate thickness goes to zero (that is the small parameter in this case), numerical "locking" occurs, i.e., if a careless method is used, the computed solution goes to zero (a wrong limit). In due time we shall explain why this happens. . . .

Several numerical methods try to somehow overcome these and other difficulties related to asymptotic limits. Some methods perform well for a certain asymptotic range, for instance by assuming $\varepsilon << 1$. Some other methods try to be performing for a broader range of parameters. See for instance [**11**], [**21**], [**24**], [**36**].

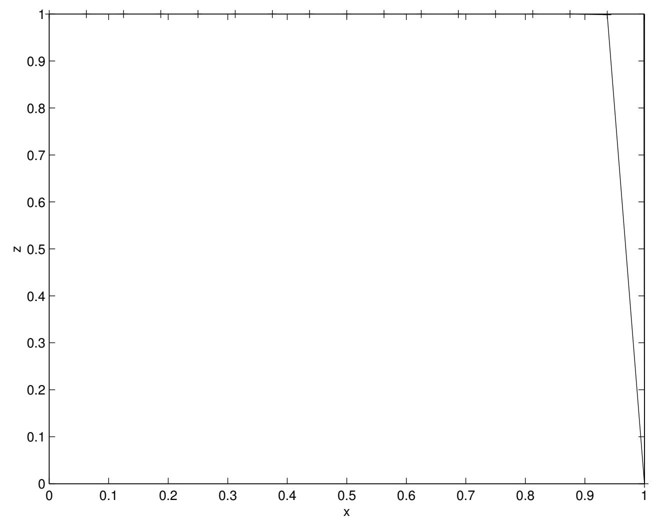


FIGURE 9. Finite difference approximation for $\varepsilon = 0.0001$ and $h = 1/15$

Looking at these difficulties (and their corresponding solutions!) it becomes more clear that it is important to have a full understanding of the solution's behavior. This is useful not only to help designing new numerical methods, but also to Analise and estimate old ones. A valuable analysis tool is the method of *matching asymptotics*, where the exact solution for a given singular perturbed PDE is expressed in terms of a (formal) power series with respect to a small parameter.

In the example we are considering in this section, the asymptotic expansion is trivial. Indeed the exact solution

$$u^\epsilon(x) = 1 - \frac{e^{x/\varepsilon} - 1}{e^{1/\varepsilon} - 1}$$

is the sum of a "regular function" (the unit function in this case), plus a "boundary layer" term. The regular part is independent of ε, and the boundary layer term, also called "boundary corrector", becomes exponentially small in the interior of the domain. This is a typical behavior of singular perturbed problems.

We shall explain how an asymtotic expansion can be developed by looking at a simple, still one-dimensional, example.

1.2. A singular perturbed general second order ODE

Consider the differential operator

$$\mathcal{L}^\epsilon\, u^\epsilon = -\varepsilon \frac{d^2 u^\epsilon}{dx^2} + b(x)\frac{du^\epsilon}{dx} + c(x)u^\epsilon,$$

and the problem

$$\mathcal{L}^\epsilon\, u^\epsilon = f \quad \text{in } (0,1), \tag{1.2.1}$$

$$u^\epsilon(0) = u^\epsilon(1) = 0. \tag{1.2.2}$$

We assume that b, c, and f are smooth and that b is always positive.

In 1.2.1 we shall develop asymptotic expansion for u^ϵ, and in 1.2.2 we show error estimates. Then, in 1.2.3, we shall make some comments about what can happen if b vanishes at a particular point. This is the "turning point problem."

1.2.1. Formal development of the asymptotic expansion. In this subsection, we follow the outline of [**36**].

Consider the series

$$u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \dots$$

and formally substitute it in (1.2.1). Then

$$b(x)\frac{du^0}{dx} + c(x)u^0 + \varepsilon\left(-\frac{d^2u^0}{dx^2} + b(x)\frac{du^1}{dx} + c(x)u^1\right) + \dots$$
$$+ \varepsilon^i\left(-\frac{d^2u^{i-1}}{dx^2} + b(x)\frac{du^i}{dx} + c(x)u^i\right) + \dots = f.$$

By comparing the different powers of ε, it is natural to require that

$$\mathcal{L}^0\, u^0 = f, \quad \mathcal{L}^0\, u^1 = \frac{d^2u^0}{dx^2}, \dots, \quad \mathcal{L}^0\, u^i = \frac{d^2u^{i-1}}{dx^2}, \dots, \tag{1.2.3}$$

where $\mathcal{L}^0\, v = b(x)dv/dx + c(x)v$.

From (1.2.2), it would be natural to impose $u^i(0) = u^i(1) = 0$. However, the equations in (1.2.3) are of first order, and only one boundary condition is to be imposed. We set then

$$u^i(0) = 0.$$

We correct this discrepancy by introducing the boundary corrector U. We would like to have

$$\mathcal{L}^\epsilon\, U = 0, \quad U(0) = 0, \quad U(1) = u^0(1) + \varepsilon u^1(1) + \varepsilon^2 u^2(1) + \dots$$

Note that if we make the change of coordinates $\hat{\rho} = \varepsilon^{-1}(1-x)$, and set $\hat{U}(\hat{\rho}) = U(1 - \varepsilon\hat{\rho})$, then

$$-\frac{d^2\hat{U}}{d\hat{\rho}^2}(\hat{\rho}) - b(1-\varepsilon\hat{\rho})\frac{d\hat{U}}{d\hat{\rho}}(\hat{\rho}) + \varepsilon c(1-\varepsilon\hat{\rho})\hat{U}(\hat{\rho}) = 0,$$
$$\hat{U}(0) = u^0(1) + \varepsilon u^1(1) + \varepsilon^2 u^2(1) + \dots$$

Going one step further, we develop the Taylor expansions

$$b(1-\varepsilon\hat{\rho}) = b(1) - \varepsilon\hat{\rho}\frac{db}{dx}(1) + \frac{\varepsilon^2\hat{\rho}^2}{2}\frac{d^2b}{dx^2}(1) - \dots,$$
$$c(1-\varepsilon\hat{\rho}) = c(1) - \varepsilon\hat{\rho}\frac{dc}{dx}(1) + \frac{\varepsilon^2\hat{\rho}^2}{2}\frac{d^2c}{dx^2}(1) - \dots.$$

Finally, assuming the asymptotic expansion

$$\hat{U} \sim \hat{U}^0 + \varepsilon\hat{U}^1 + \varepsilon^2\hat{U}^2 + \dots,$$

we gather that

$$-\frac{d^2\hat{U}^0}{d\hat{\rho}^2} - b(1)\frac{d\hat{U}^0}{d\hat{\rho}} = 0,$$
$$U^0(0) = u^0(1).$$

Noting that we need another boundary condition for $\hat{U}^0$, we try to ensure a "local behavior" by imposing

$$\lim_{\hat{\rho}\to\infty} \hat{U}^0(\hat{\rho}) = 0.$$

Hence, $\hat{U}^0(\hat{\rho}) = u(1)\exp\bigl(-b(1)\hat{\rho}\bigr)$.

Similarly,

$$-\frac{d^2\hat{U}^1}{d\hat{\rho}^2} - b(1)\frac{d\hat{U}^1}{d\hat{\rho}} = -\hat{\rho}\frac{db}{dx}(1)\frac{d\hat{U}^0}{d\hat{\rho}} - c(1)\hat{U}^0,$$
$$U^1(0) = u^1(1), \qquad \lim_{\hat{\rho}\to\infty} U^1(\hat{\rho}) = 0,$$

etc. It is possible to show that for all positive integers i there exist ε-independent positive constants C and α such that

$$U^i(\hat{\rho}) \leq C\exp(-\alpha\hat{\rho}). \tag{1.2.4}$$

So, putting everything together, we have that

$$u^\epsilon(x) \sim u^0(x)+\varepsilon u^1(x)+\varepsilon^2 u^2(x)+\cdots-\hat{U}^0(\varepsilon^{-1}(1-x))-\varepsilon\hat{U}^1(\varepsilon^{-1}(1-x))-\varepsilon^2\hat{U}^2(\varepsilon^{-1}(1-x))-\dots. \tag{1.2.5}$$

By construction, the above infinite power series *formally* solves the ODE (1.2.1). We did not make any comment regarding convergence of the above expansion. Actually, what we will prove is that a *truncated expansion* approximates well the exact solution.

REMARK. Note that each term u^i in the series (1.2.5) is independent of ε. Each boundary corrector terms $\hat{U}^i$ depends on ε but only up to a change of coordinates.

REMARK. Note that the U^i does not satisfy the boundary condition at $x = 0$, but for ε small enough, this error is exponentially small.

1.2.2. Truncation Error analysis. We start by developing here an analysis quite similar to that of Subsection 1.1.2. We assume that $b(x) > b_0 > 0$ and $2c - db/dx \geq 0$. Let

$$a(u,v) = \varepsilon\int_0^1 \frac{du}{dx}\frac{dv}{dx}\,dx + \int_0^1 b\frac{du}{dx}v\,dx + \int_0^1 cuv\,dx$$

To obtain a coercivity estimate, first note that

$$a(v,v) = \varepsilon\int_0^1 \left(\frac{dv}{dx}\right)^2 dx + \int_0^1 b\frac{dv}{dx}v\,dx + \int_0^1 c|v|^2\,dx$$

Integrating by parts yields

$$\int_0^1 b\frac{dv}{dx}v\,dx = -\frac{1}{2}\int_0^1 \frac{db}{dx}|v|^2\,dx \quad \text{for all } v \in H_0^1(0,1).$$

Thus,

$$a(v,v) = \varepsilon\int_0^1 \left(\frac{dv}{dx}\right)^2 dx + \int_0^1 \left(c - \frac{1}{2}\frac{db}{dx}\right)|v|^2\,dx \geq \varepsilon\int_0^1 \left(\frac{dv}{dx}\right)^2 dx \geq C\varepsilon\|v\|^2_{H^1(0,1)} \quad \text{for all } v \in H_0^1(0,1). \tag{1.2.6}$$

We again used Poincaré's inequality at the last estimate.

LEMMA 1.2.1. If $\mathcal{L}^\epsilon v = f$ weakly, and $v \in H_0^1(0,1)$, then

$$\|v\|_{H^1(0,1)} \le C\varepsilon^{-1}\|f\|_{H^{-1}(0,1)}.$$

PROOF. From (1.2.6), we conclude that

$$\|v\|_{H^1(0,1)}^2 \le C\varepsilon^{-1}a(v,v) = C\varepsilon^{-1}(f,v) \le C\varepsilon^{-1}\|f\|_{H^{-1}(0,1)}\|v\|_{H_0^1(0,1)},$$

where $(\cdot,\cdot)$ denotes the $L^2(0,1)$ inner product. $\square$

COROLLARY 1.2.2. If $w \in H^1(0,1)$ is the weak solution of

$$\mathcal{L}^\epsilon w = f, \qquad w(0) = w_0, \quad w(1) = w_1,$$

then

$$\|w\|_{H^1(0,1)} \le C\varepsilon^{-1}\big(\|f\|_{H^{-1}(0,1)} + |w_0| + |w_1|\big).$$

PROOF. Let w_{bc} be such that $w_{bc}(0) = w_0$, and $w_{bc} = w_1$, with $\|w_{bc}\|_{H^1(0,1)} \le C\big(|w_0| + |w_1|\big)$. Then Lemma 1.2.1 with $v = w - w_{bc}$ yields the result. $\square$

With the aid of Corollary 1.2.2, we are ready to estimate how well the asymptotic expansion approximates the exact solution of (1.2.1)–(1.2.2). Let

$$e_N(x) = u^\epsilon(x) - \sum_{i=0}^N \varepsilon^i u^i(x) + \sum_{i=0}^N \varepsilon^i U^i(\varepsilon^{-1}(1-x)), \tag{1.2.7}$$

From its construction, $e_N \in H^1(0,1)$, and

$$\mathcal{L}^\epsilon e_N = \varepsilon^{N+1}\frac{du^i}{dx^2}, \qquad e_N(0) = \sum_{i=0}^N \varepsilon^i U^i(\varepsilon^{-1}), \quad e_N(1) = 0. \tag{1.2.8}$$

Using now Corollary 1.2.2, equation (1.2.8), and estimate (1.2.4), we gather that

$$\|e_N\|_{H^1(0,1)} \le C\varepsilon^N.$$

This estimate is not sharp. We improve it by adding and subtracting the $(N+1)$th term of the expansion:

$$\|e_N\|_{H^1(0,1)} \le \|e_{N+1}\|_{H^1(0,1)} + \|e_N - e_{N+1}\|_{H^1(0,1)} \le C\varepsilon^{N+1/2}.$$

Estimates in other norms can be obtained in a similar fashion:

$$\|e_N\|_{L^2(0,1)} \le \|e_{N+1}\|_{H^1(0,1)} + \|e_N - e_{N+1}\|_{L^2(0,1)} \le C\varepsilon^{N+1}.$$

We obtained then the following important result.

THEOREM 1.2.3. *Let u^ϵ be the solution of the ODE* (1.2.1), *and let e_N be as in* (1.2.7). *Then, for every nonnegative integer N, there exists a constant C such that*

$$\|e_N\|_{H^1(0,1)} \le C\varepsilon^{N+1/2}. \qquad \|e_N\|_{L^2(0,1)} \le C\varepsilon^{N+1}.$$

The constant C might depend on N, and on Sobolev norms of f, b, and c, but not on ε.

REMARK. Theorem 1.2.3 does not imply convergence of the power series as N goes to infinity, since the constants that appear in the right hand side of the estimates depend on N. What the theorem provides is a convergence in ε, i.e., if ε is quite small, then the asymptotic expansion truncation error gets small.

To derive estimates in higher order norms, it is enough to gather from (1.2.8) and the definition of $\mathcal{L}^\epsilon$ that

$$\|e_N\|_{H^k(0,1)} \le \varepsilon^{-1}\|e_N\|_{H^{k-1}(0,1)}.$$

Hence, by induction we have that

$$\|e_N\|_{H^k(0,1)} \le C\varepsilon^{N+1/2-k}.$$

REMARK. The asymptotic *rule of the thumb* works quite well: the error estimates present in Theorem 1.2.3 are of the same order as the terms left out of the truncated asymptotic expansion.

1.2.3. Problems with "turning point". Here, we again follow the outline of [**36**]. Consider the problem

$$-\varepsilon\frac{d^2u^\epsilon}{dx^2} + xb(x)\frac{du^\epsilon}{dx} + c(x)u^\epsilon = f \quad \text{in } (-1,1), \tag{1.2.9}$$

$$u^\epsilon(-1) = u^\epsilon(1) = 0. \tag{1.2.10}$$

We assume that $b(x) \ne 0$, $c(x) \ge 0$, and $c(0) \ne 0$.

From what we have seen, if $b(x)$ is positive, then $xb(x) < 0$ at $x = -1$, and $xb(x) > 0$ at $x = 1$. Hence we can expect boundary layers at -1 and 1. The reduced problem is

$$xb(x)\frac{du^0}{dx} + c(x)u^0 = f \quad \text{in } (-1,1),$$

no further boundary conditions are necessary. Indeed, since $c(0) \ne 0$, then $u^0(0) = f(0)/c(0)$. An example is given by

$$x\frac{du^0}{dx} + u^0 = 2x \quad \text{in } (-1,1).$$

The solution is simply $u^0 = x$.

If however b is negative, then the boundary layers occur at $x = 0$ only! In this case, the reduced problem spits in two:

$$\begin{aligned} xb(x)\frac{du^0}{dx} + c(x)u^0 = f \quad &\text{in } (-1,0), \qquad u^0(-1) = 0,\\ xb(x)\frac{du^0}{dx} + c(x)u^0 = f \quad &\text{in } (0,1), \qquad u^0(1) = 0. \end{aligned}$$

Note from the above equations that u^0 is continuous, since we still have $u^0(0) = f(0)/c(0)$. An interesting example is given by the problem

$$xb\frac{du^0}{dx} + cu^0 = bx^k \quad \text{in } (-1,1), \tag{1.2.11}$$

where $b < 0$ and $c > 0$ are constants. The exact solution is

$$u^0(x) = \begin{cases} \frac{1}{k-\lambda}(|x|^k - |x|^\lambda) & \text{if } \lambda \ne k,\\ x^k \ln|x| & \text{otherwise,} \end{cases}$$

where $\lambda = -c/b$. See figures 10, 11.

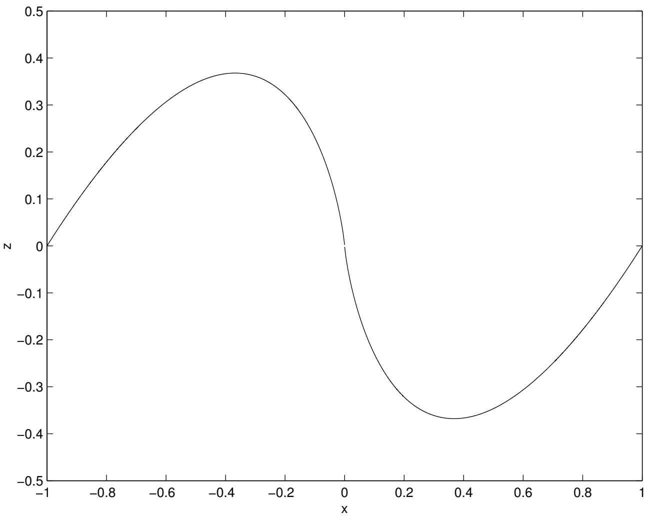


FIGURE 10. Exact solution of (1.2.11) for $k = -c/b = 1$

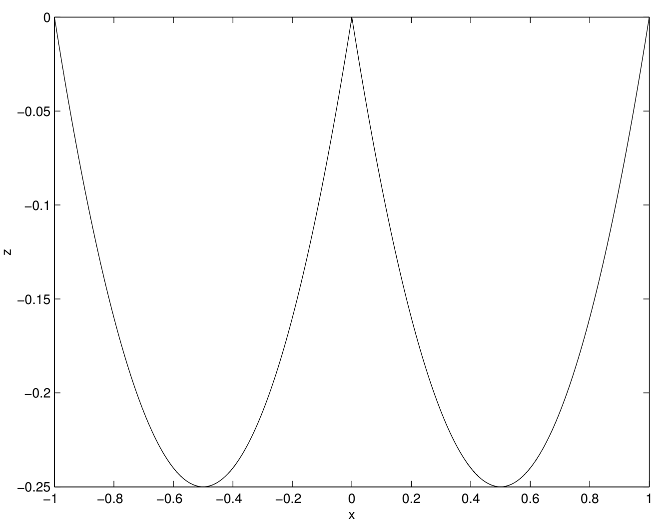


FIGURE 11. Exact solution of (1.2.11) for $k = -c/b = 1$

One last point for discussion is if $c(0) = 0$. Then u^0 might be discontinuous, and u^ϵ has an interior boundary layer. We are still assuming that b is non-negative. For instance,

$$x\frac{du^0}{dx} = x$$

with homogeneous boundary conditions has as solution

$$u^0(x) = \begin{cases} -1 - x & \text{for } x \in (-1, 0), \\ 1 - x & \text{for } x \in (0, 1). \end{cases}$$

CHAPTER 2

Asymptotic Analysis for Two Dimensional Reaction–Diffusion Equation

We now investigate two-dimensional domains, and consider a reaction–diffusion problem. We first develop an asymptotic expansion for the solution, and this time we show how to deal with the boundary layer in a two-dimensional problem. This will be important to devise efficient numerical methods.

2.1. Asymptotic Expansion

Consider the singuar perturbed Reaction–Diffusion problem

$$\begin{gathered}\mathcal{L}^\epsilon u := -\varepsilon^2 \Delta u + \sigma u = f \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{2.1.1}$$

where Ω is a smooth two-dimensional bounded domain, ε is a positive constant and σ is a positive constant. Also assume that f is smooth.

Consider the series

$$u^0 + \varepsilon^2 u^2 + \varepsilon^4 u^4 + \dots$$

and formally substitute it in (2.1.1). Then

$$\sigma u^0 + \varepsilon^2\big(-\Delta u^0 + \sigma u^2\big) + \cdots + \varepsilon^{2i}\big(-\Delta u^{2i-2} + \sigma u^{2i}\big) + \cdots = f.$$

By comparing the different powers of ε, it is natural to require that

$$u^0 = \frac{f}{\sigma}, \quad u^2 = \frac{\Delta u^0}{\sigma}, \dots, \quad u^{2i} = \frac{\Delta u^{2i-2}}{\sigma}, \dots. \tag{2.1.2}$$

Since the u^i are already well-defined, we cannot impose the zero Dirichlet boundary condition. We again correct this by introducing boundary correctors. We would like to have

$$\mathcal{L}^\epsilon U = 0, \quad \text{in } \Omega, \qquad U = u^0 + \varepsilon^2 u^2 + \varepsilon^4 u^4 + \dots \quad \text{on } \partial\Omega, \tag{2.1.3}$$

and formally expand

$$U \sim U^0 + \varepsilon U^1 + \varepsilon^2 U^2 + \dots. \tag{2.1.4}$$

Motivated by the one-dimensional problem, we expect the boundary correctors to have only a "local" influence, and we introduce for that purpose boundary-fitted coordinates. We digress now to introduce these coordinates, following the notation of Chen [**13**]. Suppose that $\partial\Omega$ is arc-length parametrized by $\mathbf{z}(\theta) = (X(\theta), Y(\theta))$. Let $\mathbf{s} = (X', Y')$, $\mathbf{n} = (Y', -X')$ denote the tangent and the outward normal of $\partial\Omega$, and define the sub-domain $\Omega_b \subset \Omega$,

$$\Omega_b = \big\{\mathbf{z} - \rho\mathbf{n} \,:\, \mathbf{z} \in \partial\Omega,\, 0 < \rho < \rho_0\big\},$$

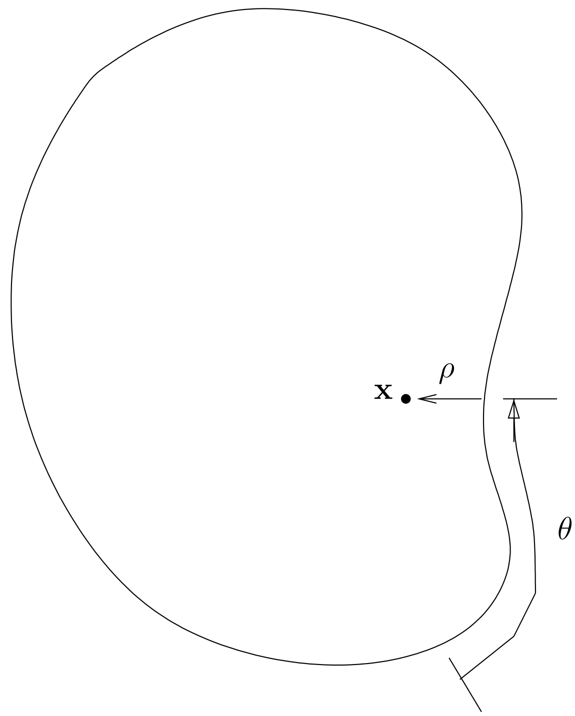


FIGURE 1. Boundary-fitted coordinates

where ρ_0 is a positive number smaller than the minimum radius of curvature of $\partial\Omega$. With L denoting the arc-length of $\partial\Omega$, then

$$\mathbf{x} : (0, \rho_0) \times \mathbb{R}/L \longrightarrow \Omega_b,$$

where

$$\mathbf{x}(\rho, \theta) = \mathbf{z}(\theta) - \rho\mathbf{n}(\theta),$$

is a diffeomorphism. See figure 1 (based on a figure by Arnold and Falk [**4**]).

REMARK. The smoothness of the domain is important here. In particular for a polygon, the transformation above is not a diffeomorphism.

From Frenet Formula, we have that $\mathbf{z}'' = -\kappa\mathbf{n}$, where κ is the curvature of $\partial\Omega$. We also gather the identity

$$\kappa(\theta) = -(\mathbf{z}'' \times \mathbf{z}') \cdot \mathbf{e_3} = Y'X'' - X'Y''. \tag{2.1.5}$$

Note then that

$$\underset{\approx}{\nabla}_{(\rho,\theta)}\mathbf{x} = \begin{pmatrix} \partial_\rho \boldsymbol{\psi} & \partial_\theta \boldsymbol{\psi} \end{pmatrix} = \begin{pmatrix} -\mathbf{n}(\theta) & \mathbf{z}'(\theta) - \rho\mathbf{n}(\theta) \end{pmatrix} = \begin{pmatrix} -Y' & X' - \rho Y'' \\ X' & Y' + \rho X'' \end{pmatrix},$$

and using (2.1.5), and $\rho < \rho_0$, we obtain that $\det \underset{\approx}{\nabla}_{(\rho,\theta)}\mathbf{x} = 1 - \rho\kappa > 0$. Inverting the above matrix, we have that

$$\underset{\approx}{\nabla}_{\mathbf{x}} \begin{pmatrix} \rho \\ \theta \end{pmatrix} = \begin{pmatrix} \underset{-}{\nabla}^T_{\mathbf{x}} \rho \\ \underset{-}{\nabla}^T_{\mathbf{x}} \theta \end{pmatrix} = \frac{1}{J} \begin{pmatrix} -Y' - \rho X'' & X' - \rho Y'' \\ X' & Y' \end{pmatrix},$$

where $J(\rho,\theta) = 1 - \rho\kappa(\theta)$. Hence,

$$\underset{-}{\nabla}_{\mathbf{x}}\,\rho = -\frac{1}{J}\mathbf{n} - \frac{\rho}{J}z'' = \frac{1}{J}(-1+\rho\kappa)\mathbf{n} = -\mathbf{n}.$$

Also,

$$\underset{-}{\nabla}_{\mathbf{x}}\,\theta = \frac{1}{J}\mathbf{s}.$$

Finally, the change of coordinates yields

$$\partial_\alpha f = \partial_\theta f \partial_\alpha \theta + \partial_\rho f \partial_\alpha \rho, \qquad \text{for } \alpha = 1,\, 2,$$

for an arbitrary function f.

The expression for the Laplacian in these new coordinates follows:

$$\begin{aligned}(\partial_{11}+\partial_{22})U &= \partial_{\rho\rho}U - \frac{\kappa}{J}\partial_\rho U + \frac{1}{J^2}\partial_{\theta\theta}U + \frac{\rho\kappa'}{J^3}\partial_\theta U \\ &= \partial_{\rho\rho}U + \sum_{j=0}^{\infty}\rho^j\left(a_1^j\partial_\rho U + a_2^j\partial_{\theta\theta}U + a_3^j\partial_\theta U\right),\end{aligned} \tag{2.1.6}$$

where we formally replace each coefficient with its respective Taylor expansion [**3**], and

$$a_1^j = -[\kappa(\theta)]^{j+1}, \quad a_2^j = (j+1)[\kappa(\theta)]^j, \quad a_3^j = \frac{j(j+1)}{2}[\kappa(\theta)]^{j-1}\kappa'(\theta).$$

Defining the new variable $\hat\rho = \varepsilon^{-1}\rho$ and using the same name for functions different only up to this change of coordinates, we have from (2.1.6) that

$$(\partial_{11}+\partial_{22})U = \varepsilon^{-2}\partial_{\hat\rho\hat\rho}U + \sum_{j=0}^{\infty}(\varepsilon\hat\rho)^j\left(a_1^j\varepsilon^{-1}\partial_{\hat\rho}U + a_2^j\partial_{\theta\theta}U + a_3^j\partial_\theta U\right). \tag{2.1.7}$$

Aiming to solve (2.1.3) we formally use (2.1.4) and (2.1.7), collect together terms with same order of ε and for $k \geq 2$, pose the following sequence of problems parametrized by θ:

$$\begin{aligned}-\partial_{\hat\rho\hat\rho}U^k + U^k &= F_k \qquad \text{in } \mathbb{R}^+, \\ U^k(0,\theta) &= u^k(0,\theta),\end{aligned} \tag{2.1.8}$$

with the convention that $u^k = 0$ for k odd, and

$$F_0 = 0, \qquad F_1 = a_1^0\partial_{\hat\rho}U^0,$$

$$F_k = \sum_{j=0}^{k-1}\hat\rho^j a_1^j\partial_{\hat\rho}U^{k-j-1} + \sum_{j=0}^{k-2}\hat\rho^j\left(a_2^j\partial_{\theta\theta}U^{k-j-2} + a_3^j\partial_\theta U^{k-j-2}\right), \quad \text{for } k \geq 2.$$

With the boundary layer terms defined, we gather that the asymptotic expansion is given by

$$\begin{aligned}u^\epsilon(\mathbf{x}) \sim{}& u^0(\mathbf{x}) + \varepsilon^2 u^(\mathbf{x}) + \varepsilon^4 u^4(\mathbf{x}) + \dots \\ &- \chi(\rho)\big[U^0(\varepsilon^{-1}\rho,\theta) + \varepsilon U^1(\varepsilon^{-1}\rho,\theta) + \varepsilon^2 U^2(\varepsilon^{-1}\rho,\theta) + \dots\big],\end{aligned} \tag{2.1.9}$$

where χ is a smooth cutoff function identically one if $0 \leq \rho \leq \rho_0/3$ and identically zero if $\rho \geq 2\rho_0/3$.

REMARK. The presence of a cutoff function is essential, since the boundary fitted coordinates are only defined in a neighborhood of $\partial\Omega$. Since the boundary correctors decay exponentially to zero, the cutoff functions will introduce only a exponentially small, thus negligible, error.

2.2. Error Estimates for the Asymptotic Expansion

Here we estimate how close a truncated asymptotic expansion approximates the exact solution. We shall assume that C is a constant that might depend on the domain, and the right hand side f.

We first estimate the boundary correctors. For every s and positive integer i, there exist ε-independent constants C and α such that

$$\|u^{2i}\|_{H^s(\Omega)} \le C, \quad U^i(\hat{\rho}) \le C\exp(-\alpha\hat{\rho}). \tag{2.2.1}$$

Although (2.1.9) is a formal expansion, a rigorous error estimate shows that the difference between the exact solution and a truncated asymptotic expansion is of the same order of the first term omitted in the expansion. In fact, define

$$e_{2N}(\mathbf{x}) = u^\epsilon(\mathbf{x}) - \sum_{k=0}^{N} \varepsilon^{2k} u^{2k}(\mathbf{x}) + \chi(\rho)\sum_{k=0}^{2N} \varepsilon^k U^k(\varepsilon^{-1}\rho, \theta). \tag{2.2.2}$$

In the theorem below we bound the $H^1(\Omega)$ norm of e_{2N}.

THEOREM 2.2.1. *For any positive integer N, there exists a constant C such that the difference between the truncated asymptotic expansion and the original solution measured in the original domain is bounded as follows:*

$$\|e_{2N}\|_{H^1(\Omega)} \le C\varepsilon^{2N+1/2}. \tag{2.2.3}$$

Before we prove Theorem 2.2.1, we develop some other estimates. For instance, in the $L^2(\Omega)$ norm, we have from the triangle inequality that

$$\|e_{2N}\|_{L^2(\Omega)} \le \|e_{2N+2}\|_{H^1(\Omega)} + \|e_{2N+2} - e_{2N}\|_{L^2(\Omega)}.$$

Since

$$(e_{2N+2} - e_{2N})(\mathbf{x}) = -\varepsilon^{2N+1}\chi(\rho)U^{2N+1}(\varepsilon^{-1}\rho, \theta) + \varepsilon^{2N+2}\big[u^{2N+2}(\mathbf{x}) - \chi(\rho)U^{2N+2}(\varepsilon^{-1}\rho, \theta)\big], \tag{2.2.4}$$

we conclude that

$$\|e_{2N}\|_{L^2(\Omega)} = O(\varepsilon^{2N+3/2}),$$

for N nonnegative.

Using similar arguments, it is possible to compute *interior estimates*, which achieve better convergence in regions “far away” from the lateral boundary of the plate. The reason for the improvement in such subdomains is that the influence of the boundary layer is negligible. The table below presents these interior and various other error estimates. We assume that f is a sufficiently smooth function and we show only the order of the norms with respect to ε. “BL” stands for “Boundary Layer” and the “Relative Error” column presents the norm of e_{2N} divided by the norm of u^ϵ. In parentheses are the interior estimates, when these are better than the global estimates.

The remainder of this section contains a proof of Theorem 2.2.1.

TABLE 1. Order with respect to ε of the exact solution, the first term of the boundary layer expansion, and the difference between the solution and a truncated asymptotic expansion in various norms.

norm	u^ϵ	BL	$e_{2N},\ N \geq 0$	Relative Error
$\\|\cdot\\|_{L^2(\Omega)}$	1	$\varepsilon^{1/2}$	$\varepsilon^{2N+3/2}\big(\varepsilon^{2N+2}\big)$	$\varepsilon^{2N+3/2}\big(\varepsilon^{2N+2}\big)$
$\\|\partial_\rho \cdot\\|_{L^2(\Omega)}$	$\varepsilon^{-1/2}(1)$	$\varepsilon^{-1/2}$	$\varepsilon^{2N+1/2}\big(\varepsilon^{2N+2}\big)$	$\varepsilon^{2N+1}\big(\varepsilon^{2N+2}\big)$
$\\|\partial_\theta \cdot\\|_{L^2(\Omega)}$	1	$\varepsilon^{1/2}$	$\varepsilon^{2N+3/2}\big(\varepsilon^{2N+2}\big)$	$\varepsilon^{2N+3/2}\big(\varepsilon^{2N+2}\big)$
$\\|\cdot\\|_{H^1(\Omega)}$	$\varepsilon^{-1/2}(1)$	$\varepsilon^{-1/2}$	$\varepsilon^{2N+1/2}\big(\varepsilon^{2N+2}\big)$	$\varepsilon^{2N+1}\big(\varepsilon^{2N+2}\big)$

DEFINITION 2.2.2. *Set*

$$u_{2N}(\mathbf{x}) = \sum_{k=0}^{N} \varepsilon^{2k} u^{2k}(\mathbf{x}), \quad U_{2N}(\mathbf{x}) = \sum_{k=0}^{2N} \varepsilon^{k} U^{k}(\varepsilon^{-1}\rho, \theta, x_3).$$

Some results regarding the boundary layer terms are collected below.

LEMMA 2.2.3. For any positive integer N, there exists positive constants C and α such that

$$\|\chi' U_{2N}\|_{L^2(\Omega)} + \varepsilon\|\chi'\partial_\rho U_{2N}\|_{L^2(\Omega)} \leq C\exp(-\alpha\varepsilon^{-1}). \tag{2.2.5}$$

Also, for all $v \in H_0^1(\Omega)$,

$$\left|\int_\Omega \underline{\nabla}\, U_{2N}\, \underline{\nabla}(\chi v) + U_{2N}\chi v\, d\mathbf{x}\right| \leq C\varepsilon^{2N}\|v\|_{H^1(\Omega)}. \tag{2.2.6}$$

PROOF. The inequalities (2.2.5) follow from a change of coordinates, (2.2.1), and the definition of χ. To see that (2.2.6) holds, first rewrite (2.1.6) as a finite series, using Taylor expansion with remainders. Then the result follows from the definition of U_{2N}, (2.1.8), and (2.2.1). □

PROOF (OF THEOREM 2.2.1). Let $v \in H_0^1(\Omega)$. If we define

$$E(2N, v) = \int_\Omega \underline{\nabla}(u^\epsilon - u_{2N})\, \underline{\nabla}\, v + (u^\epsilon - u_{2N})v\, d\mathbf{x},$$

then, by construction of the asymptotic expansion, we have

$$E(2N, v) = \int_\Omega f v\, d\mathbf{x} - \sum_{k=0}^{N} \varepsilon^{2k} \int_\Omega \big(\underline{\nabla}\, u^{2k}\, \underline{\nabla}\, v + u^{2k} v\big)\, d\mathbf{x} = -\varepsilon^{2N} \int_\Omega \underline{\nabla}\, u^{2N}\, \underline{\nabla}\, v\, d\mathbf{x},$$

and we conclude that

$$|E(2N, v)| \leq C\varepsilon^{2N}\|v\|_{H^1(\Omega)}. \tag{2.2.7}$$

We also have

$$\left|\int_\Omega \underline{\nabla}(\chi U_{2N})\, \underline{\nabla}\, v - \underline{\nabla}\, U_{2N}\, \underline{\nabla}(\chi v)\, d\mathbf{x}\right| \leq \big(\|\chi' U_{2N}\|_{L^2(\Omega)} + \|\chi'\partial_\rho U_{2N}\|_{L^2(\Omega)}\big)\|v\|_{H^1(\Omega)}.$$

Hence, by Lemma 2.2.3

$$\left|\int_\Omega \left[\underset{\sim}{\nabla}(\chi U_{2N})\,\underset{\sim}{\nabla}\, v + \chi U_{2N} v\right] d\mathbf{x}\right| \le C\varepsilon^{2N}\|v\|_{H^1(\Omega)}. \tag{2.2.8}$$

Finally, since e_{2N} vanishes on $\partial\Omega$,

(2.2.9)

$$\|e_{2N}\|^2_{H^1(\Omega)} = \int_\Omega |\underset{\sim}{\nabla}\, e_{2N}|^2 + (e_{2N})^2\, d\mathbf{x} = E(2N, e_{2N}) + \int_\Omega \left[\underset{\sim}{\nabla}(\chi U_{2N})\,\underset{\sim}{\nabla}\, e_{2N} + \chi U_{2N} e_{2N}\right] d\mathbf{x}$$
$$\le C\varepsilon^{2N}\|e_{2N}\|_{H^1(\Omega)},$$

from (2.2.7) and (2.2.8). The estimate (2.2.9) is not sharp yet, so we use the triangle inequality:

$$\|e^\epsilon_{2N}\|_{H^1(\Omega)} \le \|e_{2N+2} - e_{2N}\|_{H^1(\Omega)} + O(\varepsilon^{2N+2}),$$

and then the result follows from (2.2.4). □

2.3. Estimates for Non Smooth Domain

We consider now estimates for problem (2.1.1) when the domain Ω is not necessarily smooth, for example, if Ω is polygonal.

We shall need the following interpolation inequality:

$$\|g\|^u_{s+v,\Omega} \le \|g\|^{u-v}_{s,\Omega}\|g\|^v_{s+u,\Omega}, \quad s \ge 0,\ u \ge v \ge 0. \tag{2.3.1}$$

Furthermore, for $g \in L^2(\Omega)$, let $\Delta^{-1} g$ be the unique function in $H^2(\Omega) \cap \mathring{H}^1(\Omega)$ whose Laplacian is equal to g. Then

$$C^{-1}\|\,\Delta^{-1} g\|_{s+2,\Omega} \le \|g\|_{s,\Omega} \le C\|\,\Delta^{-1} g\|_{s+2,\Omega}, \quad s \ge 0. \tag{2.3.2}$$

See [**3**] for further details.

From [**2**], we have the following result. We reproduce here the proof, in some detail.

LEMMA 2.3.1. Let $f \in H^1(\Omega)$, and u be the solution of (2.1.1). Then there exists a constant that might depend on Ω and σ such that

$$\varepsilon^2\|\,\nabla u\|^2_{0,\Omega} + \|u - f/\,\sigma\,\|^2_{0,\Omega} \le C\big(\varepsilon\|f/\,\sigma\,\|^2_{0,\partial\Omega} + \varepsilon^2\|f/\,\sigma\,\|^2_{1,\Omega}\big).$$

PROOF. Multiplying the differential equation by $-\Delta u$, and integrating by parts yields

$$\varepsilon^2\|\,\Delta u\|^2_{0,\Omega} + \sigma\,\|\,\nabla u\|^2_{0,\Omega} = \int_\Omega \nabla f\,\nabla u\, d\mathbf{x} - \int_{\partial\Omega} f\frac{\partial u}{\partial n}\, ds.$$

Note that using the trace inequality, and (2.3.1) with $u = 1$, $v = 1/2$, and $s = 1$, we find that

$$\|\frac{\partial u}{\partial n}\|_{0,\partial\Omega} \le \|u\|_{3/2,\Omega} \le \|u\|^{1/2}_{1,\Omega}\|u\|^{1/2}_{2,\Omega}.$$

Hence,

$$\|\frac{\partial u}{\partial n}\|^2_{0,\partial\Omega} \le C\big(\varepsilon^{-1}\|u\|^2_{1,\Omega} + \varepsilon\|u\|^2_{2,\Omega}\big).$$

So, for any $\delta_1 > 0$,

$$\left|\int_{\partial\Omega} f \frac{\partial u}{\partial n}\, ds\right| \le \|f\|_{0,\partial\Omega} \|\frac{\partial u}{\partial n}\|_{0,\partial\Omega} \le C_{\delta_1}\varepsilon^{-1}\|f\|^2_{0,\partial\Omega} + \delta_1\varepsilon\|\frac{\partial u}{\partial n}\|^2_{0,\partial\Omega}$$

$$\le C_{\delta_1}\varepsilon^{-1}\|f\|^2_{0,\partial\Omega} + C\delta_1\big(\|u\|^2_{1,\Omega} + \varepsilon^2\|u\|^2_{2,\Omega}\big) \le C_{\delta_1}\varepsilon^{-1}\|f\|^2_{0,\partial\Omega} + C\delta_1\big(\|u\|^2_{1,\Omega} + \varepsilon^2\|\Delta u\|^2_{0,\Omega}\big),$$

where the norm equivalence (2.3.2) was used in the last inequality above. Similarly, for any $\delta_2 > 0$,

$$\left|\int_{\Omega} \nabla f\, \nabla u\, d\mathbf{x}\right| \le \|f\|_{1,\Omega}\|u\|_{1,\Omega} \le C_{\delta_2}\|f\|^2_{1,\Omega} + \delta_2\|u\|^2_{1,\Omega}.$$

It follows from these estimates, and careful choices of δ_1 and δ_2 that

$$\varepsilon^2\|\Delta u\|^2_{0,\Omega} + \sigma\|\nabla u\|^2_{0,\Omega} \le C(\varepsilon^{-1}\|f\|^2_{0,\partial\Omega} + \|f\|^2_{1,\Omega}).$$

We finally multiply the above inequality by ε^2, and use that

$$\varepsilon^2\,\Delta u = u - f$$

to conclude the proof. □

Hence, if $f \in H^1(\Omega)$, the solution u converges to f/σ in $L^2(\Omega)$ as $\varepsilon \to 0$.

CHAPTER 3

Finite Element Approximations for Reaction–Diffusion Equation

Here we present an introdutory discussion on how to use finite element techniques to approximate the singuar perturbed Reaction–Diffusion problem

$$\begin{aligned} \mathcal{L}^{\epsilon} u &:= -\varepsilon^2 \Delta u + \sigma u = f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.0.3}$$

where Ω is a two-dimensional bounded, Lipschitz domain, ε is a positive constant and σ is a positive constant.

In what follows, we consider a partition $\mathcal{T} = \{K_j\}$ of Ω into "quadrilateral elements." Some usual restrictions apply:

$$K_i \cap K_j = \emptyset \quad \text{if } i \neq j, \qquad \cup_j \bar{K}_j = \bar{\Omega},$$

and for $i \neq j$, the intersection $\partial K_i \cap \partial K_j$ is either a common edge or a vertex.

Finally, let $P^1(\Omega)$, be the space of continuous functions in Ω that are bilinear polynomials in each quadrilateral, and define $P_0^1(\Omega) = P^1(\Omega) \cap H_0^1(\Omega)$.

In Section 3.1, we ilustrate the pitfalls of the Classical Galerkin approximation. Then, in Section 3.2, we propose a new scheme that is "asymptotically correct," in the sense that performs well as the small parameter of the problem approaches zero.

3.1. Classical Galerkin Approximation

We briefly describe a Galerkin approximation for (3.0.3), and show numerical results displaying its limitations. Basically, the same phenomena that we described in Section 1.1 of Chapter 1 occurs. The error analysis is also similar, with the same shortcomings, hence we do not repeat it here.

In the Galerkin formulation, we seek $u^h \in P_0^1(\Omega)$, such that

$$a(u^h, v^h) = (f, v^h) \quad \text{for all } v^h \in P_0^1(\Omega).$$

We include an example to show how the Galerkin method fails to approximate boundary layer, given in a unrefined mesh. Consider $\Omega = (0,1) \times (0,1)$, $f = 1$ with $u = 0$ on $\partial\Omega$. The domain description follows in figure 1.

The Galerkin approximation for $\varepsilon^2 = 10^{-6}$ is depicted in figure 2.

3.2. Toward Multiscale Functions: Enriching Finite Element Spaces

In this section we describe the work developed in [**21**]. We are interested in finding a finite element discretization for (3.0.3) that is stable and coarse mesh accurate for all ε. We use the approach of enriching the finite element space. The idea is to add special functions to the usual polynomial spaces to stabilize and improve accuracy of the Galerkin

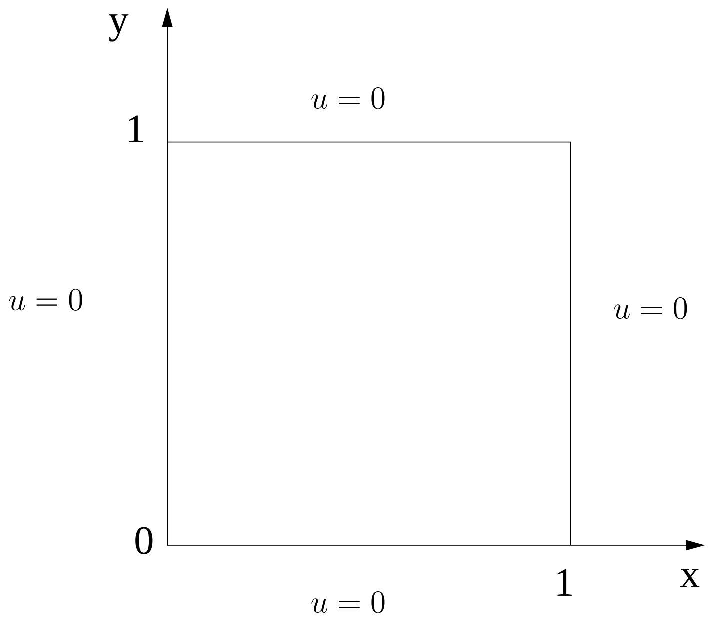


FIGURE 1. Problem Description

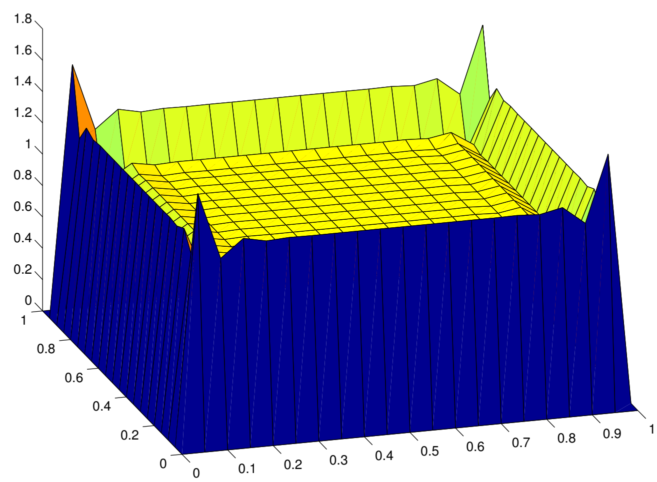


FIGURE 2. Galerkin Solution for $\varepsilon^2 = 10^{-6}$

method. This goes along the philosophy of the *Residual Free Bubbles (RFB)* method [**9**] (see also [**10**], [**18**], [**19**], [**20**]), and actually extends it.

We use a Petrov–Galerkin formulation, i.e., the space of test functions differs from the trial space. We shall choose the space of test functions as polynomial plus bubbles, but we use a different trial space.

3.2.1. New Enriched Choice. Consider

$$U^h = P_0^1(\Omega) \oplus E^*(\Omega),$$

as the trial space, where $E^*(\Omega)$ is yet to be defined. As the test space, we set

$$P_0^1(\Omega) \bigoplus_{K\in\mathcal{T}} H_0^1(K),$$

In our Petrov–Galerkin formulation, we seek $u^h = u^1 + u^* \in U^h$, where $u^1 \in P_0^1(\Omega)$ and $u^* \in E^*(\Omega)$, and

$$a(u^h, v^h) = (f, v^h) \quad \text{for all } v^h \in P_0^1(\Omega), \tag{3.2.1}$$

$$a(u^h, v) = (f, v) \quad \text{for all } v \in H_0^1(K) \text{ and all } K \in \mathcal{T}. \tag{3.2.2}$$

From (3.2.2), we conclude that, for every K,

$$\mathcal{L}\, u^* = f - \mathcal{L}\, u^1 \quad \text{in } K. \tag{3.2.3}$$

The usual residual-free bubble formulation subjects u^* to a homogeneous Dirichlet element boundary condition, i.e., $u^* = 0$ on ∂K, for all elements K. Herein, we replace this condition by a more sophisticated choice, based on ideas by Hou and Wu [**23**].

To determine u^* uniquely, we impose the boundary conditions

$$u^* = 0 \quad \text{on } \partial K, \text{ if } \partial K \subset \partial\Omega, \qquad \mathcal{L}_{\partial K}\, u^* = \mathcal{R}(f - \mathcal{L}\, u^1) \quad \text{on } \partial K, \text{ if } \partial K \not\subset \partial\Omega, \tag{3.2.4}$$

$$u^* = 0 \quad \text{on all vertices of } K, \tag{3.2.5}$$

where $\mathcal{R}$ is the trace operator, and we choose

$$\mathcal{L}_{\partial K}\, v = -\varepsilon^2 \partial_{ss} v + \sigma\, v, \tag{3.2.6}$$

where s denotes a variable that runs along ∂K. Note that the restriction of f to K must be regular enough so that its trace on ∂K makes sense. Henceforth, we assume that $f \in P^1(\Omega)$.

The choice of (3.2.6) is ad hoc, and by no means unique. But it can be justified under the light of asymptotic analysis. Indeed this is the equation satisfied by the boundary correctors, *in the direction of the boundary layers*, see (2.1.8). Hence, we are enriching the space of polynomials with functions that have the same behavior as the correctors. In some sense, the polynomial part of the approximation (u^1 in our case) "captures" the smooth behavior of the exact solution. The local, "multiscale behavior" is seen by the enrichment functions (u^* in our case), that adds its contribution to the final formulation, without making the method expensive. In other words, it is possible to describe the multiscale characteristics of a solution for a singular perturbed PDE, without having to resolve all the fine scales with a refined mesh.

We can formally write the solution of (3.2.3)–(3.2.6) as

$$u^* = \mathcal{L}_*^{-1}(f - \mathcal{L}_{\mathcal{T}}\, u^1) \in L^2(\Omega), \quad \text{where } \mathcal{L}_{\mathcal{T}} = \sum_{K\in\mathcal{T}} \chi_K\, \mathcal{L}, \tag{3.2.7}$$

and χ_K is the characteristic function of K. We finally set $E^*(\Omega) = \mathcal{L}_*^{-1}\, P^1(\Omega)$.

Substituting (3.2.7) in (3.2.1), we gather that

$$a((I - \mathcal{L}_*^{-1}\, \mathcal{L}_{\mathcal{T}})u^1, v^h) = (f, v^h) - a(\mathcal{L}_*^{-1}\, f, v^h) \quad \text{for all } v^h \in P_0^1(\Omega). \tag{3.2.8}$$

Finally, $u^h = (I - \mathcal{L}_*^{-1}\, \mathcal{L}_{\mathcal{T}})u^1 + \mathcal{L}_*^{-1}\, f$. Note nevertheless that, because of (3.2.5), $u^h = u^1$ at the nodal points, as in the usual polynomial Galerkin formulation.

Remark. Note that our particular choice of test space allowed the *static condensation* procedure, i.e., we were able to write u^* with respect to u^1 and f, as in (3.2.7).

The matrix formulation can be obtained as follows. Under the assumption that $f \in P^1(\Omega)$, we write

$$f = \sum_{j\in J} f_j \psi_j, \quad u^1 = \sum_{j\in J_0} u_j^1 \psi_j$$

where J and J_0 are the set of indexes of total and interior nodal points, $\{\psi_j\}_{j\in J}$ form a basis of $P^1(\Omega)$, and $\{\psi_j\}_{j\in J_0}$ form a basis of $P_0^1(\Omega)$. Substituting in (3.2.7), we have that

$$u^* = \sum_{j\in J} \frac{f_j}{\sigma} \mathcal{L}_*^{-1} \mathcal{L}_{\mathcal{T}} \psi_j - \sum_{j\in J_0} u_j^l \mathcal{L}_*^{-1} \mathcal{L}_{\mathcal{T}} \psi_j, \tag{3.2.9}$$

where we used that

$$\mathcal{L}_{\mathcal{T}} \psi_j = \sigma \psi_j. \tag{3.2.10}$$

To write the variational formulation in an explicit form, it is convenient to define $\lambda_j = (\sigma I - \mathcal{L}_*^{-1} \mathcal{L}_{\mathcal{T}})\psi_j$. Hence, (3.2.8) reads as

$$\sum_{j\in J_0} a(\lambda_j, \psi_i) u_j^1 = \sum_{j\in J} \big[(\psi_j, \psi_i) - a(\mathcal{L}_*^{-1} \psi_j, \psi_i)\big] f_j \quad \text{for all } i \in J_0. \tag{3.2.11}$$

Using the definition of the bilinear form $a(\cdot,\cdot)$, and (3.2.10), yields

$$\sum_{j\in J_0} a(\lambda_j, \psi_i) u_j^1 = \sum_{j\in J} \big[(\lambda_j, \psi_i) - \frac{\varepsilon^2}{\sigma}(\underset{\sim}{\nabla} \psi_j, \underset{\sim}{\nabla} \psi_i) + \frac{\varepsilon^2}{\sigma}(\underset{\sim}{\nabla} \lambda_j, \underset{\sim}{\nabla} \psi_i)\big] f_j \quad \text{for all } i \in J_0. \tag{3.2.12}$$

Concrete computations of the matrix formulation follows.

3.2.2. Solving Local Problems. A core and troublesome issue in the present method is solving the local problems. From its definition, λ_j solves

$$\begin{gathered} \mathcal{L} \lambda_j = 0 \quad \text{in } K, \\ \mathcal{L}_{\partial K} \lambda_j = 0 \quad \text{on } \partial K, \qquad \lambda_j = \begin{cases} 1 & \text{on the jth vertice of } \mathcal{T}, \\ 0 & \text{on the other vertices of } \mathcal{T}, \end{cases} \end{gathered} \tag{3.2.13}$$

Remark. In fact, we have that $\mathcal{L}_{\partial K} \lambda_j = \mathcal{L}_{\partial K} \psi_j - \mathcal{L} \psi_j$ on ∂K. Since we are assuming that ψ_j is bilinear over a rectangular mesh, we have that ψ_j is still linear *over* ∂K. Hence, $\mathcal{L}_{\partial K} \psi_j = \sigma \psi_j$.

If we take a particular node $I \in J_0$, and look at all elements connected to this node, then the equation (3.2.13) can be used to illustrate the nodal shape functions λ_I. Fixing $\sigma = 1$, we obtain for $\varepsilon = 1$, 10^{-1}, 10^{-3}, the shape functions λ_I, depicted in figures 3 and 4. Note that as ε approaches zero, the usual pyramid is squeezed in its domain of influence in the neighborhood around the node I. The functions plotted below were computed using the formulas described in subsection 3.2.3.

Actually, the functions used to enrich the finite element space, i.e., the functions in $E^*(\Omega)$ are as in Figures 5 and 6.

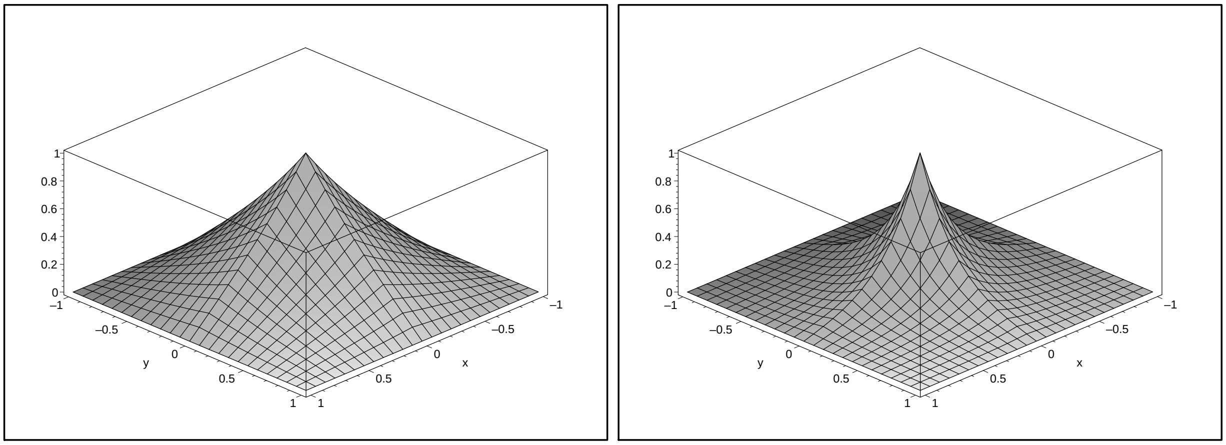


FIGURE 3. The function λ for $\varepsilon = 1,\, 10^{-1}$

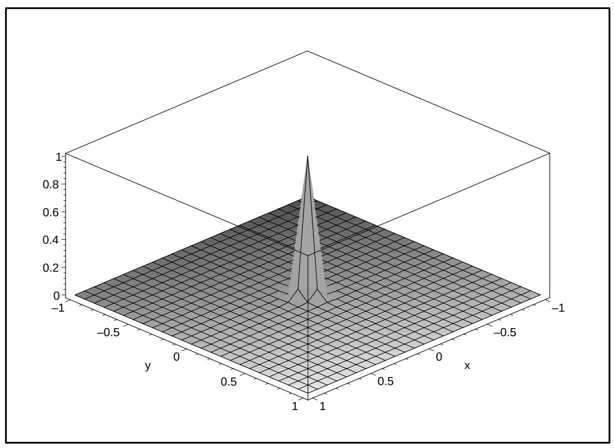


FIGURE 4. The function λ for 10^{-3}

3.2.3. Straight Bilinear Element Case. Consider now a rectangular straight mesh. Our goal is to find λ_j. Without loss of generality, consider a rectangle K with vertices $1, \cdots, 4$ at $(0,0)$, $(h_x, 0)$, (h_x, h_y), and $(0, h_y)$. So, again without loss of generalization, we want to find λ_1. We have that

$$-\varepsilon^2 \,\Delta\, \lambda_1 + \sigma\, \lambda_1 = 0 \quad \text{in } K. \tag{3.2.14}$$

On the side $y = 0$, we have that

$$-\varepsilon^2 \partial_{xx} \lambda_1 + \sigma\, \lambda_1 = 0 \quad \text{for } x \text{ in } (0,1),$$
$$\lambda_1(0,0) = 1, \quad \lambda_1(h_x, 0) = 0.$$

Hence,

$$\lambda_1(x, 0) = \mu_x(x) := -\frac{\sinh\bigl(\varepsilon^{-1}\sqrt{\sigma}(x - h_x)\bigr)}{\sinh\bigl(\varepsilon^{-1}\sqrt{\sigma} h_x\bigr)}. \tag{3.2.15}$$

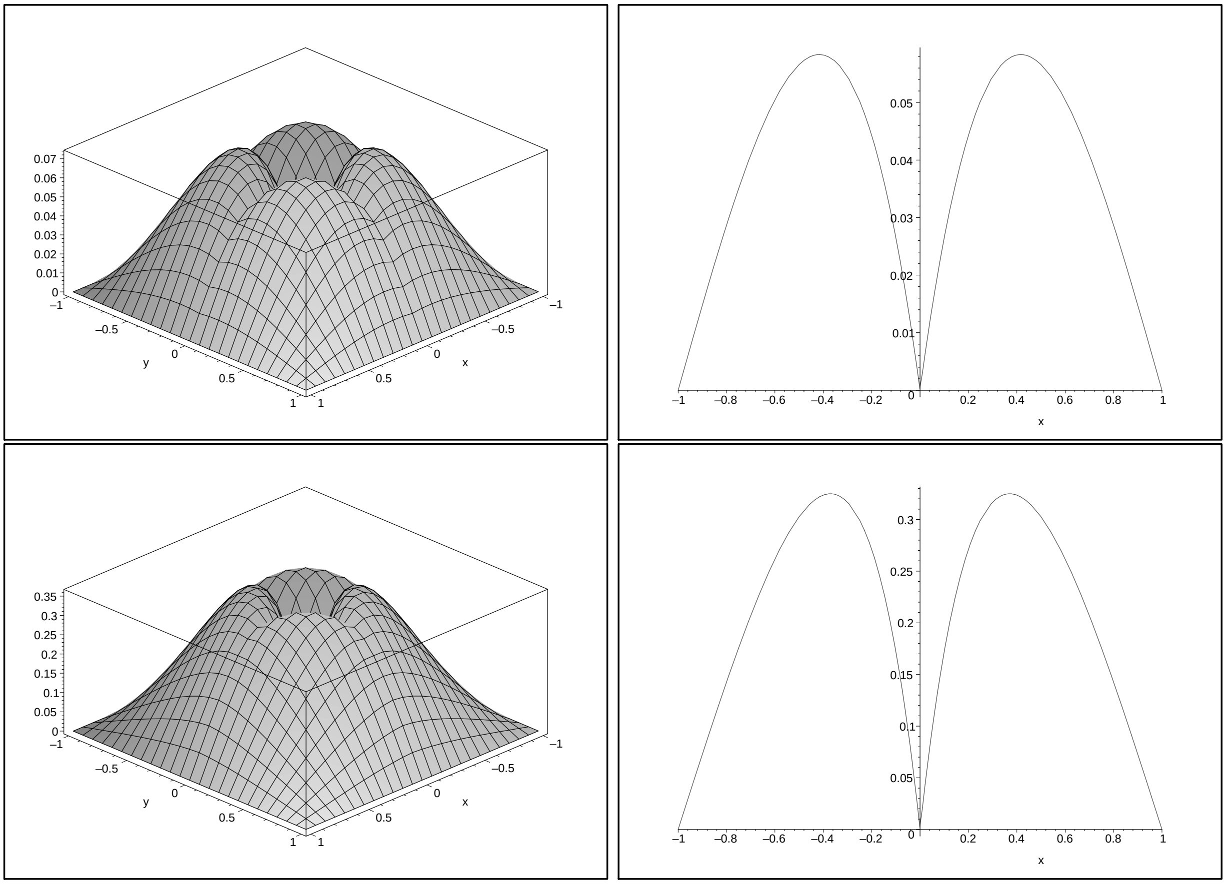


FIGURE 5. Enriching functions for $\varepsilon = 1$ and $\varepsilon^2 = 0.1$.

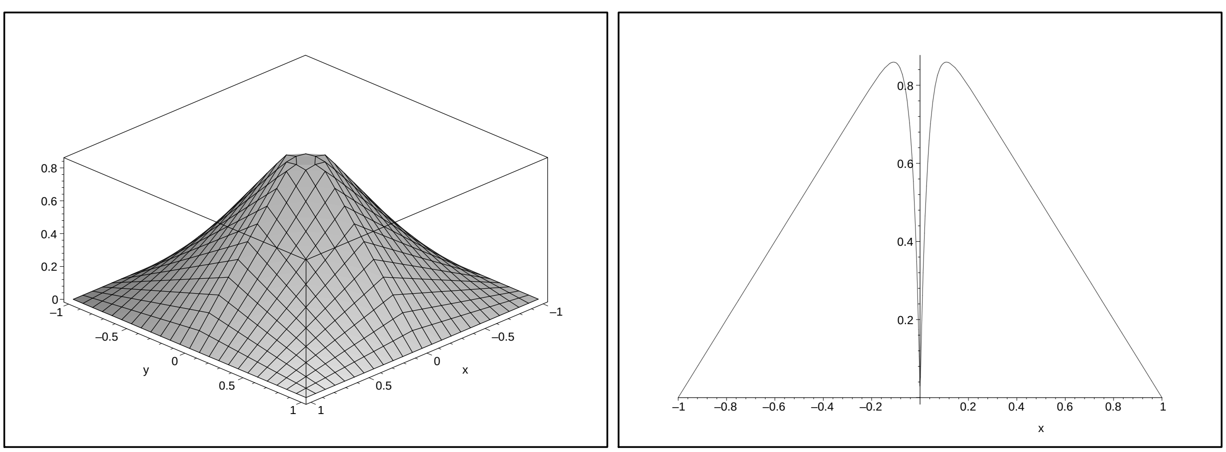


FIGURE 6. Enriching functions for $\varepsilon^2 = 10^{-3}$.

Similarly,

$$\lambda_1(0,y) = \mu_y(y) := -\frac{\sinh\big(\varepsilon^{-1}\sqrt{\sigma}(y-h_y)\big)}{\sinh\big(\varepsilon^{-1}\sqrt{\sigma}h_y\big)}, \qquad \lambda_1(h_x,y) = \lambda_1(x,h_y) = 0. \tag{3.2.16}$$

We propose two simple closed forms for λ_1, none of which satisfy (3.2.14)–(3.2.16) exactly. If we set $\lambda_1(x,y) = \mu_x(x)\mu_y(y)$, then (3.2.15)–(3.2.16) holds, but

$$-\varepsilon^2 \, \Delta \, \lambda_1 + 2 \, \sigma \, \lambda_1 = 0 \quad \text{in } K,$$

thus (3.2.14) is *not* satisfied.

If we let

$$\lambda_1(x,y) = \frac{\sinh\left(\varepsilon^{-1}\sqrt{\frac{\sigma}{2}}(x-h_x)\right)\sinh\left(\varepsilon^{-1}\sqrt{\frac{\sigma}{2}}(y-h_y)\right)}{\sinh\left(\varepsilon^{-1}\sqrt{\frac{\sigma}{2}}h_x\right)\sinh\left(\varepsilon^{-1}\sqrt{\frac{\sigma}{2}}h_y\right)},$$

then (3.2.14) holds, but the boundary conditions at $x = 0$ and $y = 0$ do not hold.

3.2.4. A Numerical Test: Source Problem. As in Section 3.1, consider the unit source problem ($f = 1$) defined on the unit square depicted in Figure 1, subject to a homogeneous Dirichlet boundary condition. For a fixed $\sigma = 1$, and small ε, boundary layers appear close to the domain boundary. Figure 7 shows, for $\varepsilon^2 = 10^{-6}$, the solutions of three different methods, Galerkin, Residual Free Bubble, and the present enriched method. It is clear that the current method performs better than the other two methods. Examining the solutions profiles, see Figure 8, it becomes clear that the current method is superior to other methods. For $\varepsilon^2 = 10^{-3}$ and $\varepsilon = 1$, all methods have comparable performance, see Figures 9, 10 .

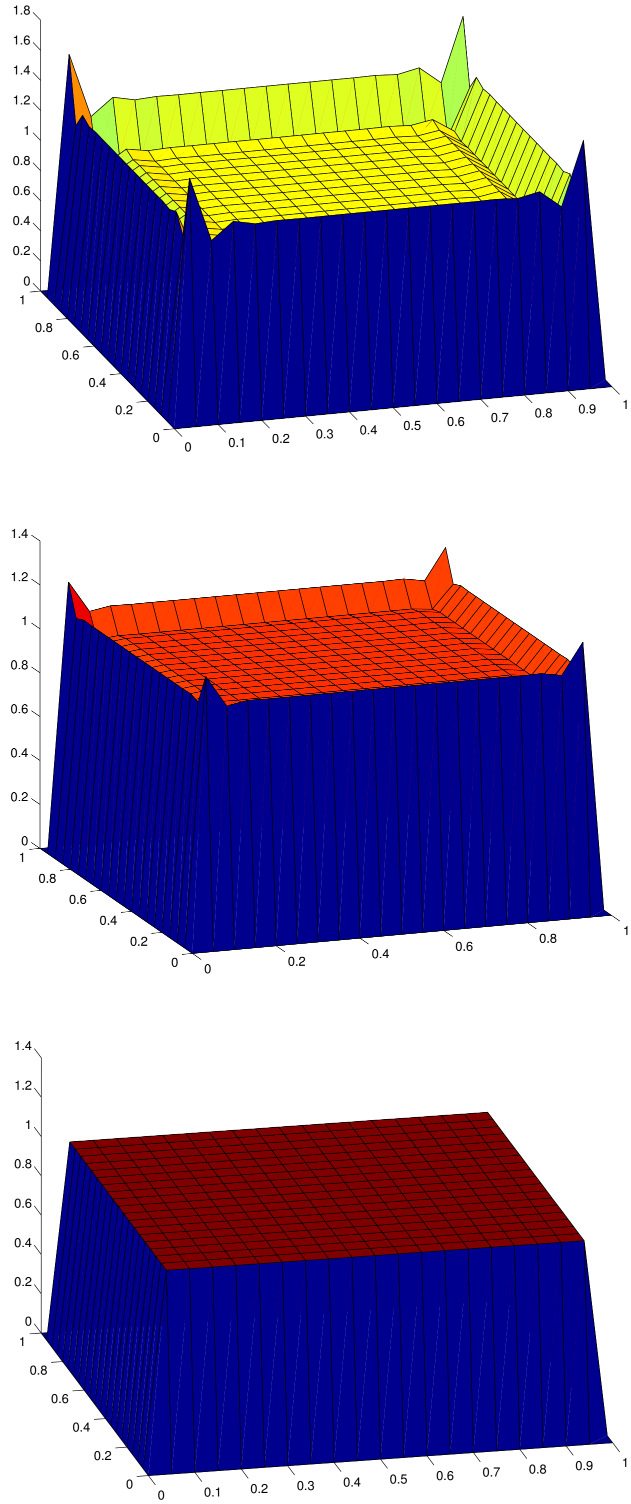


Figure 7. Comparison among Galerkin, Residual Free Bubble, and the enriched methos for $\varepsilon^2 = 10^{-6}$.

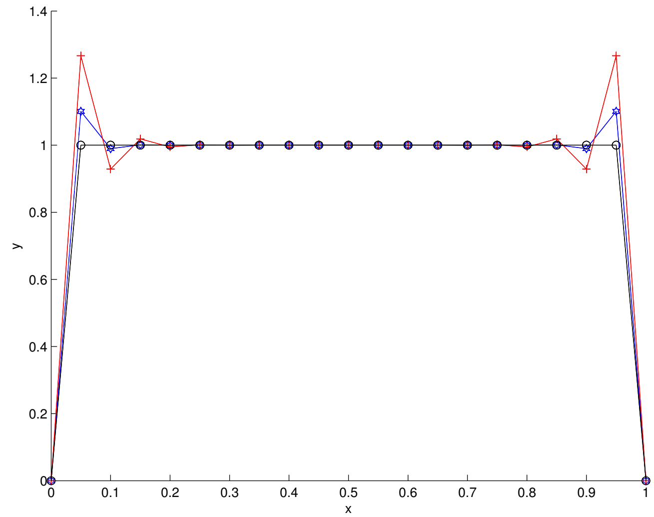


FIGURE 8. Profile of solutions at $x = 0.5$ ($\varepsilon^2 = 10^{-6}$).

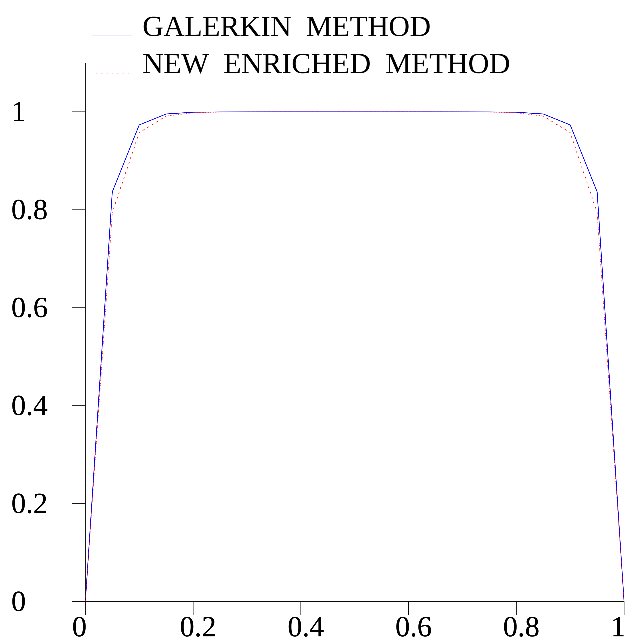


FIGURE 9. Profile of solutions at $x = 0.5$ ($\varepsilon^2 = 10^{-3}$).

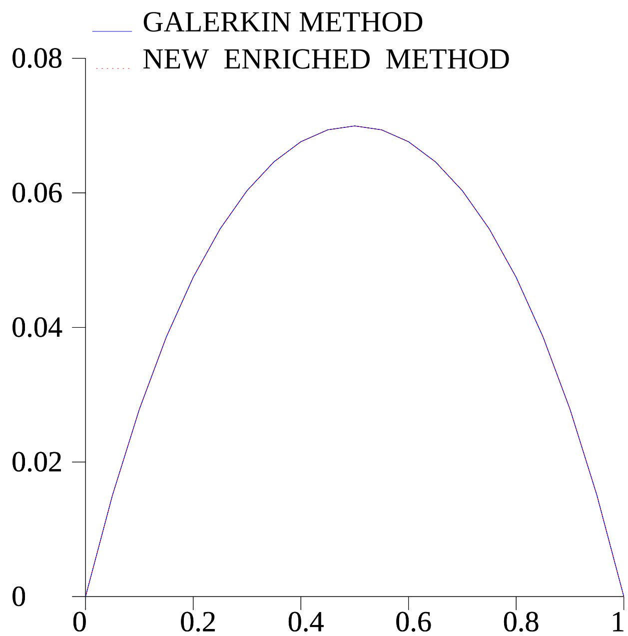


FIGURE 10. Profile of solutions at $x = 0.5$ ($\varepsilon^2 = 10^{-3}$).

CHAPTER 4

Modeling PDEs in domains with Rough Boundaries

We are interested in PDEs defined in domains where at least part of the boundary is rugous. The goal is to avoid the expensive discretization of the rough domain, and replace the original PDE by another, in a domain that is easier to discretize. Asymptotic Expansions play a key role here, motivating the development of models, and helping in deriving error estimates.

4.1. Asymptotic Expansion Definition

Let $\Omega^\epsilon \subset \mathbb{R}^2$ be the domain depicted in figure 1. It is basically a square with sides of length one, where the bottom is rugged.

The bottom of Ω^ϵ, containing the wrinkles, is described by $\boldsymbol{\psi}^\epsilon(x_1) = (x_1, \varepsilon\psi_r(\varepsilon^{-1}x_1))$, where $x_1 \in [0,1]$. The function $\psi_r : \mathbb{R} \to \mathbb{R}$ is independent of ε, Lipschitz-continuous with $\psi_r(0) = 0$, and periodic of period 1. Without loss of generality, we can assume $\|\psi_r\|_{L^\infty(\mathbb{R})} = 1$.

We now consider the problem

$$
\begin{aligned}
-\,\Delta\, u^\epsilon &= f \quad \text{in } \Omega^\epsilon, \\
u^\epsilon &= 0 \quad \text{on } \partial\Omega^\epsilon.
\end{aligned}
\tag{4.1.1}
$$

It is clear that the solution $u\varepsilon$ depends in a nontrivial way on the small parameter ε. It is our goal to make clear how is this dependence, and how this can be used to develop models for (4.1.1). We shall search for a formal asymptotic expansion in the general form,

$$
u^\epsilon \sim u^0 + \varepsilon v^1(\varepsilon) + \varepsilon \Psi^1(\varepsilon) + \varepsilon^2 v^2(\varepsilon) + \varepsilon^2 \Psi^2(\varepsilon) + \cdots\,, \tag{4.1.2}
$$

where the $\Psi^i(\varepsilon)$ are boundary correctors. The terms $v^i(\varepsilon)$, and $\Psi^i(\varepsilon)$ depend on ε, as the notation indicates. Although this appears strange at first sight, it actually renders the

Figure 1. Description of the domain Ω^ϵ.

development of the expansion easier. At a second stage, it is possible to reorder (4.1.2) as a formal power series where all the terms are independent of ε.

Our procedure to find out the terms in the expansion has a "domain decomposition flavor", and uses the following result.

LEMMA 4.1.1. Let Ω be a Lipschitz bounded domain of $\mathbb{R}^2$ and Γ an interface that divide Ω into two sub-domains Ω^- and Ω^+. Assume that e satisfies $e = 0$ on $\partial\Omega^-\backslash\Gamma \cup \partial\Omega^+\backslash\Gamma$. Then there exists a constant c such that

$$\|e\|_{H^1(\Omega^-)} + \|e\|_{H^1(\Omega^+)} \le c\big(\| \Delta e\|_{0,\Omega^-} + \| \Delta e\|_{0,\Omega^+} + \|[e]\|_{\frac{1}{2},\Gamma} + \|[\partial e/\partial n]\|_{-\frac{1}{2},\Gamma}\big) \tag{4.1.3}$$

where $[\cdot]$ represent the jump function over Γ. The constant c is independent of Ω^-.

PROOF. We first define

$$e^- = e|_{\Omega^-}, \qquad e^+ = e|_{\Omega^+}.$$

It follows from Green's identity that

$$\int_{\Omega^-} |\underset{\sim}{\nabla} e^-|^2 \, d\mathbf{x} = -\int_{\Omega^-} e^- \Delta e^- \, d\mathbf{x} - \int_\Gamma e^- \frac{\partial e^-}{\partial n^+} \, d\Gamma,$$
$$\int_{\Omega^+} |\underset{\sim}{\nabla} e^+|^2 \, d\mathbf{x} = -\int_{\Omega^+} e^+ \Delta e^+ \, d\mathbf{x} + \int_\Gamma e^+ \frac{\partial e^+}{\partial n^+} \, d\Gamma,$$

where n^+ indicates the outward normal vector with respect to Ω^+. Combining both identities, and adding and subtracting $\int_\Gamma e^- \partial e^+/\partial n^+ \, d\Gamma$, we gather that

$$\begin{aligned}
&|e^-|^2_{H^1(\Omega^-)} + |e^+|^2_{H^1(\Omega^+)} \\
&= -\int_{\Omega^-} e^- \Delta e^- \, d\mathbf{x} - \int_{\Omega^-} e^+ \Delta e^+ \, d\mathbf{x} + \int_\Gamma e^- \left(\frac{\partial e^+}{\partial n^+} - \frac{\partial e^-}{\partial n^+}\right) d\Gamma + \int_\Gamma (e^+ - e^-) \frac{\partial e^+}{\partial n^+} \, d\Gamma \\
&\le \|e^-\|_{L^2(\Omega^-)} \| \Delta e^-\|_{L^2(\Omega^-)} + \|e^+\|_{L^2(\Omega^+)} \| \Delta e^+\|_{L^2(\Omega^+)} \\
&+ \|e^-\|_{H^{1/2}(\Gamma)} \|[\frac{\partial e}{\partial n}]\|_{H^{-1/2}(\Gamma)} + \|[e]\|_{H^{1/2}(\Gamma)} \|\frac{\partial e}{\partial n}^+\|_{H^{-1/2}(\Gamma)} \\
&\le (\| \Delta e^-\|_{L^2(\Omega^-)} + \| \Delta e^+\|_{L^2(\Omega^+)}) \|e\|_{L^2(\Omega)} \\
&\qquad + \|e^-\|_{H^1(\Omega^-)} \|[\frac{\partial e}{\partial n}]\|_{H^{-1/2}(\Gamma)} + \|[e]\|_{H^{1/2}(\Gamma)} \|e^+\|_{H^1(\Omega^+)}.
\end{aligned}$$

To obtain the inequalities above, we used Cauchy–Schwartz and trace inequalities, and the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. The lemma follows now from an application of the Poincaré inequality. □

We define now sub-domains of Ω^ϵ, one "close" to the rugosity, the other one far from it. See figure 2. Let $\delta = c_0\varepsilon$, where $c_0 > 1$. Let

$$\Omega_s = \{ \mathbf{x} = (x_1, x_2) \in \Omega^\epsilon : x_2 > \delta \}, \quad \Omega_r^\epsilon = \{ \mathbf{x} = (x_1, x_2) \in \Omega^\epsilon : x_2 < \delta \},$$
$$\Gamma = \{ \mathbf{x} = (x_1, x_2) \in \Omega^\epsilon : x_2 = \delta \}.$$

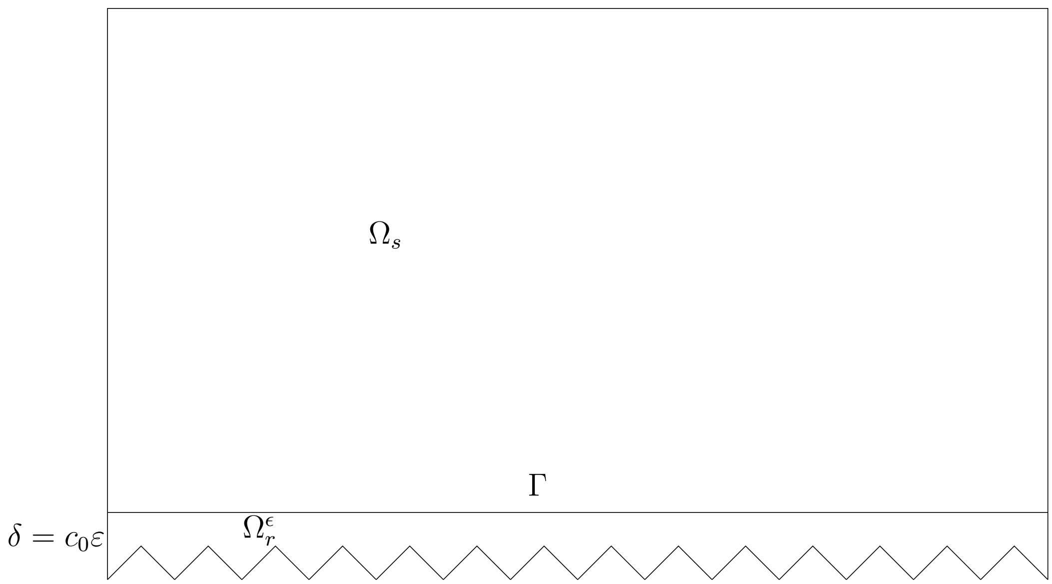


FIGURE 2. Domain decomposition

It is natural to define the first term of the asymptotics such that

$$\begin{gathered} -\Delta u^0 = f \quad \text{in } \Omega_s, \\ u^0 = 0 \quad \text{on } \partial\Omega_s, \qquad u^0 = 0 \quad \text{on } \Omega_r^\epsilon. \end{gathered} \tag{4.1.4}$$

Applying Lemma 4.1.1 with $e = u^\epsilon - u^0$, we see that the source of error is the x_2-derivative jump $[\partial_{x_2} u^0]$:

$$\|e\|_{H^1(\Omega_r^\epsilon)} + \|e\|_{H^1(\Omega_s)} \le c\|\partial_{x_2} u^0\|_{H^{-1/2}(\Gamma)}.$$

We next remedy this. Let $\phi^0(x_1) = \partial_{x_2} u^0(x_1, \delta)$, i.e., ϕ^0 is the restriction of $\partial_{x_2} u^0$ on Γ, and χ_r^ϵ be the characteristic function of the set Ω_r^ϵ, i.e.,

$$\chi_r^\epsilon(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega_r^\epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

We add then the function $-\chi_r^\epsilon(\mathbf{x}) x_2 \phi^0(x_1)$ to the asymptotic expansion, to correct the jump of the x_2-derivative. Since the zero Dirichlet boundary condition at Γ_r^ϵ is not satisfied, we add a corrector to the asymptotics: $\Psi^1(\varepsilon) - \chi_r^\epsilon Z^1(\varepsilon)$. The boundary corrector $\Psi^1(\varepsilon)$ is such that

$$-\Delta \Psi^1(\varepsilon) = -\chi_r^\epsilon \, \Delta(\varepsilon^{-1} x_2 \phi^0 + Z^1(\varepsilon)) \quad \text{in } \Omega^\epsilon, \tag{4.1.5}$$

$$\Psi^1(\varepsilon) = \varepsilon^{-1} x_2 \phi^0 + Z^1(\varepsilon) \quad \text{on } \Gamma_r^\epsilon. \tag{4.1.6}$$

The introduction of $Z^1(\varepsilon)$ is necessary to guarantee an exponential decay of $\Psi^1(\varepsilon)$ to zero in the x_2-direction. Both $\Psi^1(\varepsilon)$ and $Z^1(\varepsilon)$ depend on ε, and the are not even well-defined yet.

In fact, we shall set, in general, $\Psi^i(\varepsilon)$ and $Z^i(\varepsilon)$ as formal expansions in ε,

$$\Psi^i(\varepsilon)(\mathbf{x}) \sim \psi^0(\varepsilon^{-1}\mathbf{x})\phi^{i-1}(x_1) + \varepsilon\psi^1(\varepsilon^{-1}\mathbf{x})\partial_{x_1}\phi^{i-1}(x_1) + \varepsilon^2\psi^2(\varepsilon^{-1}\mathbf{x})\partial_{x_1x_1}\phi^{i-1}(x_1) + \cdots, \tag{4.1.7}$$

$$Z^i(\varepsilon)(\mathbf{x}) \sim z^0\phi^{i-1}(x_1) + \varepsilon z^1\partial_{x_1}\phi^{i-1}(x_1) + \varepsilon^2 z^2\partial_{x_1x_1}\phi^{i-1}(x_1) + \cdots. \tag{4.1.8}$$

In the above, ψ^i is periodic with period ε^{-1} in the x_1 direction, and z^i are ε-independent constants that depend on the geometry of the wrinkles only. We postpone the precise definition of the terms for now.

So, with the error function $e = u^\epsilon - \left[u^0 - \chi_r^\epsilon x_2\phi^0 - \varepsilon\chi_r^\epsilon Z^1(\varepsilon) + \varepsilon\Psi^1(\varepsilon)\right]$, we have

$$\|e\|_{1,\Omega_r^\epsilon} + \|e\|_{1,\Omega_s} \le \varepsilon\|Z^1(\varepsilon) + c_0\phi^0\|_{\frac{1}{2},\Gamma}.$$

We continue to define the terms of the expansion. Set

$$\begin{gathered} -\Delta u^1 = 0 \quad \text{in } \Omega_s, \\ u^1 = -(c_0 + z^0)\phi^0 \quad \text{on } \Gamma, \qquad u^1 = 0 \quad \text{on } \partial\Omega_s\backslash\Gamma, \qquad u^1 = 0 \quad \text{on } \Omega_r^\epsilon. \end{gathered} \tag{4.1.9}$$

So, if $e = u^\epsilon - \left[u^0 - x_2\chi_r^\epsilon\phi^0 - \varepsilon Z^1(\varepsilon)\chi_r^\epsilon + \varepsilon\Psi^1(\varepsilon) + \varepsilon u^1\right]$, then

$$\|e\|_{1,\Omega_r^\epsilon} + \|e\|_{1,\Omega_s} \le \varepsilon\|\partial_{x_2}u^1\|_{-\frac{1}{2},\Gamma} + \varepsilon^2\|z^1\partial_{x_1}\phi^0 + \varepsilon z^2\partial_{x_1x_1}\phi^0 + \cdots\|_{\frac{1}{2},\Gamma}.$$

We define $\phi^1 = \partial_{x_2}u^1|_\Gamma$, and add $-\varepsilon\chi_r^\epsilon x_2\phi^1 + \varepsilon^2\Psi^2(\varepsilon) - \varepsilon^2\chi_r^\epsilon Z^2(\varepsilon)$ to the expansion, where

$$\begin{gathered} -\Delta\Psi^2(\varepsilon) = -\chi_r^\epsilon\,\Delta(\varepsilon^{-1}x_2\phi^1 + Z^2(\varepsilon)) \quad \text{in } \Omega^\epsilon, \\ \Psi^2(\varepsilon) = \varepsilon^{-1}x_2\phi^1 + Z^2(\varepsilon) \quad \text{on } \Gamma_r^\epsilon. \end{gathered}$$

So far,

$$e = u^\epsilon - \left[u^0 - x_2\chi_r^\epsilon\phi^0 - \varepsilon Z^1(\varepsilon)\chi_r^\epsilon + \varepsilon\Psi^1(\varepsilon) + \varepsilon u^1 - \varepsilon x_2\chi_r^\epsilon\phi^1 + \varepsilon^2\Psi^2(\varepsilon) - \varepsilon^2\chi_r^\epsilon Z^2(\varepsilon)\right],$$

and

$$\|e\|_{1,\Omega_r^\epsilon} + \|e\|_{1,\Omega_s} \le \varepsilon^2\|Z^2(\varepsilon) + c_0\phi^1 + z^1\partial_{x_1}\phi^0 + \varepsilon z^2\partial_{x_1x_1}\phi^0 + \cdots\|_{\frac{1}{2},\Gamma}.$$

We proceed one more step by defining $\phi^2 = \partial_{x_2}u^2|_\Gamma$, and adding

$$\varepsilon^2u^2 - \varepsilon^2x_2\chi_r^\epsilon\partial_{x_2}u^2 + \varepsilon^3\Psi^3(\varepsilon) - \varepsilon^3\chi_r^\epsilon Z^3(\varepsilon)$$

to the expansion, where

$$\begin{gathered} -\Delta u^2 = 0 \quad \text{in } \Omega_s, \\ u^2 = -(c_0 + z^0)\phi^1 + z^1\partial_{x_1}\phi^0 \quad \text{on } \Gamma, \qquad u^2 = 0 \quad \text{on } \partial\Omega_s\backslash\Gamma, \qquad u^2 = 0 \quad \text{on } \Omega_r^\epsilon, \end{gathered} \tag{4.1.10}$$

and

$$\begin{gathered} -\Delta\Psi^3(\varepsilon) = -\chi_r^\epsilon\,\Delta(\varepsilon^{-1}x_2\phi^2 + Z^3(\varepsilon)) \quad \text{in } \Omega^\epsilon, \\ \Psi^3(\varepsilon) = \varepsilon^{-1}x_2\phi^2 + Z^3(\varepsilon) \quad \text{on } \Gamma_r^\epsilon. \end{gathered} \tag{4.1.11}$$

The asymptotic error is now

$$\begin{aligned} e = u^\epsilon - \big[u^0 &- x_2\chi_r^\epsilon\phi^0 + \varepsilon\Psi^1(\varepsilon) - \varepsilon\chi_r^\epsilon Z^1(\varepsilon) + \varepsilon u^1 - \varepsilon x_2\chi_r^\epsilon\phi^1 + \varepsilon^2\Psi^2(\varepsilon) - \varepsilon^2\chi_r^\epsilon Z^2(\varepsilon) \\ &+ \varepsilon^2u^2 - \varepsilon^2x_2\chi_r^\epsilon\partial_{x_2}u^2 + \varepsilon^3\Psi^3(\varepsilon) - \varepsilon^3\chi_r^\epsilon Z^3(\varepsilon)\big], \end{aligned}$$

and

$$\|e\|_{1,\Omega_r^\epsilon} + \|e\|_{1,\Omega_s} \le \varepsilon^3\|z^2\partial_{x_1x_1}\phi^0 + z^1\partial_{x_1}\phi^1 + z^0\phi^2 + \varepsilon\cdots\|_{\frac{1}{2},\Gamma}.$$

The expansion pattern should be clear by now, and the successive terms are defined in similar manner.

4.2. The boundary Corrector problem

We now analise the boundary corrector problem in more details. Consider the problem of finding Ψ, and Z such that

$$-\Delta\Psi = -\chi_r^\epsilon\,\Delta(\varepsilon^{-1}x_2\phi + Z) \quad \text{in } \Omega^\epsilon, \tag{4.2.1}$$

$$\Psi = \varepsilon^{-1}x_2\phi + Z \quad \text{on } \Gamma_r^\epsilon. \tag{4.2.2}$$

Here, while ϕ is a given function of x_1 only, Z is unknown a priori, and it is introduced to guarantee that Ψ decays exponentially to zero in the x_2 direction. The solutions Ψ and Z will depend on ε in a nontrivial way, so we assume that formally

$$\Psi \sim \Psi^0 + \varepsilon\Psi^1 + \varepsilon^2\Psi^2 + \cdots, \quad Z \sim Z^0 + \varepsilon Z^1 + \varepsilon^2 Z^2 + \cdots, \tag{4.2.3}$$

The functions involved in (4.2.1), (4.2.2) are not periodic in general. Nevertheless, we try to make use of the periodicity of the wrinkles and recast the corrector problem as a sequence of problems in periodic domains. Furthermore, motivated by the "rapidly" variations of the wrinkles, we make use of the stretched coordinates

$$\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2) = (\varepsilon^{-1}x_1, \varepsilon^{-1}x_2),$$

and assume that a first approximation for Ψ and Z is given by

$$\Psi^0(\mathbf{x}) = \psi^0(\hat{\mathbf{x}})\phi(x_1), \quad Z^0(x_1) = z^0\phi(x_1), \tag{4.2.4}$$

where ψ^0 is $\hat{x}_1$-periodic, z^0 is a constant, and both Ψ^0 and z^0 are to be determined.

Remark. At this stage, there are no good arguments indicating that z^0 is only a constant, but notice that the simpler Z^0 is, the better. And we will show, *a posteriori*, that the form we are assuming for Z^0 suffices to make Ψ^0 exponentially decaying.

It follows from a straightforward computation that the laplacian of a function in the form $u(\hat{x}_1, \hat{x}_2)v(x_1)$ is

$$-(\partial_{11} + \partial_{22})(uv) = -\varepsilon^{-2}\big(\partial_{\hat{x}_2\hat{x}_2}u + \partial_{\hat{x}_1\hat{x}_1}u\big)v - 2\varepsilon^{-1}\partial_{\hat{x}_1}wv' - wv''. \tag{4.2.5}$$

Hence,

$$\begin{gathered} -\Delta(\psi^0\phi) = -\varepsilon^{-2}\big(\partial_{\hat{x}_2\hat{x}_2}\psi^0 + \partial_{\hat{x}_1\hat{x}_1}\psi^0\big)\phi - 2\varepsilon^{-1}\partial_{\hat{x}_1}\psi^0\phi' - \psi^0\phi'', \\ -\Delta(\hat{x}_2\phi) = -\hat{x}_2\phi'', \qquad -\Delta(z^0\phi) = -z^0\phi''. \end{gathered} \tag{4.2.6}$$

Based on (4.2.1), (4.2.3), (4.2.4), and (4.2.6), and formally equating the same powers of ε, we gather that ψ^0 is harmonic. The boundary condition over the wrinkles comes from (4.2.2), (4.2.3), and (4.2.4). The final problem determining the function ψ^0 and the constant z^0 is then

$$\begin{gathered} \partial_{\hat{x}_1\hat{x}_1}\psi^0 + \partial_{\hat{x}_2\hat{x}_2}\psi^0 = 0 \quad \text{in } \Omega_r, \qquad \psi^0 = \hat{x}_2 + z^0 \quad \text{on } \Gamma_r^-, \\ \psi^0 \text{ is } \hat{x}_1\text{-periodic}, \qquad \lim_{\hat{x}_2\to\infty}\psi^0 = 0. \end{gathered}$$

The domain Ω_r occupies the semi-infinite region limited by straight lateral boundaries at $\hat{x}_1 = 0$ and $\hat{x}_1 = 1$, and by the lower boundary $\Gamma_r^- = \{\,(\hat{x}_1, \psi_r(\hat{x}_1)) : \hat{x}_1 \in (0,1)\,\}$.

Hence we have that

$$-\Delta[(\psi^0 - \chi_r^\epsilon \hat{x}_2 + \chi_r^\epsilon z^0)\phi] = -2\varepsilon^{-1}\partial_{\hat{x}_1}\psi^0\phi' - \psi^0\phi'' + \chi_r^\epsilon \hat{x}_2\phi'' - \chi_r^\epsilon z^0\phi'' \quad \text{in } \Omega^\epsilon,$$
$$\psi^0 - \hat{x}_2 + z^0 = 0 \quad \text{on } \Gamma_r^\epsilon.$$

We set then $\Psi^1 = \psi^1\phi'$, $Z^1 = z^1\phi'$ where

$$-[\partial_{\hat{x}_2\hat{x}_2}\psi^1 + \partial_{\hat{x}_1\hat{x}_1}\psi^1] = 2\partial_{\hat{x}_1}\psi^0 \quad \text{in } \Omega_r, \qquad \psi^1 = z^1 \quad \text{on } \Gamma_r^-.$$

Hence,

$$\begin{aligned} -\Delta[(\psi^0 - \chi_r^\epsilon \hat{x}_2 &+ \chi_r^\epsilon z^0)\phi + \varepsilon(\psi^1 + \chi_r^\epsilon z^1)\phi'] \\ &= -\psi^0\phi'' + \chi_r^\epsilon \hat{x}_2\phi'' - \chi_r^\epsilon z^0\phi'' - 2\partial_{\hat{x}_1}\psi^1\phi'' - \varepsilon\psi^1\phi''' - \varepsilon\chi_r^\epsilon z^1\phi''' \quad \text{in } \Omega^\epsilon, \\ &(\psi^0 - \chi_r^\epsilon \hat{x}_2 + \chi_r^\epsilon z^0)\phi + \varepsilon(\psi^1 + \chi_r^\epsilon z^1)\phi' = 0 \quad \text{on } \Gamma_r^\epsilon. \end{aligned}$$

Now, $\Psi^2 = \psi^2\phi''$, and $Z^2 = z^2\phi''$ where

$$-[\partial_{\hat{x}_2\hat{x}_2}\psi^2 + \partial_{\hat{x}_1\hat{x}_1}\psi^2] = \psi^0 - \chi_r\hat{x}_2 + \chi_r z^0 + 2\partial_{\hat{x}_1}\psi^1 \quad \text{in } \Omega_r, \qquad \psi^2 = z^2 \quad \text{on } \Gamma_r^-,$$

where $\chi_r(\hat{x}_2) = 1$ if $\hat{x}_2 \le c_0$, and $\chi_r(\hat{x}_2) = 0$ otherwise. It is easy to see that the right hand sides of the equations become more involved as we proceed. The crucial point is to note that in the above cases, the equations did not involve ϕ or their derivatives.

We finally conclude that

$$\Psi(\varepsilon) \sim \psi^0\phi + \varepsilon\psi^1\phi' + \varepsilon^2\psi^2\phi'' + \cdots, \qquad Z(\varepsilon) \sim z^0\phi + \varepsilon z^1\phi' + \varepsilon^2 z^2\phi'' + \cdots. \tag{4.2.7}$$

4.3. Derivation of wall-laws

Our goal is to approximate u^ϵ using finite elements (or finite differences), without having to discretize the rough boundary. One first step would be to try to approximate u^ϵ in Ω_s only, since this is a smooth domain. We consider the series

$$u^0 + \varepsilon\psi^0\phi^0 + \varepsilon u^1.$$

The functions u^0, u^1 are defined by (4.1.4), (4.1.9), and ψ^0 is defined in the previous section. We would like to approximate u^ϵ by the above sum, but without solving all the problems that define each term. Heuristically, we consider only the functions that actually have influence in the interior of the domain, i.e.,

$$u \approx u^0 + \varepsilon u^1.$$

Hence, over Γ, $u \approx u^0 + \varepsilon u^1 = -\varepsilon(c_0 + z^0)\phi^0$, and $\partial_{x_2} u \approx \phi^0 + \varepsilon\phi^1$, so

$$u + \varepsilon(c_0 + z^0)\partial_{x_2} u \approx \varepsilon^2(c_0 + z^0)\phi^1,$$

and this amount can be small enough for certain applications.

So we define $\bar{u}$ approximating u^ϵ in Ω_s by

$$\begin{aligned} -\Delta\,\bar{u} &= f \quad \text{in } \Omega_s, \\ \bar{u} + \varepsilon(c_0 + z^0)\partial_{x_2}\bar{u} &= 0 \quad \text{on } \Gamma, \qquad \bar{u} = 0 \quad \text{on } \partial\Omega_s\backslash\Gamma. \end{aligned} \tag{4.3.1}$$

The error estimates follow by developing asymptotic expansions for $\bar{u}$. In fact, it is easy to see that

$$\bar{u} \sim \bar{u}^0 + \varepsilon\bar{u}^1 + \varepsilon^2\bar{u}^2 + \cdots,$$

TABLE 1. Relative error convergence rates for order 0 model.

quantity	$L^2(\Omega_s)$ error	$L^2(\mathring{\Omega}_s)$ norm error
u	$O(\varepsilon)$	$O(\varepsilon)$
∇u	$O(\varepsilon^{1/2})$	$O(\varepsilon)$

TABLE 2. Relative error convergence rates for order 1 model.

quantity	$L^2(\Omega_s)$ error	$L^2(\mathring{\Omega}_s)$ norm error
u	$O(\varepsilon^2)$	$O(\varepsilon^2)$
∇u	$O(\varepsilon^{1/2})$	$O(\varepsilon^2)$

where

$$
\begin{gathered}
-\Delta \bar{u}^i = \delta_{i,0} f \quad \text{in } \Omega_s, \\
\bar{u}^i = -(c_0 + z^0)\partial_{x_2}\bar{u}^{i-1} \quad \text{on } \Gamma, \qquad \bar{u}^i = 0 \quad \text{on } \partial\Omega_s \backslash \Gamma.
\end{gathered} \tag{4.3.2}
$$

Also,

$$
\left\| \bar{u} - \sum_{i=0}^{n} \varepsilon^i \bar{u}^i \right\|_{H^k(\Omega_s)} \leq c\varepsilon^{n+1}. \tag{4.3.3}
$$

The modeling error estimates are then as follows.

$$
\begin{aligned}
\|u^\epsilon - \bar{u}\|_{H^1(\Omega_s)} &\leq \|u^\epsilon - u^0 - \varepsilon u^1\|_{H^1(\Omega_s)} + \|\bar{u} - \bar{u}^0 - \varepsilon \bar{u}^1\|_{H^1(\Omega_s)} \\
&\leq \|u^\epsilon - u^0 - \varepsilon u^1 - \varepsilon \psi^0 \phi^0\|_{H^1(\Omega_s)} + \varepsilon \|\psi^0 \phi^0\|_{H^1(\Omega_s)} + c\varepsilon^2 \leq c\varepsilon^{1/2},
\end{aligned}
$$

where in the first inequality we used the triangle inequality, and the identities $u^0 = \bar{u}^0$ and $u^1 = \bar{u}^1$.

In a similar fashion, we can consider other norms, for instance

$$
\|u^\epsilon - \bar{u}\|_{L^2(\Omega_s)} \leq c\varepsilon^2,
$$

Another important measure is how well our model approximates the exact solution in the interior of the domain, i.e., consider $\mathring{\Omega}_s \subset \Omega_s$ such that $\overline{\mathring{\Omega}_s} \cap \overline{\Gamma} = \emptyset$. Hence

$$
\|u^\epsilon - \bar{u}\|_{L^2(\Omega_s)} + \|u^\epsilon - \bar{u}\|_{H^k(\mathring{\Omega}_s)} \leq c\varepsilon^2.
$$

This improved convergence is due to the fact that the boundary layer has no influence far from the wrinkles.

We also compare in tables 1 and 2 the approximability properties of our model and a simple minded model, which approximates u^ϵ by u^0 only. We call such model as “order zero model,” and our model as “order one model.” Note the improved convergence rate for most of the norms, with the exception of the H^1 norm. This is due to the fact that neither approaches captures the boundary layer exactly.

CHAPTER 5

Hierarchical Modeling of the Heat Equation in a Thin Plate

Much investigation has been done in the recent and not so recent past to take advantage of the small thickness to solve or approximate elliptic problems in thin domains. Indeed it is tempting to use *dimension reduction*, i.e., to pose and solve a modified problem in a region with one less dimension and then extend the reduced solution to the more general domain. It is reasonable to expect that the new problem will be simpler than the original one, but it is not easy to predict how far apart are the two solutions. Here, we analyze the approximation properties of some classes of models for elliptic problems in thin domains, not only as the thickness of the domain goes to zero, but also as the "degree" of the models increases, in a sense that we will make clear.

In this chapter we use a slightly different notation for vectors. We use one underbar for 3-dimnesional vectors, and one undertilde for 2-dimnesional vectors. We can then decompose 3-vectors as follows:

$$\underline{u} = \begin{pmatrix} \underset{\sim}{u} \\ u_3 \end{pmatrix}.$$

5.1. Introduction

We assume that the thin domain is a three-dimensional plate of the form $P^\epsilon = \Omega \times (-\varepsilon, \varepsilon)$, where Ω is a two-dimensional smoothly bounded region and $\varepsilon < 1$ is a small positive quantity. For simplicity, we analyze the Poisson problem with vanishing Dirichlet boundary condition on the lateral boundary $\partial\Omega \times (-\varepsilon, \varepsilon)$, despite the fact that other equations and conditions are also of interest. Let $\partial P_L^\epsilon = \partial\Omega \times (-\varepsilon, \varepsilon)$ be the lateral boundary of the plate and $\partial P_\pm^\epsilon = \Omega \times \{-\varepsilon, \varepsilon\}$ its top and bottom. We define then $u^\epsilon \in H^1(P^\epsilon)$ as the weak solution of

$$\begin{aligned} -\Delta u^\epsilon &= f^\epsilon \qquad \text{in } P^\epsilon, \\ \frac{\partial u^\epsilon}{\partial n} &= g^\epsilon \qquad \text{on } \partial P_\pm^\epsilon, \\ u^\epsilon &= 0 \qquad \text{on } \partial P_L^\epsilon, \end{aligned} \tag{5.1.1}$$

where $f^\epsilon : P^\epsilon \to \mathbb{R}$ and $g^\epsilon : P_\pm^\epsilon \to \mathbb{R}$. We denote a typical point in P^ϵ by $\underline{x}^\epsilon = (\underset{\sim}{x}, x_3^\epsilon)$, with $\underset{\sim}{x} = (x_1, x_2) \in \Omega$.

In general, the solution of (5.1.1) will depend on ε in a nontrivial way. In fact the above problem is a singularly perturbed one, and as ε goes to zero it "loses" ellipticity. This results in the onset of boundary layers, as we make clear below.

Projecting the exact solution of (5.1.1) into the space of functions with polynomial dependence in the transverse direction results in a whole hierarchy of models that approximate the original problem with increasing accuracy as the semi-discrete space gets richer, and maintain the lower dimensional character. For symmetric elliptic problems, one possibility is to use a Ritz projection [**38**], deriving the *minimum energy models*. See also [**37**], [**6**], [**7**], [**27**].

Characterizing the solution of (5.1.1) as the minimizer of the associated energy functional, i.e.,

$$u^\epsilon = \operatorname*{arg\,min}_{v\in V(P^\epsilon)} \mathcal{J}(v), \text{ where } \mathcal{J}(v) = \frac{1}{2}\int_{P^\epsilon} |\underline{\nabla} v|^2 \, d\underline{x} - \int_{P^\epsilon} f^\epsilon v \, d\underline{x} - \int_{\partial P^\epsilon_\pm} g^\epsilon v \, d\underset{\sim}{x},$$

and $V(P^\epsilon) = \big\{ v \in H^1(P^\epsilon) \, : \, v = 0 \text{ on } \partial P^\epsilon_L \big\}$, we aim to find a "good" approximation for u^ϵ searching for

$$u^\epsilon(p) = \operatorname*{arg\,min}_{v\in \mathring{H}^1(\Omega;\mathbb{P}_p(-\varepsilon,\varepsilon))} \mathcal{J}(v), \tag{5.1.2}$$

where $\mathring{H}^1(\Omega;\mathbb{P}_p(-\varepsilon,\varepsilon))$ is the space of polynomials of degree p in $(-\varepsilon,\varepsilon)$ with coefficients in $\mathring{H}^1(\Omega)$. It immediately follows from its definition that $u^\epsilon(p)$ is the Ritz projection of u^ϵ into $\mathring{H}^1(\Omega;\mathbb{P}_p(-\varepsilon,\varepsilon))$, and (5.1.2) characterizes a minimum energy model. Observe that using higher polynomial degrees, i.e., higher order models, we obtain a hierarchy of models that furnish increasingly better solutions.

As an example, we write the model explicitly for $p = 1$. Note that (5.1.2) is equivalent to

$$\int_{P^\epsilon} \underline{\nabla} u(1) \cdot \underline{\nabla} v \, d\underline{x} = \int_{P^\epsilon} f^\epsilon v \, d\underline{x} + \int_{\partial P^\epsilon_\pm} g^\epsilon v \, d\underline{x} \qquad \text{for all } v \in \mathring{H}^1(\Omega;\mathbb{P}_1(-\varepsilon,\varepsilon)). \tag{5.1.3}$$

Assuming that $u^\epsilon(1)(\underline{x}^\epsilon) = \omega_0(\underset{\sim}{x}) + \omega_1(\underset{\sim}{x})x_3^\epsilon$, and integrating the in the x_3^ϵ direction, we conclude that

$$\int_\Omega \underset{\sim}{\nabla}\, \omega_0(\underset{\sim}{x}) \cdot \underset{\sim}{\nabla}\, v(\underset{\sim}{x}) \, d\underset{\sim}{x} = \int_\Omega f^0(\underset{\sim}{x}) v(\underset{\sim}{x}) \, d\underset{\sim}{x} + \varepsilon^{-1} \int_\Omega g^0(\underset{\sim}{x}) v(\underset{\sim}{x}) \, d\underset{\sim}{x} \qquad \text{for all } v \in \mathring{H}^1(\Omega),$$

where

$$f^0(\underset{\sim}{x}) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\epsilon} f^\epsilon(\underset{\sim}{x}, x_3^\epsilon) \, dx_3^\epsilon, \qquad g^0(\underset{\sim}{x}) = \frac{1}{2}\big[g^\epsilon(\underset{\sim}{x},\varepsilon) + g^\epsilon(\underset{\sim}{x},-\varepsilon)\big].$$

Hence,

$$\Delta_{2D}\, \omega_0 = -f^0 - \varepsilon^{-1} g^0, \qquad \omega_0 = 0 \qquad \text{on } \partial\Omega, \tag{5.1.4}$$

where $\Delta_{2D} = \partial_{11} + \partial_{22}$.

Similarly, assuming test function of the form $v(\underset{\sim}{x})x_3^\epsilon$ in (5.1.3), we have that

$$\frac{\varepsilon^2}{3}\int_\Omega \underset{\sim}{\nabla}\, \omega_1(\underset{\sim}{x}) \cdot \underset{\sim}{\nabla}\, v(\underset{\sim}{x}) \, d\underset{\sim}{x} = \int_\Omega f^1(\underset{\sim}{x}) v(\underset{\sim}{x}) \, d\underset{\sim}{x} + \varepsilon^{-1} \int_\Omega g^1(\underset{\sim}{x}) v(\underset{\sim}{x}) \, d\underset{\sim}{x} \qquad \text{for all } v \in \mathring{H}^1(\Omega),$$

where

$$f^1(\underset{\sim}{x}) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\epsilon} f^\epsilon(\underset{\sim}{x}, x_3^\epsilon) x_3^\epsilon \, dx_3^\epsilon, \qquad g^1(\underset{\sim}{x}) = \frac{1}{2}\big[g^\epsilon(\underset{\sim}{x},\varepsilon) - g^\epsilon(\underset{\sim}{x},-\varepsilon)\big].$$

Hence,

$$\frac{\varepsilon^2}{3}\,\Delta_{2D}\, \omega_1 - \omega_1 = -f^1 - g^1 \qquad \text{in } \Omega, \qquad \omega_1 = 0 \qquad \text{on } \partial\Omega, \tag{5.1.5}$$

Note that the two differential equations in (5.1.4) and (5.1.5) are independent of each other. We can express in a unique way any function defined on P^ϵ as a sum of its even and odd parts with respect to x_3^ϵ. The even parts of f^ϵ, g^ϵ appear only in the equation for ω_0, and the respective odd parts show up in the equation for ω_1. Also, the equation determining ω_1 is singularly perturbed, but this is not the case for the equation determining ω_0. If higher order methods were used, we would have two independent singularly perturbed systems of equations, corresponding to the even and odd parts of $u^\epsilon(p)$. A similar splitting also occurs for the linearly elastic isotropic plate, where the equations decouple into two independent problems corresponding to bending and stretching of the plate.

The natural question of how close $u^\epsilon(p)$ is to u^ϵ is not easy to answer due to the complex influence of ε in both the original and model solutions. We resolve this not by comparing the exact and model solutions directly, but rather by first looking at the difference between the solutions and their truncated asymptotic expansions, and then comparing the asymptotic expansions. This is possible because the projection used to define each model can be used to find terms of the asymptotic expansion of the model. This allows the comparison between corresponding terms of the expansions. Schematically, this is how it works:

$$
\begin{array}{ccl}
u^\epsilon & \longleftrightarrow & \begin{array}{l}\text{Asymptotic}\\ \text{Expansion of } u^\epsilon\end{array} \\
\updownarrow & & \qquad\updownarrow \\
u^\epsilon(p) & \longleftrightarrow & \begin{array}{l}\text{Asymptotic}\\ \text{Expansion of } u^\epsilon(p)\end{array}
\end{array}
$$

Several authors investigated various aspects of this and other related problems. For a review of the literature, see [**29**]. It is worth mentioning nevertheless the work of of Vogelius and Babuška. In a series of three remarkable papers [**38**], [**39**], [**40**], they investigated various aspects of minimum energy methods for scalar elliptic homogeneous problems in a N-dimensional plate, with Neumann boundary condition on the top and bottom of the domain. They started by showing how to optimally choose the semidiscrete subspace that characterizes each model. This space depends only on the coefficients of the differential equation, and a truncated outer asymptotic expansion (i.e., ignoring boundary layer terms) of the exact solution belongs to it. Then they estimated the rate of convergence of the model solution (with respect to the thickness, in the energy norm). To do this they assumed that the volume loads vanished and that the surface loading was such that boundary layer layer effects were of higher order than the first truncated term of the outer expansion. They then estimated the difference between the exact solution and the truncated expansion. As this quantity is certainly bigger than the error of the minimum energy model in the energy norm, they the obtained an upper bound for the modeling error. This procedure was extended by Miara [**32**] to linearly elastic plates, again with strong restrictions on the volume and surface loads. In this case the optimal subspace might depend on the data, a clear disadvantage. Recent work by Ovaskainen and Pitkäranta [**34**] used similar ideas to obtain more refined estimates for minimum energy methods applied to a thin linearly elastic strip under traction. Some limitations of this approach are that it is not clear how to extend it to models that are not energy minimizers, nor how to obtain sharp estimates in norms other than energy or in the interior of the domain.

Our approach differs significantly from the aforementioned ones. Although we also use asymptotic expansion techniques, we do not rely on the fact that our solution minimizes the potential energy. In fact our arguments work for saddle point models as well [**29**]. In addition to the flexibility to tackle different models, we are also able to obtain sharp estimates in different norms and interior estimates.

We consider the Poisson problem as it contains the same basic characteristics and difficulties of more complex elliptic equations, but is still simple enough so that technicalities do not overshadow the main aspects of our analysis. We avoid nonetheless using specificities of the problem, and the arguments employed here extend in a natural way to the analysis of hierarchical models for linearly elastic plates. Indeed, based on the asymptotic expansions developed by Monique Dauge and her collaborators [**14**], [**15**], a similar kind of study can be performed.

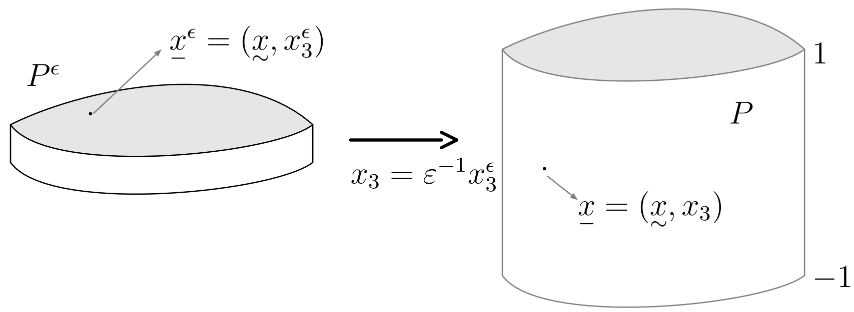
We now briefly introduce and explain some basic notation that we use throughout this chapter. For an integer p and a positive real number a, we define $\mathbb{P}_p(-a,a)$ as the space of polynomials of degree p in $(-a,a)$. If s is a real number and D an open set, then $H^s(D)$ is the Sobolev space of order s, and $\mathring{H}^s(D)$ is the closure in $H^s(D)$ of the set of smooth functions with compact support in D. For $m \in \mathbb{N}$ and a separable Hilbert space E, we denote $H^m(D;E)$ as the space of functions defined on D with values in E such that the E-norm of all partial derivatives of order less or equal to m are in $L^2(D)$. Also, $\hat{L}^2(-a,a)$ is the set of square integrable functions with mean value zero in the domain $(-a,a)$, for a positive number a. Finally $\mathcal{D}(D)$ denotes the space of $\mathcal{C}^\infty$ functions in D with compact support, while $\mathcal{D}'(D)$ denotes the space of distributions.

We denote by lowercase c a generic constant (not necessarily the same in all occurences) which is independent not only of ε and p, but also of f and g, while we use uppercase C when the constant may depend on f and g but not ε and p.

Next, we outline the contents of this chapter. In Section 5.2 we develop an asymptotic expansion for the solution of (5.1.1), presenting upper bounds for the difference between the exact solution and truncated asymptotic expansions. The following section contains the same sort of development, but this time concerning the hierarchical model solution. We present modeling error estimates in Section 5.4. Finally, many issues involving the boundary correctors are discussed in the appendix, including existence, uniqueness and exponential decay of solutions, approximation by polynomials, and corner singularities.

5.2. Asymptotic Expansions for the Exact Solution

We start this section by developing an asymptotic expansion for u^ϵ. As in [**12**], we define an ε-independent domain $P = \Omega \times (-1,1)$. A point $\underline{x} = (\underset{\sim}{x}, x_3)$ in P is related to a point $\underline{x}^\epsilon$ in P^ϵ by $x_3 = \varepsilon^{-1}x_3^\epsilon$. We accordingly define $\partial P_L = \partial\Omega \times (-1,1)$, and $\partial P_\pm = \Omega \times \{-1,1\}$.



In this new domain we define $u(\varepsilon)(\underset{-}{x}) = u^\epsilon(\underset{-}{x}^\epsilon)$, $f(\underset{-}{x}) = f^\epsilon(\underset{-}{x}^\epsilon)$, and $g(\underset{-}{x}) = \varepsilon^{-1}g^\epsilon(\underset{-}{x}^\epsilon)$. We infer from (5.1.1) that

$$\begin{aligned} \Delta_{2D}\, u(\varepsilon) + \varepsilon^{-2}\partial_{33}u(\varepsilon) &= -f \qquad \text{in } P,\\ \frac{\partial u(\varepsilon)}{\partial n} &= \varepsilon^2 g \qquad \text{on } \partial P_\pm,\\ u(\varepsilon) &= 0 \qquad \text{on } \partial P_L. \end{aligned} \tag{5.2.1}$$

We assume that f, g are ε-independent, but this restriction could be relaxed, for instance by assuming that f and g can be represented as a power series in ε, plus a small remainder [**30**]. Also, how exactly f and g scales with respect to ε is immaterial since we are considering a linear problem and the final rates of convergence are in relative norms.

Consider the asymptotic expansion

$$u^0 + \varepsilon^2 u^2 + \varepsilon^4 u^4 + \cdots, \tag{5.2.2}$$

and formally substitute it for $u(\varepsilon)$ in (5.2.1). Grouping together terms with same power in ε we have

$$\varepsilon^{-2}\partial_{33}u^0 + \big(\Delta_{2D}\, u^0 + \partial_{33}u^2\big) + \varepsilon^2\big(\Delta_{2D}\, u^2 + \partial_{33}u^4\big) + \cdots = -f, \tag{5.2.3}$$

$$\frac{\partial u^0}{\partial n} + \varepsilon^2\frac{\partial u^2}{\partial n} + \varepsilon^4\frac{\partial u^4}{\partial n} + \cdots = \varepsilon^2 g \qquad \text{on } \partial P_\pm. \tag{5.2.4}$$

It is then natural to require that

$$\partial_{33}u^0 = 0, \tag{5.2.5}$$

$$\partial_{33}u^2 = -f - \Delta_{2D}\, u^0, \tag{5.2.6}$$

$$\partial_{33}u^{2k} = -\,\Delta_{2D}\, u^{2k-2}, \qquad \text{for all } k > 1, \tag{5.2.7}$$

along with the boundary conditions

$$\frac{\partial u^{2k}}{\partial n} = \delta_{k1} g \text{ on } \partial P_\pm, \qquad \text{for all } k \in \mathbb{N}. \tag{5.2.8}$$

Equations (5.2.5)–(5.2.8) define a sequence of Neumann problems on the interval $x_3 \in (-1, 1)$ parametrized by $\underset{\sim}{x} \in \Omega$. If the data for these problems is compatible then the solution can be written as

$$u^{2k}(\underset{-}{x}) = \hat{u}^{2k}(\underset{-}{x}) + \zeta^{2k}(\underset{\sim}{x}), \qquad \text{for all } k \in \mathbb{N}, \tag{5.2.9}$$

where

$$\int_{-1}^{1} \hat{u}^{2k}(\underset{\sim}{x}, x_3)\, dx_3 = 0, \tag{5.2.10}$$

with $\hat{u}^{2k}$ uniquely determined, but ζ^{2k} an arbitrary function of $\underset{\sim}{x}$ only. As we shall see, the ζ^{2k} will be determined using the condition of compatibility of the data for the Neumann problems. From the Dirichlet boundary condition in (5.2.1), it would be natural to require that $u^{2k} = 0$ on ∂P_L. This is equivalent to imposing

$$\zeta^{2k} = 0 \qquad \text{on } \partial\Omega, \tag{5.2.11}$$

$$\hat{u}^{2k} = 0 \qquad \text{on } \partial P_L. \tag{5.2.12}$$

However, in general, only (5.2.11) can be imposed and (5.2.12) will not hold. We shall correct this discrepancy latter. Now we show that the functions ζ^{2k}, $\hat{u}^{2k}$ (and so u^{2k}) are uniquely determined from (5.2.5)–(5.2.11). In fact, (5.2.5) and (5.2.8) yields $\hat{u}^0 = 0$. From the compatibility of (5.2.6) and (5.2.8) we see that

$$\Delta_{2D}\, \zeta^0(\underset{\sim}{x}) = -\frac{1}{2}\int_{-1}^{1} f(\underset{\sim}{x}, x_3)\, dx_3 - \frac{1}{2}[g(\underset{\sim}{x}, 1) + g(\underset{\sim}{x}, -1)], \tag{5.2.13}$$

which together with (5.2.11), determines ζ^0 and then, from (5.2.9), u^0. In view of the compatibility condition (5.2.13), $\hat{u}^2$ is fully determined by (5.2.6), (5.2.8), and (5.2.10). Next, the Neumann problem (5.2.7), (5.2.8) admits a solution for $k > 1$ if and only if $\Delta_{2D}\, \zeta^{2k-2} = 0$. But in view of (5.2.11), this means $\zeta^{2k-2} = 0$, for $k > 1$, and then $\hat{u}^{2k}$ is uniquely determined from (5.2.7), (5.2.8). Note that $u^0 = \zeta^0$ and $u^{2k} = \hat{u}^{2k}$ for $k \geq 1$.

Observe that $u^0 = 0$ on the lateral boundary of P, since $\hat{u}^0 = 0$ and so (5.2.12) holds for $k = 0$. However, u^2, u^4, etc, will not in general vanish on ∂P_L (although their vertical integrals do). Thus (5.2.2) does not give a complete asymptotic expansion of $u(\varepsilon)$ and we seek a boundary corrector U, which should satisfy

$$\begin{gathered} \Delta_{2D}\, U + \varepsilon^{-2}\partial_{33} U = 0 \qquad \text{in } P, \\ \frac{\partial U}{\partial n} = 0 \qquad \text{on } \partial P_{\pm}, \quad U \sim \varepsilon^2 u^2 + \varepsilon^4 u^4 + \cdots \qquad \text{on } \partial P_L. \end{gathered} \tag{5.2.14}$$

To study this singular perturbation problem, we again use a system of boundary-fitted horizontal coordinates. See Chapter 2 for details.

Defining the new variable $\tilde{\rho} = \varepsilon^{-1}\rho$ and using the same name for functions different only up to this change of coordinates, we have from (2.1.6) that

$$\Delta_{2D}\, U = \varepsilon^{-2}\partial_{\tilde{\rho}\tilde{\rho}} U + \sum_{j=0}^{\infty} (\varepsilon\tilde{\rho})^j \left(a_1^j \varepsilon^{-1}\partial_{\tilde{\rho}} U + a_2^j \partial_{\theta\theta} U + a_3^j \partial_\theta U\right), \tag{5.2.15}$$

Aiming to solve (5.2.14), we insert the asymptotic expansion

$$U \sim \varepsilon^2 U^2 + \varepsilon^3 U^3 + \varepsilon^4 U^4 + \cdots, \tag{5.2.16}$$

in (5.2.15), and collect together terms with same order of ε. This leads us to pose a sequence of problems in the semi-infinite strip $\Sigma = \mathbb{R}^+ \times (-1,1)$, for $k \geq 2$:

$$
\begin{aligned}
(\partial_{\tilde\rho\tilde\rho} + \partial_{33})U^k &= F_k \qquad \text{in } \Sigma,\\
\frac{\partial U^k}{\partial n} &= 0 \qquad \text{on } \mathbb{R}^+ \times \{-1,1\},\\
U^k(0,\theta,x_3) &= u^k(0,\theta,x_3) \qquad \text{for } x_3 \in (-1,1),
\end{aligned}
\tag{5.2.17}
$$

where

$$F_k = \sum_{j=0}^{k-2} \tilde\rho^j \left(a_1^j \partial_{\tilde\rho} U^{k-j-1} + a_2^j \partial_{\theta\theta} U^{k-j-2} + a_3^j \partial_\theta U^{k-j-2} \right),$$

with the convention that $u^k = 0$ for k odd and $U^0 = U^1 = 0$. Note that the problem described by (5.2.17)—we show that it is well–defined further ahead—is parametrized by θ, and that the geometry of Ω plays an important role through the coefficients a_1^j, a_2^j, a_3^j.

Combining (5.2.2) and (5.2.16) we obtain the formal asymptotic expansion

$$u^\epsilon(\underline{x}^\epsilon) \sim \sum_{k=0}^{\infty} \varepsilon^{2k} u^{2k}(\underset{\sim}{x}, \varepsilon^{-1} x_3^\epsilon) - \chi(\rho) \sum_{k=2}^{\infty} \varepsilon^k U^k(\varepsilon^{-1}\rho, \theta, \varepsilon^{-1} x_3^\epsilon). \tag{5.2.18}$$

Here $\chi(\rho)$, is a smooth cutoff function which is identically one if $0 \leq \rho \leq \rho_0/3$ and identically zero if $\rho \geq 2\rho_0/3$. (This does not turn out to be a significant source of error since U^k decays exponentially to zero in the normal direction.)

Although our reasoning has been formal so far, we shall rigorously justify this asymptotic expansion in Theorem 5.2.3. Before doing that, we first study the terms entering into the expansion.

We use the following notation:

$$\|v\|_{(m,n,P)} = \|v\|_{H^m(\Omega; H^n(-1,1))}, \quad |||(f,g)|||_{m,P} = \|f\|_{(m,0,P)} + \|g\|_{H^m(\partial P_\pm)}.$$

In the lemma below, the bounds follow from standard regularity estimates for equations (5.2.13), (5.2.6)–(5.2.8).

LEMMA 5.2.1. Suppose that f and g are smooth functions on P and $\partial P_\pm$, respectively. Then the functions u^0, $u^2, \cdots$ on P are uniquely determined by (5.2.5)–(5.2.11), and $u^0(\underline{x}) = \zeta^0(\underset{\sim}{x})$ is independent of x_3. Moreover, for m a nonnegative integer and s a real number such that $s \geq 2$, there exists a constant c independent of f and g such that

$$
\begin{gathered}
\|\zeta^0\|_{H^{m+1}(\Omega)} \leq c|||(f,g)|||_{m-1,P},\\
\|u^2(\underset{\sim}{x},\cdot)\|_{H^s(-1,1)} \leq c\big(\|f(\underset{\sim}{x},\cdot)\|_{H^{s-2}(-1,1)} + |g(\underset{\sim}{x},-1)| + |g(\underset{\sim}{x},1)|\big),\\
\|u^2\|_{(m,s,P)} \leq c\big(\|f\|_{(m,s-2,P)} + \|g\|_{H^m(\partial P_\pm)}\big).
\end{gathered}
$$

The next lemma, whose proof we postpone to the appendix, guarantees the existence, uniqueness, and exponential decay of solutions for (5.2.17).

LEMMA 5.2.2. Assume, for a fixed positive integer k, that u^k is defined as above. Then, for each θ, there exists a unique weak solution $U^k(\cdot,\theta,\cdot) \in H^1(\Sigma)$ to (5.2.17). Also, there exist positive constants C and α such that

$$\int_t^\infty \int_{-1}^1 (U^k)^2 + (\partial_{\tilde\rho} U^k)^2 + (\partial_3 U^k)^2 \, dx_3 \, d\tilde\rho \le C e^{-\alpha t}, \tag{5.2.19}$$

for every nonnegative real number t. The constant α may depend on Ω and k, but is independent of f and g, while the constant C may depend on Ω, k, f, and g.

Remark. The dependence of the constant C in the lemma on f and g may be specified explicitly in terms of finitely many Sobolev norm of these functions. However the dependence is quite complicated and so we shall not do so. See [**29**] for such expressions in a 2D version of this problem.

Although (5.2.18) is a formal expansion, a rigorous error estimate shows that the difference between the exact solution and a truncated asymptotic expansion is of the same order of the first term omitted in the expansion. In fact, define

$$e^\epsilon_{2N}(\underline{x}^\epsilon) = u^\epsilon(\underline{x}^\epsilon) - \sum_{k=0}^{N} \varepsilon^{2k} u^{2k}(\underset{\sim}{x}, \varepsilon^{-1} x_3^\epsilon) + \chi(\rho) \sum_{k=2}^{2N} \varepsilon^k U^k(\varepsilon^{-1}\rho, \theta, \varepsilon^{-1} x_3^\epsilon). \tag{5.2.20}$$

In the theorem below we bound the $H^1(P^\epsilon)$ norm of e^ϵ_{2N}.

Theorem 5.2.3. *For any positive integer N, there exists a constant C such that the difference between the truncated asymptotic expansion and the original solution measured in the original domain is bounded as follows:*

$$\|e^\epsilon_0\|_{H^1(P^\epsilon)} \le C\varepsilon^{3/2}, \quad \|e^\epsilon_{2N}\|_{H^1(P^\epsilon)} \le C\varepsilon^{2N+1}. \tag{5.2.21}$$

Since the domain P^ϵ depends on ε, the interpretation of the convergence estimates given in Theorem 5.2.3 is not straightforward. The relative error may be more informative in this case. For this we may use (5.2.21) and the triangle inequality to obtain a lower bound on the $H^1(P^\epsilon)$ norm of the solution. The leading term of the asymptotic expansion for u^ϵ is ζ^0, unless ζ^0 is identically zero, that is, unless the quantity

$$-\frac{1}{2}\int_{-1}^1 f(\underset{\sim}{x}, x_3)\, dx_3 - \frac{1}{2}[g(\underset{\sim}{x}, 1) + g(\underset{\sim}{x}, -1)] \tag{5.2.22}$$

appearing on the right-hand side of (5.2.13), vanishes. Assuming momentarily that ζ^0 does not vanish, then we easily conclude from (5.2.21) and the triangle inequality that there exists a strictly positive constant C depending on f and g, such that $\|u^\epsilon\|_{H^1(P^\epsilon)} \ge C\varepsilon^{1/2}$ for all ε sufficiently small. If, on the other hand ζ^0 vanishes, but f and g do not both vanish identically, then it can be seen from (5.2.6) and (5.2.8) that u^2 does not vanish. Applying the second estimate of (5.2.21) with $N = 1$ and using the triangle inequality, we conclude $\|u^\epsilon\|_{H^1(P^\epsilon)} \ge C\varepsilon^{3/2}$ in this case. Thus in any case (as long as f and g do not both vanish identically) we have

$$\|u^\epsilon\|_{H^1(P^\epsilon)} \ge C\nu\varepsilon^{1/2}, \quad \text{where } \nu = \begin{cases} 1, & \zeta^0 \not\equiv 0, \\ \varepsilon, & \zeta^0 \equiv 0. \end{cases}$$

Thus, for any positive integer N,

$$\frac{\|e_0^\epsilon\|_{H^1(P^\epsilon)}}{\|u^\epsilon\|_{H^1(P^\epsilon)}} = O(\nu^{-1}\varepsilon), \qquad \frac{\|e_{2N}^\epsilon\|_{H^1(P^\epsilon)}}{\|u^\epsilon\|_{H^1(P^\epsilon)}} = O(\nu^{-1}\varepsilon^{2N+1/2}).$$

It is easy to estimate the convergence in some other norms as well. For instance, in the $L^2(P^\epsilon)$ norm, we have from the triangle inequality that

$$\|e_{2N}^\epsilon\|_{L^2(P^\epsilon)} \le \|e_{2N+2}^\epsilon\|_{H^1(P^\epsilon)} + \|e_{2N+2}^\epsilon - e_{2N}^\epsilon\|_{L^2(P^\epsilon)}.$$

Since

$$(e_{2N+2}^\epsilon - e_{2N}^\epsilon)(\underline{x}^\epsilon) = -\varepsilon^{2N+1}U^{2N+1}(\varepsilon^{-1}\rho,\theta,\varepsilon^{-1}x_3^\epsilon) + \varepsilon^{2N+2}\big[u^{2N+2}(\underset{\sim}{x},\varepsilon^{-1}x_3^\epsilon) - U^{2N+2}(\varepsilon^{-1}\rho,\theta,\varepsilon^{-1}x_3^\epsilon)\big], \tag{5.2.23}$$

we easily conclude from a scaling argument that $\|e_{2N}^\epsilon\|_{L^2(P^\epsilon)} = O(\varepsilon^{2N+2})$, for N positive.

Using similar arguments, it is possible to compute *interior estimates*, which achieve better convergence in regions "far away" from the lateral boundary of the plate. The reason for the improvement in such subdomains is that the influence of the boundary layer is negligible. The table below presents these interior and various other error estimates. We assume that f and g are sufficiently smooth functions and we show only the order of the norms with respect to ε. "BL" stands for "Boundary Layer" and the "Relative Error" column presents the norm of e_{2N}^ϵ divided by the norm of u^ϵ. In parentheses are the interior estimates, when these are better than the global estimates.

TABLE 1. Order with respect to the thickness of the exact solution, the first term of the boundary layer expansion, and the difference between the solution and a truncated asymptotic expansion in various norms.

norm	u^ϵ	BL	e_{2N}^ϵ, $N \ge 1$	Relative Error
$\|\cdot\|_{L^2(P^\epsilon)}$	$\nu^2\varepsilon^{1/2}$	ε^3	$\varepsilon^{2N+2}(\varepsilon^{2N+5/2})$	$\nu^{-2}\varepsilon^{2N+3/2}(\nu^{-2}\varepsilon^{2N+2})$
$\|\partial_\rho\cdot\|_{L^2(P^\epsilon)}$	$\nu^{3/2}\varepsilon^{1/2}(\nu^2\varepsilon^{1/2})$	ε^2	$\varepsilon^{2N+1}(\varepsilon^{2N+5/2})$	$\nu^{-3/2}\varepsilon^{2N+1/2}(\nu^{-2}\varepsilon^{2N+2})$
$\|\partial_\theta\cdot\|_{L^2(P^\epsilon)}$	$\nu^2\varepsilon^{1/2}$	ε^3	$\varepsilon^{2N+2}(\varepsilon^{2N+5/2})$	$\nu^{-2}\varepsilon^{2N+3/2}(\nu^{-2}\varepsilon^{2N+2})$
$\|\partial_{x_3^\epsilon}\cdot\|_{L^2(P^\epsilon)}$	$\varepsilon^{3/2}$	ε^2	$\varepsilon^{2N+1}(\varepsilon^{2N+3/2})$	$\varepsilon^{2N-1/2}(\varepsilon^{2N})$
$\|\cdot\|_{H^1(P^\epsilon)}$	$\nu\varepsilon^{1/2}$	ε^2	$\varepsilon^{2N+1}(\varepsilon^{2N+3/2})$	$\nu^{-1}\varepsilon^{2N+1/2}(\nu^{-1}\varepsilon^{2N+1})$

The remainder of this section contains the proof of Theorem 5.2.3. In our demonstration, we follow the basic steps of a similar proof for an elasticity problem [**14**].

DEFINITION 5.2.4. *Set*

$$u_{2N}(\underline{x}) = \sum_{k=0}^{N} \varepsilon^{2k}u^{2k}(\underline{x}), \qquad U_{2N}(\underline{x}) = \sum_{k=2}^{2N} \varepsilon^k U^k(\varepsilon^{-1}\rho,\theta,x_3).$$

Some results regarding the boundary layer terms are collected below.

LEMMA 5.2.5. For any positive integer N, there exists positive constants C and α such that

$$\|\chi'(\rho)U_{2N}\|_{L^2(P)} + \varepsilon\|\chi'\partial_\rho U_{2N}(\rho)\|_{L^2(P)} \le C\varepsilon^{5/2}\exp(-\alpha\varepsilon^{-1}). \tag{5.2.24}$$

Also, for all $v \in H^1(P)$ that vanishes on ∂P_L,

$$\left|\int_P \underset{\sim}{\nabla} U_{2N}\, \underset{\sim}{\nabla}(\chi v) + \varepsilon^{-2}\partial_3 U_{2N}\partial_3(v\chi)\, d\underline{x}\right| \le C\varepsilon^{2N}\|v\|_{H^1(P)}. \tag{5.2.25}$$

PROOF. The inequalities (5.2.24) follow from a change of coordinates, Lemma 5.2.2, and the definition of χ. To see that (5.2.25) holds, first rewrite (2.1.6) as a finite series, using Taylor expansion with remainders. Then the result follows from the definition of U_{2N}, (5.2.17), and Lemma 5.2.2 $\square$

We obtain now a rough estimate for the asymptotic expansion error.

LEMMA 5.2.6. For any positive integer N, let $e_{2N}(\underline{x}) = e^\epsilon_{2N}(\underline{x}^\epsilon)$. Then there exists a constant C such that

$$\|e_{2N}\|_{H^1(P)} \le C\varepsilon^{2N}.$$

PROOF. We use in this proof that u^ϵ_{2N} solves the Poisson problem up to arbitrary powers of ε. First note that e_{2N} vanishes on ∂P_L. Hence, in view of the Poincaré's inequality,

$$\|e_{2N}\|^2_{H^1(P)} \le c\int_P |\underset{\sim}{\nabla} e_{2N}|^2 + \varepsilon^{-2}(\partial_3 e_{2N})^2\, d\underline{x}, \tag{5.2.26}$$

and we estimate next the right hand side of (5.2.26). Let $v \in H^1(P)$ such that $v = 0$ on ∂P_L. If we define

$$E(2N,v) = \int_P \underset{\sim}{\nabla}(u(\varepsilon) - u_{2N})\, \underset{\sim}{\nabla} v + \varepsilon^{-2}\partial_3(u(\varepsilon) - u_{2N})\partial_3 v\, d\underset{\sim}{x},$$

then, by construction of the asymptotic expansion, we have

$$\begin{aligned} E(2N,v) = \int_P fv\, d\underset{\sim}{x} + \int_{\partial P_\pm} gv\, d\underset{\sim}{x} - \sum_{k=0}^N \varepsilon^{2k}\int_P \big(\underset{\sim}{\nabla} u^{2k}\, \underset{\sim}{\nabla} v + \varepsilon^{-2}\partial_3 u^{2k}\partial_3 v\big)\, d\underset{\sim}{x} \\ = -\varepsilon^{2N}\int_P \underset{\sim}{\nabla} u^{2N}\, \underset{\sim}{\nabla} v\, d\underset{\sim}{x}, \end{aligned}$$

and we conclude that

$$|E(2N,v)| \le C\varepsilon^{2N}\|v\|_{H^1(P)}. \tag{5.2.27}$$

We also have

$$\begin{aligned} \left|\int_P \underset{\sim}{\nabla}(\chi U_{2N})\, \underset{\sim}{\nabla} v - \underset{\sim}{\nabla} U_{2N}\, \underset{\sim}{\nabla}(\chi v) + \varepsilon^{-2}\big[\partial_3(\chi U_{2N})\partial_3 v - \partial_3 U_{2N}\partial_3(\chi v)\big]\, d\underset{\sim}{x}\right| \\ \le \big(\|\chi' U_{2N}\|_{L^2(P)} + \|\chi'\partial_\rho U_{2N}\|_{L^2(P)}\big)\|v\|_{H^1(P)}. \end{aligned}$$

Hence, by Lemma 5.2.5

$$\left|\int_P \big[\underset{\sim}{\nabla}(\chi U_{2N})\, \underset{\sim}{\nabla} v + \varepsilon^{-2}\partial_3(\chi U_{2N})\partial_3 v\big]\, d\underset{\sim}{x}\right| \le C\varepsilon^{2N}\|v\|_{H^1(P)}. \tag{5.2.28}$$

Making $v = e_{2N}$ we have

$$\int_P |\underset{\sim}{\nabla} e_{2N}|^2 + \varepsilon^{-2}(\partial_3 e_{2N})^2 \, d\underset{\sim}{x} = E(2N, e_{2N}) + \int_P \big[\underset{\sim}{\nabla}(\chi U_{2N}) \underset{\sim}{\nabla} e_{2N} + \varepsilon^{-2}\partial_3(\chi U_{2N})\partial_3 e_{2N}\big] \, d\underset{\sim}{x}$$
$$\leq C\varepsilon^{2N} \|e_{2N}\|_{H^1(P)},$$

from (5.2.27) and (5.2.28), and the result follows from (5.2.26). □

The estimate in Lemma 5.2.6 is not sharp. The powers of ε can be shown to be $2N+1/2$. We make this improvement when we consider the error on the unscaled plate P^ϵ.

PROOF (OF THEOREM 5.2.3). Assume first that N is positive. From Lemma 5.2.6, we immediately obtain $\|e^\epsilon_{2N}\|_{H^1(P^\epsilon)} = O(\varepsilon^{2N-1/2})$. This result too is not sharp. To obtain a sharp result, we use the triangle inequality:

$$\|e^\epsilon_{2N}\|_{H^1(P^\epsilon)} \leq \|e^\epsilon_{2N+2} - e^\epsilon_{2N}\|_{H^1(P^\epsilon)} + O(\varepsilon^{2N+3/2}),$$

and then the result follows from (5.2.23) and a scaling argument. A similar argument holds for $N = 0$. □

5.3. Asymptotic Expansions for the Model Solution

To develop an asymptotic expansion for the solution of the hierarchical models, we reason as before, but use weak equations instead of their strong form. We start by posing a problem for the solution of the minimum energy model in the scaled domain P. If we define $u(p)(\underline{x}) = u^\epsilon(p)(\underline{x}^\epsilon)$, then

(5.3.1)
$$\int_P \underset{\sim}{\nabla} u(p) \underset{\sim}{\nabla} v + \varepsilon^{-2}\partial_3 u(p)\partial_3 v \, d\underset{\sim}{x} = \int_P fv \, d\underset{\sim}{x} + \int_{\partial P_\pm} gv \, d\underset{\sim}{x} \qquad \text{for all } v \in \mathring{H}^1(\Omega; \mathbb{P}_p(-1,1)).$$

Considering the asymptotic expansion

$$u^0(p) + \varepsilon^2 u^2(p) + \varepsilon^4 u^4(p) + \cdots, \tag{5.3.2}$$

and formally substituting it for $u(p)$ in (5.3.1), we conclude that for all $v \in \mathring{H}^1(\Omega; \mathbb{P}_p(-1,1))$,

$$\int_P \partial_3 u^0(p)\partial_3 v \, d\underset{\sim}{x} = 0, \tag{5.3.3}$$

$$\int_P \partial_3 u^2(p)\partial_3 v \, d\underset{\sim}{x} = \int_P \big(f + \Delta_{2D}\, u^0(p)\big) v \, d\underset{\sim}{x} + \int_{\partial P_\pm} gv, \tag{5.3.4}$$

$$\int_P \partial_3 u^{2k}(p)\partial_3 v \, d\underset{\sim}{x} = \int_P \Delta_{2D}\, u^{2k-2}(p) v \, d\underset{\sim}{x}, \qquad \text{for } k > 1. \tag{5.3.5}$$

Let $\hat{\mathbb{P}}_p(-1,1)$ be the space of polynomials of degree p in $(-1,1)$ with zero average. Repeating the arguments of the expansion for the exact solution, we set $u^0(p)(\underline{x}) = \zeta^0(\underset{\sim}{x})$ and $u^2(p)(\underset{\sim}{x}, \cdot)$ as the Galerkin projection of $u^2(\underset{\sim}{x}, \cdot)$ into $\hat{\mathbb{P}}_p(-1,1)$ for almost every $\underset{\sim}{x} \in \Omega$, i.e.,

$$\begin{aligned}\int_{-1}^1 \partial_3 u^2(p)(\underset{\sim}{x}, x_3)\partial_3 v(x_3) \, dx_3 = \int_{-1}^1 [f(\underset{\sim}{x}, x_3) + \Delta_{2D}\, \zeta^0(\underset{\sim}{x})] v(x_3) \, dx_3 \\ + g(\underset{\sim}{x}, -1)v(-1) + g(\underset{\sim}{x}, 1)v(1), \qquad \text{for all } v \in \hat{\mathbb{P}}_p(-1,1).\end{aligned} \tag{5.3.6}$$

For any integer $k \geq 2$, we define $u^{2k}(p)(\underset{\sim}{x}, \cdot) \in \hat{\mathbb{P}}_p(-1,1)$ by

$$\int_{-1}^{1} \partial_3 u^{2k}(p)(\underset{\sim}{x}, x_3)\partial_3 v(x_3)\, dx_3 = \int_{-1}^{1} \Delta_{2D}\, u^{2k-2}(p)(\underset{\sim}{x}, x_3) v(x_3)\, dx_3, \tag{5.3.7}$$

for all $v \in \hat{\mathbb{P}}_p(-1,1)$, and for almost every $\underset{\sim}{x} \in \Omega$.

The ansatz (5.3.2) does not satisfy the Dirichlet boundary conditions at ∂P_L and we use then boundary correctors $U^k(p)$. These functions are polynomial in the transverse direction, and are defined in the semi-infinite strip Σ. We need to define the spaces

$$V(\Sigma, p) = \left\{ v \in \mathcal{D}'(\mathbb{R}^+; \mathbb{P}_p(-1,1)) \,:\, \|\underset{\sim}{\nabla} v\|_{L^2(\Sigma)} + \|v(0,\cdot)\|_{L^2(-1,1)} < \infty \right\},$$
$$V_0(\Sigma, p) = \left\{ v \in V(\Sigma, p) \,:\, v(0,\cdot) = 0 \right\}.$$

For any positive integer k, let $U^k(p) \in V(\Sigma, p)$ be the solutions of

$$\begin{gathered} \int_{\Sigma} \underset{\sim}{\nabla} U^k(p) \cdot \underset{\sim}{\nabla} v \, d\tilde{\rho}\, dx_3 = \int_{\Sigma} F_k(p) v \, d\tilde{\rho}\, dx_3 \qquad \text{for all } v \in V_0(\Sigma, p), \\ U^k(p)(0, \theta, x_3) = u^k(p)(0, \theta, x_3) \qquad \text{for all } x_3 \in (-1,1), \\ F_k(p) = \sum_{j=0}^{k-1} \tilde{\rho}^j \left(a_1^j \partial_{\tilde{\rho}} U^{k-j-1}(p) + a_2^j \partial_{\theta\theta} U^{k-j-2}(p) + a_3^j \partial_{\theta} U^{k-j-2}(p) \right), \end{gathered} \tag{5.3.8}$$

where $u^k = 0$ for k odd and $U^0(p) = U^1(p) = 0$.

A result analogous to Lemma 5.2.2 holds for $U^k(p)$, guaranteeing existence, uniqueness and exponential decay, with the same decaying rate [**29**]. This implies in particular that there exist constants C and α such that

$$\int_{\Sigma} [\chi'(\varepsilon\tilde{\rho}) U^2(p)]^2 \, d\tilde{\rho}\, dx_3 \leq C \exp(-\alpha\varepsilon^{-1}). \tag{5.3.9}$$

The above inequality will be of use further on. Similarly to (5.2.18), we have that

$$u^\epsilon(p)(\underline{x}^\epsilon) \sim \zeta^0(\underset{\sim}{x}) + \sum_{k=1}^{\infty} \varepsilon^{2k} u^{2k}(p)(\underset{\sim}{x}, \varepsilon^{-1} x_3^\epsilon) - \chi(\rho) \sum_{k=2}^{\infty} \varepsilon^k U^k(p)(\varepsilon^{-1}\rho, \theta, \varepsilon^{-1} x_3^\epsilon),$$

where ζ^0 solves (5.2.13).

We present next an estimate, in the $H^1(P^\epsilon)$ norm, of $u^\epsilon(p)$ minus its truncated asymptotic expansion. Since the proofs of the previous section work here with minor modifications, we refrain from repeating them. We would like to remark that this result gives a bound that is uniform in p, and that the bound is the same (up to a constant) as in Theorem 5.2.3.

THEOREM 5.3.1. *For any positive integer N, let*

$$e_{2N}^\epsilon(p)(\underline{x}^\epsilon) = u^\epsilon(p)(\underline{x}^\epsilon) - \sum_{k=0}^{N} \varepsilon^{2k} u^{2k}(p)(\underset{\sim}{x}, \varepsilon^{-1} x_3^\epsilon) + \chi(\rho) \sum_{k=2}^{2N} \varepsilon^k U^k(p)(\varepsilon^{-1}\rho, \theta, \varepsilon^{-1} x_3^\epsilon).$$

Then there exists a constant C such that $\|e_{2N}^\epsilon(p)\|_{H^1(P^\epsilon)} \leq C\varepsilon^{2N+1}$, for all $p \in \mathbb{N}$.

5.4. Estimates for the modeling error

In this section, we estimate the modeling error. As we mentioned before, this is done by comparing the asymptotic expansions of the exact and model solution. A key point is to estimate the difference between terms of the respective expansions. We need the following definitions.

DEFINITION 5.4.1. *For a nonnegative real number s, let*

$$a_s = \|f\|_{L^2(\Omega;H^s(-1,1))} + \|g\|_{L^2(\partial P_\pm)}, \quad a_s^1 = \|f\|_{H^1(\Omega;H^s(-1,1))} + \|g\|_{H^1(\partial P_\pm)},$$
$$a_s^b = \left(\int_{\partial\Omega} \|f(\underset{\sim}{x},\cdot)\|^2_{H^s(-1,1)} + |g(\underset{\sim}{x},-1)|^2 + |g(\underset{\sim}{x},1)|^2 \, d\underset{\sim}{x}\right)^{1/2}.$$

The comparison between $u^2(p)$ and u^2 is straightforward since the former is a Galerkin projection of the latter. Indeed, let $\hat{\pi}_p^1$ be the orthogonal projection operator from $H^1(-1,1)\cap \hat{L}^2(-1,1)$ to $\hat{\mathbb{P}}_p(-1,1)$, with respect to the inner product that induces the norm $|\cdot|_{H^1(-1,1)}$. The next classical result [**8**], estimates the projection error.

LEMMA 5.4.2. For any nonnegative real number s, there exists a constant C such that if $0 \le r \le 1 \le s$, then

$$\|\phi - \hat{\pi}_p^1\phi\|_{H^r(-1,1)} \le Cp^{r-s}\|\phi\|_{H^s(-1,1)} \qquad \text{for } \phi \in H^s(-1,1)\cap \hat{L}^2(-1,1). \tag{5.4.1}$$

From (5.2.6), (5.2.8)–(5.2.11), and (5.3.6), we gather that $u^2(p) = \hat{\pi}_p^1 u^2$, for all $\underset{\sim}{x} \in \Omega$. From Lemmas 5.4.2, and 5.2.1, we conclude the following result.

LEMMA 5.4.3. For any nonnegative real number s, there exists a constant c independent of ε, p, f, and g, such that

$$\|u^2 - u^2(p)\|_{L^2(P)} \le cp^{-2-s}a_s,$$
$$\|\underset{\sim}{\nabla}\, u^2 - \underset{\sim}{\nabla}\, u^2(p)\|_{L^2(P)} \le cp^{-2-s}a_s^1,$$
$$\|\partial_{x_3}u^2 - \partial_{x_3}u^2(p)\|_{L^2(P)} \le cp^{-1-s}a_s.$$

Bounding the difference $U^2 - U^2(p)$ is harder due to the presence of corner singularities. Since both U^2 and $U^2(p)$ are originally defined in the semi-infinite strip Σ, it is natural to investigate the approximation properties in this domain, and such is done in the appendix. We apply these approximation results to estimate the difference between boundary correctors in P^ϵ.

DEFINITION 5.4.4. *Let $\underset{\sim}{x} \in \partial\Omega$, and let s be a nonnegative real number. Let*

$$N(s) = \max\{n \in \mathbb{Z} \,:\, 2n < s\}. \tag{5.4.2}$$

If $\sup_{x_3\in\{-1,1\}} |g(\underset{\sim}{x},x_3)| \neq 0$, *set* $m = 1$. *If* $|g(\underset{\sim}{x},-1)| = |g(\underset{\sim}{x},1)| = 0$ *and*

$$\sup_{x_3\in\{-1,1\}} \sum_{j=2}^{N(s+5/2)} |\partial_3^{2j-3} f(\underset{\sim}{x},x_3)| \neq 0, \tag{5.4.3}$$

let m be the minimum integer in $\{2,\dots,N(s+5/2)\}$ such that $\sup_{x_3\in\{-1,1\}}|\partial_3^{2m-3}f(\underset{\sim}{x},x_3)|\neq 0$. *We define in both cases* $\mu(\underset{\sim}{x},s,\delta)=\min\{4m-2-\delta,s+3/2\}$. *If* $|g(\underset{\sim}{x},-1)|=|g(\underset{\sim}{x},1)|=0$ *and* (5.4.3) *does not hold, then define* $\mu(\underset{\sim}{x},s,\delta)=s+3/2$. *Finally, set*

$$\bar{\mu}(s,\delta)=\inf_{\underset{\sim}{x}\in\partial\Omega}\mu(\underset{\sim}{x},s,\delta).$$

REMARK. Four our purposes, the minimum value that $\bar{\mu}(s,\delta)$ can assume is $2-\delta$, since we will always impose $s>3/2$.

We pospone the proof of the next Lemma to the appendix II.

LEMMA 5.4.5. Let $Z(\underset{\sim}{x}^\epsilon)=\chi(\rho)[U^2-U^2(p)](\varepsilon^{-1}\rho,\theta,\varepsilon^{-1}x_3^\epsilon)$. For any nonnegative real number s such that $s+1/2$ is not an even integer, and for any arbitrarily small $\delta>0$, there exists a constant c independent of ε, p, f, and g, such that

$$\|\partial_\rho Z\|_{L^2(P^\epsilon)}+\|\partial_{x_3}Z\|_{L^2(P^\epsilon)}\le cp^{-\bar{\mu}(s,\delta)}a_s^b.$$

Finally, we present the convergence results for the hierarchical models. Let $P_0^\epsilon=\Omega_0\times(-\varepsilon,\varepsilon)$, here Ω_0 is an open domain such that $\bar{\Omega}_0\subset\Omega$. This is usefull to obtain interior estimates.

THEOREM 5.4.6. *For any nonnegative real numbers s and s^* such that $s^*+1/2$ is not an even integer, and for any arbitrarily small $\delta>0$, there exist constants c and C independent of ε and p, with c also independent of f and g, such that the error between u^ϵ and its approximation $u^\epsilon(p)$ is bounded as*

$$\begin{gathered}
\|u^\epsilon-u^\epsilon(p)\|_{L^2(P^\epsilon)}\le c\varepsilon^{5/2}p^{-2-s}a_s+C\varepsilon^3,\\
\|\partial_\rho[u^\epsilon-u^\epsilon(p)]\|_{L^2(P^\epsilon)}\le c\varepsilon^2p^{-\bar{\mu}(s^*,\delta)}a_{s^*}^b+C\varepsilon^{5/2},\\
\|\partial_\theta[u^\epsilon-u^\epsilon(p)]\|_{L^2(P^\epsilon)}\le c\varepsilon^{5/2}p^{-2-s}a_s^1+C\varepsilon^3,\\
\|\underset{\sim}{\nabla}u^\epsilon-\underset{\sim}{\nabla}u^\epsilon(p)\|_{L^2(P_0^\epsilon)}\le c\varepsilon^{5/2}p^{-2-s}a_s^1+C\varepsilon^{9/2},\\
\|\partial_{x_3^\epsilon}u^\epsilon-\partial_{x_3^\epsilon}u^\epsilon(p)\|_{L^2(P^\epsilon)}\le c\varepsilon^{3/2}p^{-1-s}a_s+C\varepsilon^2,\\
\|u^\epsilon-u^\epsilon(p)\|_{H^1(P^\epsilon)}\le c\varepsilon^{3/2}p^{-1-s}a_s+C\varepsilon^2.
\end{gathered}$$

Moreover, if $f\equiv 0$, then $\|u^\epsilon-u^\epsilon(p)\|_{H^1(P^\epsilon)}\le c\varepsilon^2p^{-\bar{\mu}(s^*,\delta)}a_{s^*}^b+C\varepsilon^{5/2}$.

PROOF. We prove the second estimate. Using the triangle inequality, the following holds:

$$\begin{aligned}
\|\partial_\rho[u^\epsilon-u^\epsilon(p)]\|_{L^2(P^\epsilon)}\le{}&\|e_2^\epsilon\|_{H^1(P^\epsilon)}+\|e_2^\epsilon(p)\|_{H^1(P^\epsilon)}\\
&+\varepsilon^{5/2}\|\underset{\sim}{\nabla}u^2-\underset{\sim}{\nabla}u^2(p)\|_{L^2(P)}+\varepsilon^2\|\partial_\rho Z\|_{L^2(P^\epsilon)}.
\end{aligned}\tag{5.4.4}$$

From Theorems 5.2.3 and 5.3.1, we have that $\|e_2^\epsilon\|_{H^1(P^\epsilon)}+\|e_2^\epsilon(p)\|_{H^1(P)}\le C\varepsilon^3$. The estimate for $\|\partial_\rho Z\|_{L^2(P^\epsilon)}$ comes from Lemma 5.4.5. Finally we apply Lemma 5.4.3 to bound $\|\underset{\sim}{\nabla}u^2-\underset{\sim}{\nabla}u^2(p)\|_{L^2(P)}$, and substituting in (5.4.4) we have the result. The other estimates follow from similar arguments. ☐

REMARK. In the worst case scenario, when g does not vanish identically along the boundary of $\partial P_\pm$, i.e., $\sup_{\underset{\sim}{x}\in\partial\Omega}\max_{x_3\in\{-1,1\}}|g(\underset{\sim}{x},x_3)|\neq 0$, then $\bar{\mu}(s^*,\delta)=2-\delta$.

We summarize the convergence results in the table below. We present only the leading terms of the errors and in parenthesis we show interior estimates if those are better than the global ones.

TABLE 2. Rates of convergence of the model error.

norm	$u^\epsilon - u^\epsilon(p)$	Relative Error
$\\|\cdot\\|_{L^2(P^\epsilon)}$	$\varepsilon^{5/2}p^{-2-s}a_s$	$\nu^{-2}\varepsilon^2 p^{-2-s}a_s$
$\\|\partial_\rho \cdot\\|_{L^2(P^\epsilon)}$	$\varepsilon^2 p^{-\bar{\mu}}a_s^b\ (\varepsilon^{5/2}p^{-2-s}a_s^1)$	$\nu^{-3/2}\varepsilon^{3/2}p^{-\bar{\mu}}a_s^b\ (\nu^{-2}\varepsilon^2 p^{-2-s}a_s^1)$
$\\|\partial_\theta \cdot\\|_{L^2(P^\epsilon)}$	$\varepsilon^{5/2}p^{-2-s}a_s^1$	$\nu^{-2}\varepsilon^2 p^{-2-s}a_s^1$
$\\|\partial_{x_3^\epsilon} \cdot\\|_{L^2(P^\epsilon)}$	$\varepsilon^{3/2}p^{-1-s}a_s$	$p^{-1-s}a_s$
$\\|\cdot\\|_{H^1(P^\epsilon)}$	$\varepsilon^{3/2}p^{-1-s}a_s$	$\nu^{-1}\varepsilon p^{-1-s}a_s$

The estimates of the table above indicate that the rate of convergence in ε is the same regardless of the value of p. Nonetheless, increasing p does diminish the modeling error, as expected. It is interesting to see that for the relative error norm, when $\zeta^0 \equiv 0$ there is no convergence in ε, only in p. Finally, if f is polynomial in the transverse direction, then $u^2 = u^2(p)$ in this case, for p high enough, and it is possible to obtain better convergence rates with respect to ε in all norms of Table 2, with the exception of the $L^2(P^\epsilon)$ norm of the normal derivative of the error [**29**].

5.5. Appendices

5.5.1. Appendix I. In this appendix, we discuss several issues related to the boundary correctors. Our first goal is to prove existence, uniqueness and regularity of solutions for Poisson problems in the semi-infinite strip Σ. We also prove that under certain conditions, such solutions and their approximations decay exponentially. Next, we will study the properties of a standard Galerkin approximations for the boundary corrector U^2 in spaces with polynomial dependence in the vertical direction. We show estability and convergence results. We do not use the technique of separation of variables, although it would simplify some of the proofs, because it does not generalize to the case of linear elasticity.

In this appendix, we denote a typical point in Σ by $\underset{\sim}{x} = (x_1, x_2)$. It is useful to consider the sets

$$\Sigma(t,s) = \{\underset{\sim}{x} \in \Sigma : t < x_1 < s\}, \quad \text{and} \quad \gamma_t = \{\underset{\sim}{x} \in \Sigma : x_1 = t\},$$

for $0 \le t \le s < \infty$. Let $V(\Sigma) = \{v \in \mathcal{D}'(\Sigma) : wv \in L^2(\Sigma), \underset{\sim}{\nabla} v \in \underset{\sim}{L}^2(\Sigma)\}$, where $w(\underset{\sim}{x}) = (1+x_1)^{-1}$. By means of Hardy's inequality, it is possible to show that $V(\Sigma)$ endowed with the inner product $\int_\Sigma \underset{\sim}{\nabla} u \cdot \underset{\sim}{\nabla} v\, d\underset{\sim}{x} + \int_{\gamma_0} uv\, dx_2$ is a Hilbert space [**29**]. We denote $V_0(\Sigma)$ as the set of functions in $V(\Sigma)$ that vanish on γ_0. The following well-posedness result holds.

THEOREM 5.5.1. *Assume that $w^{-\alpha} f \in L^2(\Sigma)$, where $\alpha \geq 1$, and let $U_0 \in H^{1/2}(\gamma_0)$. Then there exists unique $U \in V(\Sigma)$ such that*

$$\int_\Sigma \underset{\sim}{\nabla} U \cdot \underset{\sim}{\nabla} v \, d\underset{\sim}{x} = \int_\Sigma f \, v \, d\underset{\sim}{x} \qquad \text{for all } v \in V_0(\Sigma), \tag{5.5.1}$$

$$U = U_0 \qquad \text{on } \gamma_0. \tag{5.5.2}$$

Moreover, there exists a constant c independent of f such that

$$|U|_{H^1(\Sigma)} \leq \frac{2}{2\alpha - 1} \|w^{-\alpha} f\|_{L^2(\Sigma)} + c\|U_0\|_{H^{1/2}(\gamma_0)}.$$

With the questions of existence and uniqueness answered, we proceed to further characterize the boundary correctors. We show that they decay exponentially fast to a constant, in a sense that we will make clear. Our proof generalizes previous approaches [**22**]. It allows a nontrivial right hand side, and, more importantly, it works not only for the exact solution of (5.5.1), but also for some of its approximations. So, $\bar{U}$ does not necessarily solves (5.5.1), but it might be the projection of the solution into some particular space. Similarly, $\underset{\sim}{\bar{\sigma}}$ might be either the gradient of the solution or its approximation. In our applications, $\bar{U}$ and $\underset{\sim}{\bar{\sigma}}$ are given by Galerkin or mixed approximations. As we see below, sufficient conditions for such exponential decay are that $\bar{U} \in L^2_w(\Sigma)$, $\underset{\sim}{\bar{\sigma}} \in \underset{\sim}{L}^2(\Sigma)$, and that $\bar{U}$, $\underset{\sim}{\bar{\sigma}}$ satisfy for $0 \leq t \leq s < \infty$:

$$\int_{\Sigma(t,s)} |\underset{\sim}{\bar{\sigma}}|^2 \, d\underset{\sim}{x} = \int_{\Sigma(t,s)} f\bar{U} \, d\underset{\sim}{x} - \int_{\gamma_t} \bar{\sigma}_1 \bar{U} \, dx_2 + \int_{\gamma_s} \bar{\sigma}_1 \bar{U} \, dx_2, \tag{C1}$$

$$\int_{\Sigma(t,s)} f \, d\underset{\sim}{x} = \int_{\gamma_t} \bar{\sigma}_1 \, dx_2 - \int_{\gamma_s} \bar{\sigma}_1 \, dx_2, \tag{C2}$$

$$-\int_{\Sigma(0,t)} x_1 f \, d\underset{\sim}{x} = \int_{\gamma_0} \bar{U} \, dx_2 + \int_{\gamma_t} (t\bar{\sigma}_1 - \bar{U}) \, dx_2, \tag{C3}$$

$$\int_{\gamma_t} \bar{U}^2 \, dx_2 \leq c_W \int_{\gamma_t} \bar{\sigma}_2^2 \, dx_2 + \frac{1}{2}\left(\int_{\gamma_t} \bar{U} \, dx_2\right)^2 \qquad \text{for some } c_W \geq 0. \tag{C4}$$

The constant c_W in the condition (C4) mimics the Wirtinger inequality (the one-dimensional version of the Poincaré's inequality [**31**]).

Assume that there exist positive constants c_0 and M such that

$$\left(\int_{\Sigma(t,\infty)} f(\underset{\sim}{x})^2 \, d\underset{\sim}{x}\right)^{1/2} + \left|\int_{\Sigma(t,\infty)} (t - x_1) f(\underset{\sim}{x}) \, d\underset{\sim}{x}\right| \leq M \exp(-c_0 t) \tag{5.5.3}$$

and define

$$c_\infty(\bar{U}) = \int_\Sigma x_1 f(\underset{\sim}{x}) \, d\underset{\sim}{x} + \int_{\gamma_0} \bar{U} \, dx_2. \tag{5.5.4}$$

In the following two lemmas we show that results similar to (C1)–(C3) are valid in unbounded sections of Σ as well.

LEMMA 5.5.2. Assume that (5.5.3) holds, $wU \in L^2(\Sigma)$, $\underset{\sim}{\sigma} \in \underset{\sim}{L}^2(\Sigma)$ and that conditions (C2), (C3) are satisfied. Then for $t \geq 0$

$$\int_{\Sigma(t,\infty)} f \, d\underset{\sim}{x} = \int_{\gamma_t} \sigma_1 \, dx_2, \tag{5.5.5}$$

$$\int_{\gamma_t} U \, dx_2 = c_\infty(U) + \int_{\Sigma(t,\infty)} (t - x_1) f(\underset{\sim}{x}) \, d\underset{\sim}{x}. \tag{5.5.6}$$

PROOF. If we define $P(s) = \int_{\gamma_s} \sigma_1 \, dx_2$, then in view of (C2) we have that

$$P(s) = \int_{\gamma_t} \sigma_1 \, dx_2 - \int_{\Sigma(t,s)} f \, d\underset{\sim}{x}.$$

Thus P is a continuous function and $\lim_{s\to\infty} P(s) = d$, where d is the constant

$$d = \int_{\gamma_t} \sigma_1 \, dx_2 - \int_{\Sigma(t,\infty)} f \, d\underset{\sim}{x}.$$

Since $|\underset{\sim}{\sigma}| \in L^2(\Sigma)$, then $P(s) \in L^2(\mathbb{R}^+)$. Hence $d = 0$ and identity (5.5.5) follows. Now, to conclude (5.5.6), we use (C3) and then equations (5.5.4), (5.5.5). $\square$

The proof of the lemma below follows from similar arguments

LEMMA 5.5.3. Assume that U, $|\underset{\sim}{\sigma}| \in L^2(\Sigma)$ and that condition (C1) is satisfied. Then, for $t \geq 0$

$$\int_{\Sigma(t,\infty)} |\underset{\sim}{\sigma}|^2 \, d\underset{\sim}{x} = -\int_{\gamma_t} \sigma_1 U \, dx_2 + \int_{\Sigma(t,\infty)} f \, U \, d\underset{\sim}{x}. \tag{5.5.7}$$

We have the following results.

THEOREM 5.5.4. *Assume that* (5.5.3) *holds, that* $w\bar{U} \in L^2(\Sigma)$, $\bar{\underset{\sim}{\sigma}} \in \underset{\sim}{L}^2(\Sigma)$ *satisfy (C1)–(C4), and also that* $c_\infty(\bar{U}) = 0$. *Then there exists a constant* c *depending only on* c_0 *and* c_W *such that*

$$\int_{\Sigma(t,\infty)} \bar{U}^2 + |\bar{\underset{\sim}{\sigma}}|^2 \, d\underset{\sim}{x} \leq c \left(1 + \int_{\Sigma} |\bar{\underset{\sim}{\sigma}}|^2 \, d\underset{\sim}{x}\right) \exp(-t/c_1), \tag{5.5.8}$$

where $c_1 = \max\{1 + c_W, 1/c_0\}$.

PROOF. Let $I(t) = \int_{\gamma_t} \bar{U} \, dx_2$. Then, from Lemma 5.5.2 and equation (5.5.3),

$$|I(t)| \leq M \exp(-c_0 t). \tag{5.5.9}$$

If we define the function $E(t) = \int_{\Sigma(t,\infty)} |\bar{\underset{\sim}{\sigma}}|^2 \, d\underset{\sim}{x}$, then $E'(t) = -\int_{\gamma_t} |\bar{\underset{\sim}{\sigma}}|^2 \, dx_2$ and (C4) yields

$$\int_{\gamma_t} \bar{U}^2 \, dx_2 \leq -c_W E'(t) + \frac{I(t)^2}{2}, \tag{5.5.10}$$

(5.5.11)

$$\int_{\Sigma(t,\infty)} \bar{U}^2 \, d\underset{\sim}{x} \leq \int_t^\infty \left(c_W \int_{\gamma_{x_1}} |\bar{\underset{\sim}{\sigma}}|^2 \, dx_2 + \frac{1}{2} I(x_1)^2\right) dx_1 = c_W E(t) + \frac{1}{2} \int_t^\infty I(x_1)^2 \, dx_1.$$

We can now bound the growth of the energy. From (5.5.9) and (5.5.11), we conclude that $\bar{U} \in L^2(\Sigma)$, and using Lemma 5.5.3 we gather that:

(5.5.12)
$$\begin{aligned} E(t) &= -\int_{\gamma_t} \bar{\sigma}_1 \bar{U}\, dx_2 + \int_{\Sigma(t,\infty)} f\, \bar{U}\, d\underset{\sim}{x} \\ &\le \frac{1}{2}\int_{\gamma_t} \bar{\sigma}_1^2\, dx_2 + \frac{1}{2}\int_{\gamma_t} \bar{U}^2\, dx_2 + \frac{\alpha}{2}\int_{\Sigma(t,\infty)} \bar{U}^2\, d\underset{\sim}{x} + \frac{1}{2\alpha}\int_{\Sigma(t,\infty)} f^2\, d\underset{\sim}{x} \\ &\le -\frac{(1+c_W)}{2}E'(t) + \frac{I(t)^2}{4} + \frac{\alpha c_W}{2}E(t) + \frac{\alpha}{4}\int_t^\infty I(x_1)^2\, dx_1 + \frac{1}{2\alpha}\int_{\Sigma(t,\infty)} f^2\, d\underset{\sim}{x}, \end{aligned}$$

where (5.5.10) and (5.5.11) were used in the last inequality. Choose $\alpha = (c_W)^{-1}$ in (5.5.12) to conclude that (recall that $E'(t)$ is nonpositive):

$$c_1 E'(t) \le (1+c_W)E'(t) \le -E(t) + G(t), \tag{5.5.13}$$

where

$$c_1 = \max\{1+c_W, \frac{1}{c_0}\}, \quad G(t) = \frac{I(t)^2}{2} + \frac{1}{2c_W}\int_t^\infty I(x_1)^2\, dx_1 + c_W\int_{\Sigma(t,\infty)} f^2\, d\underset{\sim}{x}. \tag{5.5.14}$$

We estimate now the energy norm. Define $W(t)$ such that

$$W'(t) = -\frac{W(t)}{c_1} + \frac{G(t)}{c_1}, \quad W(0) = E(0).$$

Then

$$E(t) \le W(t) = \frac{1}{c_1}\exp(-t/c_1)\int_0^t \exp(x_1/c_1) G(x_1)\, dx_1 + E(0)\exp(-t/c_1). \tag{5.5.15}$$

Using (5.5.14), (5.5.3), and (5.5.9) we have that the integral in (5.5.15) is uniformly bounded and then $E(t)$ decays exponentially. Combining (5.5.9) and (5.5.11), we have the corresponding decay of $\|\bar{U}\|_{L^2(\Sigma(t,\infty))}$. □

Using the previous theorem, we can decompose a general solution as a constant term plus a exponentially decaying function, as the result below shows.

COROLLARY 5.5.5. Assume that (5.5.3) holds and that $\bar{U} \in V(\Sigma)$, $\underset{\sim}{\bar{\sigma}} \in L^2(\Sigma)$ satisfy (C1)–(C4). Defining $c_\infty(\bar{U})$ as in (5.5.4), we have the decomposition

$$\bar{U} = \frac{1}{2}c_\infty(\bar{U}) + \bar{U}^*, \tag{5.5.16}$$

where $\bar{U}^*$, $\underset{\sim}{\bar{\sigma}}$ decay to zero exponentially as in Theorem 5.5.4, i.e., (5.5.8) is satisfied with $\bar{U}$ replaced by $\bar{U}^*$.

In the rest of this appendix, we investigate how well elements of $V(\Sigma, p)$ can approximate the solution of

$$\begin{aligned} \Delta U &= 0 \qquad \text{in } \Sigma, \\ \frac{\partial U}{\partial n} &= 0 \qquad \text{on } \mathbb{R}^+ \times \{-1, 1\}, \\ U &= U_0 \qquad \text{on } \gamma_0, \end{aligned} \tag{5.5.17}$$

where $U_0 \in H^{r_0}(\gamma_0) \cap \hat{L}^2(\gamma_0)$ for some $r_0 > 3/2$. The approximation rates are severely limited by the presence of corner singularities in U. We describe these singularities explicitly and expose their influence in the convergence rates.

To describe the singular behavior of the solution of (5.5.17), we introduce in Σ two polar coordinate systems, (r_l, θ_l), $l = 1, 2$ relative to the vertices $P_1 = (0, 1)$ and $P_2 = (0, -1)$. The convention is that r_l gives the distance to P_l and the angle $\theta_l \in [0, \pi/2]$ increases counterclockwise, so points lying on γ_0 have $\theta_1 = 0$ and $\theta_2 = \pi/2$.

The next theorem [**28**], shows a decomposition of the solution U in singular and smooth parts and it is of paramount importance in future estimates.

THEOREM 5.5.6. *Let $U \in V(\Sigma)$ be the solution of* (5.5.17) *with $r_0 > 3/2$ such that $r_0 + 1/2$ is not an even integer. Then there exist constants c_j such that*

$$U = U_S + W, \quad U_S = \check{\chi} \sum_{l=1}^{2} \sum_{j=1}^{N(r_0+1/2)} c_j \partial_2^{(2j-1)} U_0\big((-1)^{l+1}\big) v_l^j, \tag{5.5.18}$$

where $\check{\chi}$ is a smooth cutoff function that equals the identity for $x_1 < 1$ and vanishes for $x_1 > 2$, N is as in (5.4.2)*, and*

$$v_1^j = \big[\theta_1 \cos\big((2j-1)\theta_1\big) + \log r_1 \sin\big((2j-1)\theta_1\big)\big] r_1^{(2j-1)},$$
$$v_2^j = \big[\big(\frac{\pi}{2} - \theta_2\big) \sin\big((2j-1)\theta_2\big) + \log r_2 \cos\big((2j-1)\theta_2\big)\big] r_2^{(2j-1)}.$$

Furthermore, $\|W\|_{H^{r_0+1/2}(\Sigma)} \le c\|U_0\|_{H^{r_0}(\gamma_0)}$ for some constant c.

REMARK. Note that $v_1^j = v_2^j = 0$ when $x_1 = 0$, and therefore U_S is identically zero at $x_1 = 0$.

REMARK. Since $U_0 \in H^{r_0}(-1, 1)$ and $2N(r_0 + 1/2) - 1 < r_0 - 1/2$, then U_S is well defined. Also, note that the singular behavior of U depends not only on the regularity of the Dirichlet data U_0 but also on how many derivatives of U_0 vanish at the endpoints -1, 1. For instance, although $U_0(y) = y$ is smooth, it gives rise to a singular solution.

Let $\hat{\pi}_p^{1(x_2)}$ be the operator that acts like $\hat{\pi}_p^1$ in each fiber, i.e., if $\phi \in L^2(\mathbb{R}^+; H^1(-1, 1) \cap \hat{L}^2(-1, 1))$, then $\hat{\pi}_p^{1(x_2)} \phi \in L^2(\mathbb{R}^+; \hat{\mathbb{P}}_p(-1, 1))$, and

$$\int_\Sigma \partial_2(\phi - \hat{\pi}_p^{1(x_2)} \phi) \partial_2 \psi \, d\underset{\sim}{x} = 0 \qquad \text{for all } \psi \in L^2((-1, 1); \hat{\mathbb{P}}_p(-1, 1)).$$

Also, let π_p be the orthogonal L^2 projection operator from $L^2(\Sigma)$ into $L^2(\mathbb{R}^+; \mathbb{P}_p(-1, 1)$, and let Π_p^1 be the orthogonal H^1 projection operator from $H^1(\Sigma)$ into $H^1(\mathbb{R}^+; \mathbb{P}_p(-1, 1)$.

To estimate the projection error of the singular function U_S, we apply the ideas of Dorr [**16**], [**17**], and the Remark 6.3 of Bernardi and Maday [**8**]. See [**29**] for a description of how such convergence rates can be obtained.

LEMMA 5.5.7. Let $v(r, \theta) = \check{\chi} r^\alpha [\xi_1(\theta) + \xi_2(\theta) \log r]$, where ξ_1, $\xi_2 \in C^\infty([0, \pi/2])$, and α is a nonnegative real number. Then, for every δ, there exists a constant c such that

$$\|v - \pi_p^{(\tilde{\rho}_2)} v\|_{H^1(\Sigma)} \le c p^{-2\alpha + \delta}.$$

The result below estimates approximations given by projection operators, based on the decomposition (5.5.18).

LEMMA 5.5.8. Assume that $U \in V(\Sigma)$ solves (5.5.17) with $r_0 > 3/2$ such that $r_0 + 1/2$ is not an even integer, and that W and U_S are as in (5.5.18). Then, there exists a constant c such that

$$\|W - \pi_p^{1(x_2)} W\|_{H^1(\Sigma)} \leq c p^{1/2 - r_0} \|U_0\|_{H^{r_0}(\gamma_0)}.$$

Also, if U_S is not the zero function then for any arbitrarily small $\delta > 0$, there exists a constant c such that

$$\|U_S - \pi_p^{(x_2)} U_S\|_{H^1(\Sigma)} \leq c p^{-4m+2+\delta} \|U_0\|_{H^{r_0}(\gamma_0)},$$

where $m \in \{1, \ldots, N(r_0 + \frac{1}{2})\}$ is the minimum integer such that

$$|\partial_2^{(2m-1)} U_0(-1)| + |\partial_2^{(2m-1)} U_0(1)| \neq 0.$$

REMARK. Using the work of Babuška and Suri [**5**], it is possible to improve the estimate of Lemma 5.5.8 slightly, replacing $p^{-4m+2+\delta}$ by $p^{-4m+2}(\log p)$, at the expense of many technicalities.

We define the rate of convergence of our approximation result below.

DEFINITION 5.5.9. *For $U_0 \in H^{r_0}(-1, 1)$, and N as in (5.4.2), if there exists an minimum integer $m \in \{1, \ldots, N(r_0 + \frac{1}{2})\}$ such that $|\partial_2^{2m-1} U_0(-1)| + |\partial_2^{2m-1} U_0(1)| \neq 0$, let $\gamma(r_0, \delta) = \min\{4m - 2 - \delta, r_0 - 1/2\}$, otherwise let $\gamma(r_0, \delta) = r_0 - 1/2$.*

We conclude now the following approximation result for U.

THEOREM 5.5.10. *Assume that U solves (5.5.17) with $r_0 > 3/2$ such that $r_0 + 1/2$ is not an even integer. Then*

$$\|U - \Pi_p^1 U\|_{H^1(\Sigma)} \leq c p^{-\gamma(r_0, \delta)} \|U_0\|_{H^{r_0}(\gamma_0)},$$

where γ is as in Definition 5.5.9. The constant c depends on r_0 and $\delta > 0$ only.

PROOF. Using the best approximation property of Π_p^1, Theorem 5.5.6, and Lemma 5.5.8, we have that

$$\|U - \Pi_p^1 U\|_{H^1(\Sigma)} \leq \|U_S - \pi_p^{(x_2)} U_S\|_{H^1(\Sigma)} + \|W - \pi_p^{1(x_2)} W\|_{H^1(\Sigma)} \leq c p^{-\gamma(r_0, \delta)} \|U_0\|_{H^{r_0}(\gamma_0)}.$$

□

Now we use the above result to estimate the errors due to the Galerkin projections.

THEOREM 5.5.11. *For any real number $r_0 > 3/2$ such that $r_0 + 1/2$ is not an even integer, and any arbitrarily small $\delta > 0$, there exists a constant c such that if $U \in V(\Sigma)$ solves (5.5.17) with $U_0 \in H^{r_0}(\gamma_0) \cap \hat{L}^2(\gamma_0)$, and if $U(p) \in V(\Sigma, p)$ solves*

$$\int_\Sigma \underset{\sim}{\nabla} U(p) \cdot \underset{\sim}{\nabla} v \, d\underset{\sim}{x} = 0 \qquad \text{for all } v \in V_0(\Sigma, p),$$

$$U(p) = \hat{\pi}_p^1 U_0 \qquad \text{on } \gamma_0,$$

then

$$|U - U(p)|_{H^1(\Sigma)} \leq c p^{-\gamma(r_0, \delta)} \|U_0\|_{H^{r_0}(\gamma_0)},$$

where γ is as in Definition 5.5.9.

PROOF. Let $\tilde{U}_0$ be the trace of $\Pi_p^1 U$ on γ_0. Then, from the trace Theorem and Theorem 5.5.10,

$$\|U_0 - \tilde{U}_0\|_{H^{1/2}(\gamma_0)} \le c\|U - \Pi_p^1 U\|_{H^1(\Sigma)} \le cp^{-\gamma(r_0,\delta)}\|U_0\|_{H^{r_0}(\gamma_0)}.$$

Also,

$$\|\hat{U}_0 - \hat{\pi}_p^1 U_0\|_{H^{1/2}(\gamma_0)} \le \|\tilde{U}_0 - U_0\|_{H^{1/2}(\gamma_0)} + \|U_0 - \hat{\pi}_p^1 U_0\|_{H^{1/2}(\gamma_0)} \le cp^{-\gamma(r_0,\delta)}\|U_0\|_{H^{r_0}(\gamma_0)}.$$

Introduce now $\tilde{U} \in V(\Sigma)$ such that

$$\int_\Sigma \underset{\sim}{\nabla}\, \tilde{U} \cdot \underset{\sim}{\nabla}\, v \, d\underset{\sim}{x} = 0 \qquad \text{for all } v \in V_0(\Sigma),$$
$$\tilde{U} = \tilde{U}_0 \qquad \text{on } \gamma_0,$$

and also $\tilde{U}(p) \in V(\Sigma, p)$ such that

$$\int_\Sigma \underset{\sim}{\nabla}\, \tilde{U}(p) \cdot \underset{\sim}{\nabla}\, v \, d\underset{\sim}{x} = 0 \qquad \text{for all } v \in V_0(\Sigma, p),$$
$$\tilde{U}(p) = \tilde{U}_0 \qquad \text{on } \gamma_0,$$

Then,

$$\begin{aligned} (5.5.19) \quad & |U - \tilde{U}|_{H^1(\Sigma)} + |\tilde{U}(p) - U(p)|_{H^1(\Sigma)} \\ & \le c\|U_0 - \tilde{U}_0\|_{H^{1/2}(\gamma_0)} + c\|\tilde{U}_0 - \hat{\pi}_p^1 U_0\|_{H^{1/2}(\gamma_0)} \le cp^{-\gamma(r_0,\delta)}\|U_0\|_{H^{r_0}(\gamma_0)}. \end{aligned}$$

Now we advance to estimate $|\tilde{U} - \tilde{U}(p)|_{H^1(\Sigma)}$. Since $\tilde{U}(p) - \Pi_p^1 U \in V_0(\Sigma, p)$, then

$$|\tilde{U}(p) - \Pi_p^1 U|^2_{H^1(\Sigma)} = \int_\Sigma \underset{\sim}{\nabla}(\tilde{U}(p) - \Pi_p^1 U) \cdot \underset{\sim}{\nabla}(\tilde{U} - \Pi_p^1 U)\, d\underset{\sim}{x} \le |\tilde{U}(p) - \Pi_p^1 U|_{H^1(\Sigma)} |\tilde{U} - \Pi_p^1 U|_{H^1(\Sigma)},$$

and therefore, $|\tilde{U}(p) - \Pi_p^1 U|_{H^1(\Sigma)} \le |\tilde{U} - \Pi_p^1 U|_{H^1(\Sigma)}$. So, using the triangle inequality

$$\begin{aligned} |\tilde{U} - \tilde{U}(p)|_{H^1(\Sigma)} & \le |\tilde{U} - \Pi_p^1 U|_{H^1(\Sigma)} + |\Pi_p^1 U - \tilde{U}(p)|_{H^1(\Sigma)} \le 2|\tilde{U} - \Pi_p^1 U|_{H^1(\Sigma)} \\ & \le 2|\tilde{U} - U|_{H^1(\Sigma)} + 2|U - \Pi_p^1 U|_{H^1(\Sigma)} \le cp^{-\gamma(r_0,\delta)}\|U_0\|_{H^{r_0}(\gamma_0)}. \end{aligned}$$

from (5.5.19) and from Theorem 5.5.10. Finally,

$$|U - U(p)|_{H^1(\Sigma)} \le |U - \tilde{U}|_{H^1(\Sigma)} + |\tilde{U} - \tilde{U}(p)|_{H^1(\Sigma)} + |\tilde{U}(p) - U(p)|_{H^1(\Sigma)} \le cp^{-\gamma(r_0,\delta)}\|U_0\|_{H^{r_0}(\gamma_0)},$$

and the result follows. □

REMARK. It is interesting to see how the corner singularities spoil an otherwise good convergence rate. For example, if $U_0(y) = y$, the Galerkin projection converges as $p^{-2+\delta}$ in $H^1(\Sigma)$, while if U_0 is still smooth but has all derivatives vanishing at the endpoints, then the convergence is faster than polynomial [**29**].

5.5.2. Appendix II. In this second part of the appendix, we include proofs of results stated throughout the chapter and which proofs use results developed in the first part of this appendix.

PROOF. (of Lemma 5.2.2) We use induction on k to prove the result and the relation

$$\int_\Sigma \tilde\rho^l \partial_{\tilde\rho} U^k \, d\tilde\rho \, dx_3 = \int_\Sigma \tilde\rho^l U^k \, d\tilde\rho \, dx_3 = 0, \qquad l = 0, 1, \cdots . \tag{5.5.20}$$

Recall that, by convention, $U^1 = 0$, and assume that the lemma and (5.5.20) hold for $k = 1, \cdots, K-1$. We show now that the same holds for $k = K$. From the definition of F_K, the hypotheses of Theorem 5.5.1 are fulfilled, and there exists a unique function U^K solving (5.2.17) such that $|\partial_{\tilde\rho} U^k| + |\partial_3 U^k| \in L^2(\Sigma)$. To conclude (5.2.19), we first note from Corollary 5.5.5 that U^K decays towards the constant $c_K(\theta) = \int_\Sigma \tilde\rho F_K(\tilde\rho, \theta, x_3) \, d\tilde\rho \, dx_3 + \int_{-1}^1 \bar u^K(0, \theta, x_3) \, dx_3$. Our goal now is to show that this constant is actually zero. Since u^K has zero average in each fiber, using the definition of F_k, it is enough to prove (5.5.20) for any positive integer l. Using the formula

$$\int_\Sigma u \, \Delta \, v \, d\tilde\rho \, dx_3 = \int_\Sigma v \, \Delta \, u \, d\tilde\rho \, dx_3 + \int_{\partial\Sigma} u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \, d\tilde\rho \, dx_3,$$

with $u = U^K$ and $v = \tilde\rho^{l+2}/[(l+2)(l+1)]$, we get

$$\int_\Sigma \tilde\rho^l U^K \, d\tilde\rho \, dx_3 = \int_\Sigma \frac{\tilde\rho^{l+2}}{(l+2)(l+1)} F_K \, d\tilde\rho \, dx_3 = 0$$

from the definition of F_K and the inductive hypothesis. Similarly, using integration by parts, we also have that $\int_\Sigma \tilde\rho^l \partial_{\tilde\rho} U^K \, d\tilde\rho \, dx_3 = 0$. Hence (5.5.20) holds and the lemma follows. ☐

PROOF. (of Lemma 5.4.5) From Theorem 5.5.11, with $u^2(0, \theta, \cdot)$ replacing U_0, we have that for each θ,

$$|U^2 - U^2(p)|_{H^1(\Sigma)} \le c p^{-\gamma(s+2,\delta)} \|u^2(0, \theta, \cdot)\|_{H^{s+2}(-1,1)}. \tag{5.5.21}$$

Changing coordinates, we have that

$$\begin{aligned} \|\partial_\rho Z\|^2_{L^2(P^\epsilon)} \le c \int_0^L \int_\Sigma |\partial_\rho [U^2 - U^2(p)]|^2 \, d\tilde\rho \, dx_3 \, d\theta + c \int_0^L \int_\Sigma |\chi'|^2 [U^2 - U^2(p)]^2 \, d\tilde\rho \, dx_3 \, d\theta \\ \le c p^{-\gamma(s+2,\delta)} \|u^2(0, \theta, \cdot)\|_{H^{s+2}(-1,1)}, \end{aligned}$$

where we used Lemma 5.2.5 and equations (5.3.9) and (5.5.21) in the last inequality. Now, from the definitions 5.4.4 and 5.5.9, $\gamma(r+2, \delta) = \bar\mu(s, \delta)$, and from Lemma 5.2.1, we have that

$$\|u^2(0, \theta, \cdot)\|_{H^{r+2}(-1,1)} \le a_s^b,$$

and the result follows. ☐

Proof of Lemma 5.4.5. For $\underset{\sim}{x} \in \partial\Omega$ fixed, let θ be such that $\underset{\sim}{z}(\theta) = \underset{\sim}{x}$. From equations (5.2.17), (5.3.8), and Theorem 5.5.11, there exists a constant c such that

$$\begin{aligned} (5.5.22) \quad \int_\Sigma |\partial_{\tilde\rho}[U^2 - U^2(p)]|^2 + |\partial_{x_3}[U^2 - U^2(p)]|^2 \, d\tilde\rho \, dx_3 &\le cp^{-\gamma(s+2,p)} \|u^2(\underset{\sim}{x}, \cdot)\|_{H^{s+2}(-1,1)} \\ &\le cp^{-\gamma(s+2,p)} (\|f(\underset{\sim}{x}, \cdot)\|_{H^s(-1,1)} + |g(\underset{\sim}{x}, -1)| + |g(\underset{\sim}{x}, 1)|). \end{aligned}$$

Replacing $U_0(\cdot)$ by $u^2(\underset{\sim}{x}, \cdot)$ in Definition 5.5.9, and using (5.2.6), (5.2.8), and (5.2.13), it is not hard to show that $\gamma(s+2, p) \le \bar\mu(s, \delta)$. Integrating (5.5.22) in θ and using Lemma 5.2.1, we conclude the proof. $\square$

CHAPTER 6

Hierarchical Modeling of Linearly Elastic Plates

This chapter presents various classes of models for a linearly elastic isotropic plate problem. The models that we consider here are based on two variational principles, the *Hellinger–Reissner principles*. This approach appeared in a joint work with Alessandrini, Arnold and Falk [**2**], which included an error analysis for one of the models, based on the two energy principles (Prager–Synge theorem).

In what follows we describe the resulting equations for some "low order" models.

6.1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain and let $\varepsilon \in (0,1]$ represent the plate thickness. The plate occupies the set $P^\epsilon = \Omega \times (-\varepsilon, \varepsilon)$. We denote its lateral side by $\partial P_L^\epsilon = \partial\Omega \times (-\varepsilon, \varepsilon)$, and the union of its top and bottom by $\partial P_\pm^\epsilon = \Omega \times \{-\varepsilon, \varepsilon\}$. We are concerned with the problem of finding the displacement $\underline{u} : P^\epsilon \to \mathbb{R}^3$ and stress $\underline{\underline{\sigma}} : P^\epsilon \to \mathbb{R}^{3\times 3}_{\text{sym}}$ (the space of 3×3 symmetric matrices) such that

$$
\begin{aligned}
&\underline{\underline{\underline{A}}}\,\underline{\underline{\sigma}}^\epsilon = \underline{\underline{e}}(\underline{u}^\epsilon), \quad \underline{\operatorname{div}}\,\underline{\underline{\sigma}}^\epsilon = -\underline{f}^\epsilon \qquad \text{in } P^\epsilon,\\
&\underline{\underline{\sigma}}^\epsilon \underline{n} = \underline{g}^\epsilon \quad \text{on } \partial P_\pm^\epsilon, \qquad \underline{u}^\epsilon = 0 \quad \text{on } \partial P_L^\epsilon,
\end{aligned}
\tag{6.1.1}
$$

where $\underline{f}^\epsilon : P^\epsilon \to \mathbb{R}^3$ and $\underline{g}^\epsilon : \partial P_\pm^\epsilon \to \mathbb{R}^3$ represent the volume and traction loads. We denote the symmetric part of the gradient of $\underline{u}$ by

$$\underline{\underline{e}}(\underline{u}^\epsilon) = \frac{1}{2}(\underline{\underline{\nabla}}\,\underline{u}^\epsilon + \underline{\underline{\nabla}}^T\,\underline{u}^\epsilon),$$

i.e., $e_{ij}(\underline{u}^\epsilon) = (\partial_i u_j^\epsilon + \partial_j u_i^\epsilon)/2$. Also, $(\underline{\operatorname{div}}\,\underline{\underline{\sigma}}^\epsilon)_i = \sum_{j=1}^3 \partial_j \sigma_{ij}^\epsilon$. The compliance tensor $\underline{\underline{\underline{A}}}$ is such that $\underline{\underline{\underline{A}}}\,\underline{\underline{\tau}} = (1+\nu)\underline{\underline{\tau}}/E - \nu \operatorname{tr}(\underline{\underline{\tau}})\underline{\underline{\delta}}/E$, where $E > 0$ is the Young's modulus, $\nu \in [0, 1/2)$ is the Poisson's ratio, and $\underline{\underline{\delta}}$ is the 3×3 identity matrix.

Extending the notation previously employed, we use one underbar for first order tensors in three variables, two underbars for second order tensors in three variables, etc. Similar notation holds with undertildes for tensors in two variables. We can then decompose 3-vectors and 3×3 matrices as follows:

$$\underline{u} = \begin{pmatrix} \underset{\sim}{u} \\ u_3 \end{pmatrix}, \quad \underline{\underline{\sigma}} = \begin{pmatrix} \underset{\approx}{\sigma} & \underset{\sim}{\sigma} \\ \underset{\sim}{\sigma}^T & \sigma_{33} \end{pmatrix}.$$

The three-dimensional elasticity problem decouples into two problems, one related to the *stretching* of the plate, another related to the *bending*. For a function k defined on P^ϵ or $\partial P_\pm^\epsilon$, there is a unique decomposition into its even and odd parts with respect to x_3^ϵ, i.e.,

$k = k^{\text{even}} + k^{\text{odd}}$ where

$$k^{\text{even}}(\underline{x}^\epsilon) = \frac{k(\underset{\sim}{x}^\epsilon, x_3^\epsilon) + k(\underset{\sim}{x}^\epsilon, -x_3^\epsilon)}{2}, \quad k^{\text{odd}}(\underline{x}^\epsilon) = \frac{k(\underset{\sim}{x}^\epsilon, x_3^\epsilon) - k(\underset{\sim}{x}^\epsilon, -x_3^\epsilon)}{2}.$$

We decompose then

$$\underline{u}^\epsilon = \underline{u}^{\varepsilon^{\text{s}}} + \underline{u}^{\varepsilon^{\text{b}}}, \quad \underline{\underline{\sigma}}^\epsilon = \underline{\underline{\sigma}}^{\varepsilon^{\text{s}}} + \underline{u}^{\varepsilon^{\text{b}}}, \quad \underline{g}^\epsilon = \underline{g}^{\varepsilon^{\text{s}}} + \underline{g}^{\varepsilon^{\text{b}}}, \quad \underline{f}^\epsilon = \underline{f}^{\varepsilon^{\text{s}}} + \underline{f}^{\varepsilon^{\text{b}}},$$

where

$$\underline{u}^{\varepsilon^{\text{s}}} = \begin{pmatrix} \underset{\sim}{u}^{\varepsilon^{\text{even}}} \\ u_3^{\varepsilon^{\text{odd}}} \end{pmatrix}, \quad \underline{\underline{\sigma}}^{\varepsilon^{\text{s}}} = \begin{pmatrix} \underset{\approx}{\sigma}^{\varepsilon^{\text{even}}} & \underset{\sim}{\sigma}^{\varepsilon^{\text{odd}}} \\ (\underset{\sim}{\sigma}^{\varepsilon^{\text{odd}}})^T & \sigma_{33}^{\varepsilon^{\text{even}}} \end{pmatrix}, \tag{6.1.2}$$

$$\underline{u}^{\varepsilon^{\text{b}}} = \begin{pmatrix} \underset{\sim}{u}^{\varepsilon^{\text{odd}}} \\ u_3^{\varepsilon^{\text{even}}} \end{pmatrix}, \quad \underline{\underline{\sigma}}^{\varepsilon^{\text{b}}} = \begin{pmatrix} \underset{\approx}{\sigma}^{\varepsilon^{\text{odd}}} & \underset{\sim}{\sigma}^{\varepsilon^{\text{even}}} \\ (\underset{\sim}{\sigma}^{\varepsilon^{\text{even}}})^T & \sigma_{33}^{\varepsilon^{\text{odd}}} \end{pmatrix}, \tag{6.1.3}$$

$$\underline{g}^{\varepsilon^{\text{s}}} = \begin{pmatrix} \underset{\sim}{g}^{\varepsilon^{\text{even}}} \\ g_3^{\varepsilon^{\text{odd}}} \end{pmatrix}, \quad \underline{g}^{\varepsilon^{\text{b}}} = \begin{pmatrix} \underset{\sim}{g}^{\varepsilon^{\text{odd}}} \\ g_3^{\varepsilon^{\text{even}}} \end{pmatrix}, \quad \underline{f}^{\varepsilon^{\text{s}}} = \begin{pmatrix} \underset{\sim}{f}^{\varepsilon^{\text{even}}} \\ f_3^{\varepsilon^{\text{odd}}} \end{pmatrix}, \quad \underline{f}^{\varepsilon^{\text{b}}} = \begin{pmatrix} \underset{\sim}{f}^{\varepsilon^{\text{odd}}} \\ f_3^{\varepsilon^{\text{even}}} \end{pmatrix}. \tag{6.1.4}$$

It is easy to see that the stretching part $\underline{u}^{\varepsilon^{\text{s}}}$, $\underline{\underline{\sigma}}^{\varepsilon^{\text{s}}}$ is the solution of (6.1.1) with $\underline{g}^\epsilon$ replaced by $\underline{g}^{\varepsilon^{\text{s}}}$ and $\underline{f}^\epsilon$ replaced by $\underline{f}^{\varepsilon^{\text{s}}}$. Similarly for the bending part $\underline{u}^{\varepsilon^{\text{b}}}$, $\underline{\underline{\sigma}}^{\varepsilon^{\text{b}}}$.

6.2. Consistency

As the plate thickeness approach zero, the membrane part of the exact solution approaches the solution of the membrane equation. Similarly, the bending solution converges to the biharmonic model.

Let ω be the solution of the biharmonic equation

$$\begin{aligned} D\varepsilon^3\, \Delta^2\, \omega(\underset{\sim}{x}) = l_K \qquad & \text{in } \Omega, \\ \omega^0(\underset{\sim}{x}) = \frac{\partial \omega}{\partial n}(\underset{\sim}{x}) = 0 \qquad & \text{on } \partial\Omega, \end{aligned} \tag{6.2.1}$$

where $D = 2E/[3(1-\nu^2)]$, and

$$\begin{aligned} l_K(\underset{\sim}{x}) = \int_{-\varepsilon}^{\varepsilon} f_3^\epsilon(\underset{\sim}{x}, x_3)\, dx_3 + \big[g_3^\epsilon(\underset{\sim}{x}, \varepsilon) + g_3^\epsilon(\underset{\sim}{x}, -\varepsilon)\big] \\ + \int_{-\varepsilon}^{\varepsilon} x_3 \operatorname{div} \underset{\sim}{f}^\epsilon(\underset{\sim}{x}, x_3)\, dx_3 + \varepsilon \operatorname{div}\big[\underset{\sim}{g}^\epsilon(\underset{\sim}{x}, \varepsilon) - \underset{\sim}{g}^\epsilon(\underset{\sim}{x}, -\varepsilon)\big]. \end{aligned}$$

Also, let $\underset{\sim}{\eta}$ solve the membrane equation

$$\begin{aligned} -\varepsilon \underset{\sim}{\operatorname{div}} \underset{\approx}{A}^{-1} \underset{\approx}{e}(\underset{\sim}{\eta}) = 2 \int_{-\varepsilon}^{\varepsilon} \underset{\sim}{f}^\epsilon(\underset{\sim}{x}, x_3)\, dx_3 + 2\big[\underset{\sim}{g}^\epsilon(\underset{\sim}{x}, \varepsilon) + \underset{\sim}{g}^\epsilon(\underset{\sim}{x}, -\varepsilon)\big] \qquad & \text{in } \Omega, \\ \underset{\sim}{\eta} = 0 \qquad \text{on } \partial\Omega. & \end{aligned} \tag{6.2.2}$$

Then $\underline{u}^\epsilon$ converges asymptotically to

$$\begin{pmatrix} \underset{\sim}{\eta}(\underset{\sim}{x}) - x_3 \underset{\sim}{\nabla}\, \omega(\underset{\sim}{x}) \\ \omega(\underset{\sim}{x}) \end{pmatrix}. \tag{6.2.3}$$

We turn back to the modeling question. Mathematically, we define a sequence of problems parametrized by the half-thickness ε, and say that a model is *consistent*, or *convergent*, if its displacements $\underline{u}^{M,\varepsilon}$ satisfy

$$\lim_{\varepsilon\to 0} \frac{\|\underline{u}^{\epsilon} - \underline{u}^{M,\varepsilon}\|_{L^2(P^\epsilon)}}{\|\underline{u}^{\epsilon}\|_{L^2(P^\epsilon)}} = 0, \tag{6.2.4}$$

for all loads such that at least one of the functions $\underset{\sim}{\eta}$ and ω are nonzero. If (6.2.4) does not hold, then the model is divergent. In [**35**], a slightly different definition of consistency is used. There, an approximation is consistent if it converges in a proper, scaled sense.

6.3. The HR models

It is possible to characterize the solution of (6.1.1) in an alternative manner. Indeed, let

$$\underline{V}(P^\epsilon) = \{\, \underline{v} \in \underline{H}^1(P^\epsilon) \;:\; \underline{v} = 0 \text{ on } \partial P^\epsilon_L \,\}, \quad \underline{\underline{S}}(P^\epsilon) = \underline{\underline{L}}^2(P^\epsilon).$$

Then the *first Hellinger-Reissner principle*, or HR for short, holds.

HR: $(\underline{u}^\epsilon, \underline{\underline{\sigma}}^\epsilon)$ is the unique critical point of

$$L(\underline{v}, \underline{\underline{\tau}}) = \frac{1}{2}\int_{P^\epsilon} \underline{\underline{\underline{\underline{A}}}}\,\underline{\underline{\tau}} : \underline{\underline{\tau}}\, d\underline{x}^\epsilon - \int_{P^\epsilon} \underline{\underline{\tau}} : \underline{\underline{e}}(\underline{v})\, d\underline{x}^\epsilon + \int_{P^\epsilon} \underline{f}^\epsilon \cdot \underline{v}\, d\underline{x}^\epsilon + \int_{\partial P^\epsilon_\pm} \underline{g}^\epsilon \cdot \underline{v}\, d\underset{\sim}{x}^\epsilon$$

on $\underline{V}(P^\epsilon) \times \underline{\underline{S}}(P^\epsilon)$.

Finding the critical point of L is equivalent to find $\underline{u}^\epsilon \in \underline{V}(P^\epsilon)$ and $\sigma^\epsilon \in \underline{\underline{S}}(P^\epsilon)$ such that

$$\int_{P^\epsilon} \underline{\underline{\underline{\underline{A}}}}\,\underline{\underline{\sigma}}^\epsilon : \underline{\underline{\tau}}\, d\underline{x} - \int_{P^\epsilon} \underline{\underline{e}}(\underline{u}^\epsilon) : \underline{\underline{\tau}}\, d\underline{x}^\epsilon = 0 \qquad \text{for all } \underline{\underline{\tau}} \in \underline{\underline{S}}(P^\epsilon), \tag{6.3.1}$$

$$\int_{P^\epsilon} \underline{\underline{\sigma}}^\epsilon : \underline{\underline{e}}(\underline{v})\, d\underline{x}^\epsilon = \int_{P^\epsilon} \underline{f} \cdot \underline{v}\, d\underline{x}^\epsilon + \int_{\partial P^\epsilon_\pm} \underline{g} \cdot \underline{v}\, d\underset{\sim}{x}^\epsilon \qquad \text{for all } \underline{v} \in \underline{V}(P^\epsilon). \tag{6.3.2}$$

A first type of models appears when we look for critical points of L in subspaces of $\underline{V}(P^\epsilon)$ and $\underline{\underline{S}}(P^\epsilon)$ that have polynomial dependence in the transverse direction. For instance, let p be a positive integer and let

$$\underline{V}(P^\epsilon, p) = \big\{ \underline{v} \in \underline{V}(P^\epsilon) \colon\ \deg_3 \underset{\sim}{v} \le p,\ \deg_3 v_3 \le p-1 \big\}, \tag{6.3.3}$$

$$\underline{\underline{S}}(P^\epsilon, p) = \big\{ \underline{\underline{\tau}} \in \underline{\underline{S}}(P^\epsilon) \colon\ \deg_3 \underset{\approx}{\tau} \le p,\ \deg_3 \underset{\sim}{\tau} \le p-1,\ \deg_3 \tau_{33} \le p-2 \big\}. \tag{6.3.4}$$

Then a critical point $(\underline{u}^\epsilon(p), \underset{\approx}{\sigma}^\epsilon(p)) \in \underline{V}(P^\epsilon, p) \times \underline{\underline{S}}(P^\epsilon, p)$ of L characterizes the $\mathrm{HR}_1(p)$ model. Carefully varying the polynomial degrees of the components for displacements and stress yields different subspaces and models. We summarize some of them in the table below. Besides the already defined $\mathrm{HR}_1(p)$, we also present the $\mathrm{HR}_2(p)$ and $\mathrm{HR}_3(p)$ models.

TABLE 1. The principle plate models based on the HR principle. The degree p is a positive integer.

model	$\deg_3 \underset{\approx}{\sigma}$	$\deg_3 \underset{\sim}{\sigma}$	$\deg_3 \sigma_{33}$	$\deg_3 \underset{\sim}{u}$	$\deg_3 u_3$
$\mathrm{HR}_1(p)$	p	$p-1$	$p-2$	p	$p-1$
$\mathrm{HR}_2(p)$	p	$p-1$	p	p	$p-1$
$\mathrm{HR}_3(p)$	p	$p+1$	p	p	$p+1$

In the case of a plate under bending, for p odd, $\mathrm{HR}_2(p)$ was called MPp in Alessandrini's thesis [**1**]. For $p = 3$ it yields the model of Lo, Christensen and Wu [**27**] (for both bending and stretching). Still considering the bending situation, the $\mathrm{HR}_3(p)$ models were denoted by $\mathrm{MP}(p+1)$ by Alessandrini [**1**], and for $p = 1$ it is also referred as the $(1, 1, 2)$ model [**6**].

Analogously to the original three-dimensional problem, the models can be equivalently characterized by a weak formulation, i.e., we shall seek $\underline{u}^\epsilon(p) \in \underline{V}(P^\epsilon, p)$ and $\underline{\underline{\sigma}}^\epsilon(p) \in \underline{\underline{S}}(P^\epsilon, p)$ such that

$$\int_{P^\epsilon} \underline{\underline{\underline{A}}}\,\underline{\underline{\sigma}}^\epsilon(p) : \underline{\underline{\tau}}\, d\underline{x} - \int_{P^\epsilon} \underline{\underline{e}}(\underline{u}^\epsilon(p)) : \underline{\underline{\tau}}\, d\underline{x}^\epsilon = 0 \quad \text{for all } \underline{\underline{\tau}} \in \underline{\underline{S}}(P^\epsilon, p), \tag{6.3.5}$$

$$\int_{P^\epsilon} \underline{\underline{\sigma}}^\epsilon(p) : \underline{\underline{e}}(\underline{v})\, d\underline{x}^\epsilon = \int_{P^\epsilon} \underline{f} \cdot \underline{v}\, d\underline{x}^\epsilon + \int_{\partial P^\epsilon_\pm} \underline{g} \cdot \underline{v}\, d\underset{\sim}{x}^\epsilon \quad \text{for all } \underline{v} \in \underline{V}(P^\epsilon, p). \tag{6.3.6}$$

With the choices of spaces presented in Table 1, the existence and uniqueness of solutions for (6.3.5) and (6.3.6) follows from $\underline{\underline{e}}(\underline{V}(P^\epsilon, p)) \subset \underline{\underline{S}}(P^\epsilon, p)$. Note also that for both the $\mathrm{HR}_2(p)$ and $\mathrm{HR}_3(p)$ models, $\underline{\underline{\underline{A}}}^{-1}\,\underline{\underline{e}}(\underline{V}(P^\epsilon, p)) \subset \underline{\underline{S}}(P^\epsilon, p)$ and it follows that the constitutive equation $\underline{\underline{\underline{A}}}\,\underline{\underline{\sigma}}^\epsilon(p) = \underline{\underline{e}}(\underline{u}^\epsilon(p))$ is satisfied exactly. As a consequence, $\underline{u}^\epsilon(p)$ is the minimizer (in $\underline{V}(P^\epsilon, p)$) of the potential energy

$$J(\underline{v}) = \frac{1}{2}\int_{P^\epsilon} \underline{\underline{\underline{A}}}^{-1}\,\underline{\underline{e}}(\underline{v}) : \underline{\underline{e}}(\underline{v})\, d\underline{x}^\epsilon - \int_{P^\epsilon} \underline{f}^\epsilon \cdot \underline{v}\, d\underline{x}^\epsilon - \int_{\partial P^\epsilon_\pm} \underline{g}^\epsilon \cdot \underline{v}\, d\underset{\sim}{x}^\epsilon,$$

i.e., $\mathrm{HR}_2(p)$ and $\mathrm{HR}_3(p)$ are minimum energy models. This sort of model is quite widespread in the literature, but a very upsetting characteristic is that its simplest version, $\mathrm{HR}_2(1)$, is worthless. In fact, Paumier and Raoult [**35**] showed that for a minimum energy model to be *consistent*, i.e. to be asymptotically convergent to the biharmonic model as $\varepsilon \to 0$, $u_3(p)$ must be at least a quadratic polynomial. $\mathrm{HR}_3(1)$ is the simplest consistent minimum energy model, but its final form is more complicated than the membrane and the Resissner–Mindlin models, having one extra equation (and unknown) in each case. The $\mathrm{HR}_1(p)$ is *not* a minimum energy model. It is convergent for $p = 1$ and it yields a membrane problem for the stretching part and a problem of Resissner–Mindlin type with shear correction factor 1 for the bending part.

Before presenting details regarding the lowest order example for each of the HR models, some notation is necessary. If we define $\underset{\approx}{\underset{\approx}{A}}\underset{\approx}{\tau} = (1+\nu)\underset{\approx}{\tau}/E - \nu\,\mathrm{tr}(\underset{\approx}{\tau})\underset{\approx}{\delta}/E$, then

$$\underline{\underline{\underline{A}}}\,\underline{\underline{\tau}} = \begin{pmatrix} \underset{\approx}{\underset{\approx}{A}}\underset{\approx}{\tau} - \frac{\nu}{E}\tau_{33}\underset{\approx}{\delta} & \frac{1+\nu}{E}\underset{\sim}{\tau} \\ \frac{1+\nu}{E}\underset{\sim}{\tau}^T & \frac{\tau_{33}}{E} - \frac{\nu}{E}\,\mathrm{tr}(\underset{\approx}{\tau}) \end{pmatrix}.$$

It is useful to know (and straightforward to check) that

$$\underset{\approx}{\underset{\approx}{A}}^{-1}\underset{\approx}{\tau} = \frac{E}{1+\nu}\left(\underset{\approx}{\tau} + \frac{\nu}{1-\nu}\,\mathrm{tr}(\underset{\approx}{\tau})\underset{\approx}{\delta}\right).$$

Let

$$\underline{f}^k(\underset{\sim}{x}^\epsilon) = \varepsilon^{-1}\int_{-\varepsilon}^{\epsilon} \underline{f}^\epsilon(\underset{\sim}{x}^\epsilon, x_3^\epsilon) Q_k(x_3^\epsilon)\, dx_3^\epsilon,$$

$$\underline{g}^0(\underset{\sim}{x}^\epsilon) = \frac{1}{2}\big[\,\underline{g}^\epsilon(\underset{\sim}{x}^\epsilon, \varepsilon) + \underline{g}^\epsilon(\underset{\sim}{x}^\epsilon, -\varepsilon)\,\big], \quad \underline{g}^1(\underset{\sim}{x}^\epsilon) = \frac{1}{2}\big[\,\underline{g}^\epsilon(\underset{\sim}{x}^\epsilon, \varepsilon) - \underline{g}^\epsilon(\underset{\sim}{x}^\epsilon, -\varepsilon)\,\big],$$

where $Q_j(z) = \varepsilon^j L_j(\varepsilon^{-1} z)$, and L_j are the Legendre polynomials in $(-1, 1)$. The first few polynomials are $L_0(z) = 1$, $L_1(z) = z$, $L_2(z) = (3z^2 - 1)/2$, $L_3(z) = (5z^3 - 3z)/2$. The constants

$$\lambda = \frac{E}{2(1+\nu)}, \quad c_1 = \frac{-E\nu^2}{12(1-\nu^2)(2\nu-1)}, \quad c_2 = \frac{2(1-\nu)}{\nu},$$

will appear in what follows.

6.3.1. The HR$_1$(1) model. We first present its final form and then show how to derive it. Writing the model solution as

$$\begin{aligned} \underline{u}(\underline{x}^\epsilon) &= \begin{pmatrix} \underset{\sim}{\eta}(\underset{\sim}{x}^\epsilon) \\ 0 \end{pmatrix} + \begin{pmatrix} -\underset{\sim}{\phi}(\underset{\sim}{x}^\epsilon) x_3^\epsilon \\ \omega(\underset{\sim}{x}) \end{pmatrix}, \\ \underline{\underline{\sigma}}(p)(\underline{x}^\epsilon) &= \begin{pmatrix} \underset{\approx}{\sigma}^0(\underset{\sim}{x}^\epsilon) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \underset{\approx}{\sigma}^1(\underset{\sim}{x}^\epsilon) x_3^\epsilon & \underset{\sim}{\sigma}^0(\underset{\sim}{x}^\epsilon) \\ (\underset{\sim}{\sigma}^0)^T(\underset{\sim}{x}^\epsilon) & 0 \end{pmatrix}, \end{aligned} \tag{6.3.7}$$

Then for the stretching part we have that $\underset{\sim}{\eta}$ satisfies the membrane equation (6.2.2). After determining $\underset{\sim}{\eta}$, we are able to find the in-plane components of the stress from

$$\underset{\approx}{\sigma}^0 = \underset{\approx}{A}^{-1}\, \underset{\approx}{e}(\underset{\sim}{\eta}). \tag{6.3.8}$$

Concerning the bending part, $\underset{\sim}{\phi}$ and ω solve the Reissner–Mindlin equation with shear correction factor 1:

$$-\frac{\varepsilon^3}{3} \underset{\sim}{\operatorname{div}}\, \underset{\approx}{A}^{-1}\, \underset{\approx}{e}(\underset{\sim}{\phi}) + \varepsilon\lambda(\underset{\sim}{\phi} - \underset{\sim}{\nabla}\, \omega) = -\varepsilon\big(\frac{1}{2}\underset{\sim}{f}^1 + \underset{\sim}{g}^1\big) \qquad \text{in } \Omega, \tag{6.3.9}$$

$$\varepsilon\lambda \operatorname{div}(\underset{\sim}{\phi} - \underset{\sim}{\nabla}\, \omega) = \frac{\varepsilon}{2} f_3^0 + g_3^0 \qquad \text{in } \Omega, \tag{6.3.10}$$

$$\underset{\sim}{\phi} = 0, \quad \omega = 0 \qquad \text{on } \partial\Omega. \tag{6.3.11}$$

The in-plane and shear stress components are found from

$$\underset{\approx}{\sigma}^1 = -\underset{\approx}{A}^{-1}\, \underset{\approx}{e}(\underset{\sim}{\phi}), \quad \underset{\sim}{\sigma}^0 = \lambda(-\underset{\sim}{\phi} + \underset{\sim}{\nabla}\, \omega). \tag{6.3.12}$$

We deduce the above equations by assuming (6.3.7), and using (6.3.5), (6.3.6). Considering the stretching problem first, we use test functions of the form

$$\underline{\underline{\tau}}(\underline{x}^\epsilon) = \begin{pmatrix} \underset{\approx}{\tau}(\underset{\sim}{x}^\epsilon) & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{where } \underset{\approx}{\tau} \in \underset{\approx}{L}^2(\Omega),$$

in (6.3.5). From

$$\underline{\underline{A}}\,\underline{\underline{\tau}} = \begin{pmatrix} \underset{\approx}{A}\,\underset{\approx}{\tau} & 0 \\ \times & -\frac{\nu}{E}\operatorname{tr}(\underset{\approx}{\tau}) \end{pmatrix},$$

and an integration in the vertical direction, equation (6.3.8) holds. Similarly, if we substitute $\underline{v}^T(\underline{x}^\epsilon) = (\underset{\sim}{v}^T, 0)(\underset{\sim}{x}^\epsilon)$, where $\underset{\sim}{v} \in \mathring{\underset{\sim}{H}}1(\Omega)$, in (6.3.6) and integrate in the vertical direction, then

$$\int_\Omega \underset{\approx}{\sigma}^0 : \underset{\approx}{e}(\underset{\sim}{v})\, d\underset{\sim}{x}^\epsilon = \int_\Omega (\frac{1}{2}\underset{\sim}{f}^0 + \varepsilon^{-1}\underset{\sim}{g}^0) \cdot \underset{\sim}{v}\, d\underset{\sim}{x}^\epsilon \qquad \text{for all } \underset{\sim}{v} \in \mathring{\underset{\sim}{H}}1(\Omega). \tag{6.3.13}$$

Equation (6.2.2) follows from (6.3.13) after an integration by parts and from (6.3.8).

The procedure to realize the bending part of $\mathrm{HR}_1(1)$ is basically the same. Assuming the test functions in (6.3.5) to be of the form

$$\underline{\underline{\tau}}(\underline{x}^\epsilon) = \begin{pmatrix} \underset{\approx}{\tau}(\underset{\sim}{x}^\epsilon)x_3^\epsilon & \underset{\sim}{\tau}(\underset{\sim}{x}^\epsilon) \\ \underset{\sim}{\tau}^T(\underset{\sim}{x}^\epsilon) & 0 \end{pmatrix}, \qquad \text{where } \underset{\approx}{\tau} \in \underset{\approx}{L}^2(\Omega) \text{ and } \underset{\sim}{\tau} \in \underset{\sim}{L}^2(\Omega),$$

and using that

$$\underline{\underline{\underline{A}}}\,\underline{\underline{\tau}} = \begin{pmatrix} \underset{\approx}{A}\underset{\approx}{\tau}x_3^\epsilon & \frac{1+\nu}{E}\underset{\sim}{\tau} \\ \frac{1+\nu}{E}\underset{\sim}{\tau}^T & -\frac{\nu}{E}\operatorname{tr}(\underset{\approx}{\tau})x_3^\epsilon \end{pmatrix},$$

we see that (6.3.12) follows. Next we use $\underline{v}^T(\underline{x}^\epsilon) = (\underset{\sim}{v}^T(\underset{\sim}{x}^\epsilon)x_3^\epsilon, 0)$ with $\underset{\sim}{v} \in \mathring{\underset{\sim}{H}}1(\Omega)$ as test functions in (6.3.6), and then

$$(6.3.14) \qquad \int_\Omega \frac{2\varepsilon^3}{3}\underset{\approx}{\sigma}^1 : \underset{\approx}{e}(\underset{\sim}{v}) + 2\varepsilon\underset{\sim}{\sigma}^0 \cdot \underset{\sim}{v}\,d\underset{\sim}{x}^\epsilon = \int_\Omega (\varepsilon \underset{\sim}{f}^1 + 2\varepsilon \underset{\sim}{g}^1)\cdot \underset{\sim}{v}\,d\underset{\sim}{x}^\epsilon \qquad \text{for all } \underset{\sim}{v} \in \mathring{\underset{\sim}{H}}1(\Omega).$$

Substituting (6.3.12) and integrating by parts yields (6.3.9). We assume then that $\underline{v}^T(\underline{x}^\epsilon) = (0, v(\underset{\sim}{x}^\epsilon))$ where $v \in \mathring{H}1(\Omega)$ and (6.3.6) yields

$$(6.3.15) \qquad \int_\Omega 2\varepsilon\underset{\sim}{\sigma}^0 \cdot \underset{\sim}{\nabla}\,v\,d\underset{\sim}{x}^\epsilon = \int_\Omega (\varepsilon f_3^0 + 2g_3^0)v\,d\underset{\sim}{x}^\epsilon \qquad \text{for all } v \in \mathring{H}1(\Omega).$$

Finally, using (6.3.12), we see that (6.3.10) holds.

The derivation for other models is analogous and become tedious as p increases. We present then the final equations for few of them.

6.3.2. The $\mathbf{HR}_2(1)$ Model. . The simplest minimum energy model, $\mathrm{HR}_2(1)$, is not consistent. It is worthwhile to mention that the same spurious mode that appears in the bending part also shows up in the stretching situation. In this case it can be shown that $\underline{u}$ is the same as in (6.3.7) (and $\underline{\underline{\sigma}}$ is given by an expression a bit more complicated which we don't report here), except that the equations (6.2.2), (6.3.9) are replaced by

$$-2\varepsilon\big[\underset{\sim}{\operatorname{div}}\,\underset{\approx}{A}^{-1}\,\underset{\approx}{e}(\underset{\sim}{\eta}) + c\,\underset{\sim}{\nabla}\,\underset{\sim}{\operatorname{div}}\,\underset{\sim}{\eta}\big] = 2\underset{\sim}{g}^0 + \underset{\sim}{f}^0 \text{ in } \Omega,$$

and

$$-\frac{2\varepsilon^3}{3}\big[\underset{\sim}{\operatorname{div}}\,\underset{\approx}{A}^{-1}\,\underset{\approx}{e}(\underset{\sim}{\phi}) + c\,\underset{\sim}{\nabla}\operatorname{div}\underset{\sim}{\phi}\big] + \varepsilon\frac{E}{(1+\nu)}(\underset{\sim}{\phi} - \underset{\sim}{\nabla}\,\omega) = -2\varepsilon\underset{\sim}{g}^1 - \varepsilon\underset{\sim}{f}^1).$$

respectively, with $c = -E\nu^2/[(1-\nu^2)(2\nu-1)]$. These additional terms are spurious, and cause the $\mathrm{HR}_2(1)$ model to be divergent as ε tends to 0 (in a sense which will be made precise in §4).

Thus the $\mathrm{HR}_2(1)$ model is incorrect. For $p \geq 3$ it can be shown that the $\mathrm{HR}_2(p)$ model is convergent. For $p = 3$, it can be shown to be identical to a method of Lo, Christensen, and Wu [**27**]. However, we feel it possible that even for larger p, the $\mathrm{HR}_2(p)$ method is both more complicated than and less accurate than the $\mathrm{HR}_1(p)$ method.

6.3.3. The HR$_3$(1) Model. . It is the simplest consistent minimum energy model. Writing the solutions as

$$\underline{u}(\underline{x}^\epsilon) = \begin{pmatrix} \underset{\sim}{\eta}(\underset{\sim}{x}^\epsilon) \\ \omega(\underset{\sim}{x}^\epsilon)x_3^\epsilon \end{pmatrix} + \begin{pmatrix} -\underset{\sim}{\phi}(\underset{\sim}{x}^\epsilon)x_3^\epsilon \\ \omega^0(\underset{\sim}{x}^\epsilon) + \omega^2(\underset{\sim}{x}^\epsilon)Q_2(x_3^\epsilon) \end{pmatrix}, \tag{6.3.16}$$

$$\begin{aligned} \underline{\sigma}(\underline{x}^\epsilon) = &\begin{pmatrix} \underset{\approx}{\sigma}^0(\underset{\sim}{x}^\epsilon) & \underset{\sim}{\sigma}^1(\underset{\sim}{x}^\epsilon)x_3^\epsilon \\ (\underset{\sim}{\sigma}^1)^T(\underset{\sim}{x}^\epsilon)x_3^\epsilon & \sigma_{33}^0(\underset{\sim}{x}^\epsilon) \end{pmatrix} \\ &+ \begin{pmatrix} \underset{\approx}{\sigma}^1(\underset{\sim}{x}^\epsilon)\underset{\sim}{x}^\epsilon & \underset{\sim}{\sigma}^0(\underset{\sim}{x}^\epsilon) + \underset{\sim}{\sigma}^2(\underset{\sim}{x}^\epsilon)Q_2(x_3^\epsilon) \\ (\underset{\sim}{\sigma}^0)^T(\underset{\sim}{x}^\epsilon) + (\underset{\sim}{\sigma}^2)^T(\underset{\sim}{x}^\epsilon)Q_2(x_3^\epsilon) & \sigma_{33}^1(\underset{\sim}{x}^\epsilon)x_3^\epsilon \end{pmatrix}, \end{aligned} \tag{6.3.17}$$

then we have for the stretching part that

$$\begin{aligned} -\varepsilon \,\underset{\sim}{\operatorname{div}}\, \underset{\approx}{A}^{-1}\, \underset{\approx}{e}(\underset{\sim}{\eta}) - 12\varepsilon c_1 \underset{\sim}{\nabla}(\operatorname{div}\underset{\sim}{\eta} + \frac{c_2}{2}\omega) &= \frac{\varepsilon}{2}\underset{\sim}{f}^0 + \underset{\sim}{g}^0 \qquad \text{in } \Omega, \\ 6c_1c_2(\operatorname{div}\underset{\sim}{\eta} + \frac{c_2}{2}\omega) - \frac{\varepsilon^2}{3}\lambda\,\Delta\,\omega &= \frac{1}{2}f_3^1 + g_3^1 \qquad \text{in } \Omega, \\ \underset{\sim}{\eta} = 0, \quad \omega = 0 \qquad &\text{on } \partial\Omega. \end{aligned}$$

Note that this model takes into account the transverse components of the load that also contribute for the stretching. These terms are not present in the HR$_1$(1) model or in the membrane equation coming from asymptotic methods. The stress components for stretching come from substituting

$$\begin{aligned} \underset{\approx}{\sigma}^0 &= \underset{\approx}{A}^{-1}\, \underset{\approx}{e}(\underset{\sim}{\eta}) + 12c_1(\operatorname{div}\underset{\sim}{\eta} + \frac{c_2}{2}\omega)\underset{\approx}{\delta}, \\ \underset{\sim}{\sigma}^1 &= \lambda\,\underset{\sim}{\nabla}\,\omega, \quad \sigma_{33}^0 = 6c_1c_2(\operatorname{div}\underset{\sim}{\eta} + \frac{c_2}{2}\omega). \end{aligned}$$

The displacement components under bending solve

$$\begin{aligned} -\frac{\varepsilon^3}{3}\,\underset{\sim}{\operatorname{div}}\,\underset{\approx}{A}^{-1}\,\underset{\approx}{e}(\underset{\sim}{\phi}) - 4\varepsilon^3\,\underset{\sim}{\nabla}\, c_1(\operatorname{div}\underset{\sim}{\phi} - \frac{3}{2}c_2\omega^2) + \varepsilon\lambda(\underset{\sim}{\phi} - \underset{\sim}{\nabla}\,\omega^0) &= -\varepsilon(\frac{1}{2}\underset{\sim}{f}^1 + \underset{\sim}{g}^1), \\ \varepsilon\lambda\operatorname{div}(\underset{\sim}{\phi} - \underset{\sim}{\nabla}\,\omega^0) &= \frac{\varepsilon}{2}f_3^0 + g_3^0, \\ \frac{\varepsilon^2}{30}\lambda\operatorname{div}\underset{\sim}{\nabla}\,\omega^2 + c_1c_2(\operatorname{div}\underset{\sim}{\phi} - \frac{3}{2}c_2\omega^2) &= \frac{\varepsilon^{-2}}{6}(-f_3^2 - \varepsilon g_3^0). \end{aligned}$$

Here we have a set of equations that are more complex than HR$_1$(1) but also include the second moment of f_3.

The equations below yield the stress components:

$$\begin{aligned} \underset{\approx}{\sigma}^1 &= -\underset{\approx}{A}^{-1}\,\underset{\approx}{e}(\underset{\sim}{\phi}) - 12c_1(\operatorname{div}\underset{\sim}{\phi} - \frac{3}{2}c_2\omega^2)\underset{\approx}{\delta}, \\ \underset{\sim}{\sigma}^0 &= \lambda(-\underset{\sim}{\phi} + \underset{\sim}{\nabla}\,\omega^0), \quad \underset{\sim}{\sigma}^2 = \lambda\,\underset{\sim}{\nabla}\,\omega^2, \\ \sigma_{33}^1 &= -6c_1c_2(\operatorname{div}\underset{\sim}{\phi} - \frac{3}{2}c_2\omega^2). \end{aligned}$$

6.4. The HR′ models

Another way to characterize the solution of (6.1.1) is by the *second Hellinger–Reissner principle*, or HR′ for short.

Define

$$\underline{V}'(P^\epsilon) = \underline{L}^2(P^\epsilon), \quad \underline{\underline{S}}'_{\underline{g}}(P^\epsilon) = \{\, \underline{\underline{\tau}} \in \underline{\underline{H}}(\operatorname{div}, P^\epsilon) \;:\; \underline{\underline{\tau}}\,\underline{n} = \underline{g} \text{ on } \partial P^\epsilon_\pm \,\}.$$

Then we have

HR′: $(\underline{u}^\epsilon, \underline{\underline{\sigma}}^\epsilon)$ is the unique critical point of

$$L'(\underline{v}, \underline{\underline{\tau}}) = \frac{1}{2}\int_{P^\epsilon} \underline{\underline{\underline{A}}}\,\underline{\underline{\tau}} \;:\; \underline{\underline{\tau}}\, d\underline{x}^\epsilon + \int_{P^\epsilon} \underline{\operatorname{div}}\, \underline{\underline{\tau}} \cdot \underline{v}\, d\underline{x}^\epsilon + \int_{P^\epsilon} \underline{f}^\epsilon \cdot \underline{v}\, d\underline{x}^\epsilon$$

on $\underline{V}'(P^\epsilon) \times \underline{\underline{S}}'_{\underline{g}}(P^\epsilon)$.

An equivalent statement is that $\underline{u}^\epsilon \in \underline{V}'(P^\epsilon)$ and $\underline{\underline{\tau}} \in \underline{\underline{S}}'_{\underline{g}}(P^\epsilon)$ satisfy

$$\int_{P^\epsilon} \underline{\underline{\underline{A}}}\,\underline{\underline{\sigma}}^\epsilon : \underline{\underline{\tau}}\, d\underline{x} + \int_{P^\epsilon} \underline{u}^\epsilon \cdot \underline{\operatorname{div}}\, \underline{\underline{\tau}}\, d\underline{x}^\epsilon = 0 \qquad \text{for all } \underline{\underline{\tau}} \in \underline{\underline{S}}'_0(P^\epsilon),$$

$$\int_{P^\epsilon} \underline{\operatorname{div}}\, \underline{\underline{\sigma}}^\epsilon \cdot \underline{v}\, d\underline{x}^\epsilon = \int_{P^\epsilon} -\underline{f} \cdot \underline{v}\, d\underline{x}^\epsilon \qquad \text{for all } \underline{v} \in \underline{V}'(P^\epsilon).$$

By seeking a critical point for L' on subspaces $\underline{V}'(P^\epsilon, p) \times \underline{\underline{S}}'_{\underline{g}}(P^\epsilon, p) \subset \underline{V}'(P^\epsilon) \times \underline{\underline{S}}'_{\underline{g}}(P^\epsilon)$, we define classes of HR′ models. The elements of these subspaces will have certain polynomial dependence in the transverse direction, and we will specify four differents classes of HR′ models in the table below.

TABLE 2. The principle plate models based on the HR′ principle. The degree p is a positive integer.

model	$\deg_3 \underset{\approx}{\sigma}$	$\deg_3 \underset{\sim}{\sigma}$	$\deg_3 \sigma_{33}$	$\deg_3 \underset{\sim}{u}$	$\deg_3 u_3$
$\mathrm{HR}'_1(p)$	p	$p-1$	p	p	$p-1$
$\mathrm{HR}'_2(p)$	p	$p+1$	p	p	$p-1$
$\mathrm{HR}'_3(p)$	p	$p+1$	p	p	$p+1$
$\mathrm{HR}'_4(p)$	p	$p+1$	$p+2$	p	$p+1$

For pure bending and p odd, Alessandrini [**1**] denoted $\mathrm{HR}'_1(p)$ by HRp.0, $\mathrm{HR}'_3(p)$ by HR$p+1.0$, and $\mathrm{HR}'_4(p)$ by HR$p+1.1$. A nice feature of some of the above models is that $\underline{\operatorname{div}}\, \underline{\underline{S}}'_0(P^\epsilon, p) = \underline{V}'(P^\epsilon, p)$ and therefore, not only

$$\underline{\operatorname{div}}\, \underline{\underline{\sigma}}^\epsilon(p) = -\underline{\pi}_{\underline{V}'} \underline{f}^\epsilon,$$

where $\underline{\pi}_{\underline{V}'} \underline{f}^\epsilon$ is the orthogonal L^2 projection of $\underline{f}^\epsilon$ into $\underline{V}'(P^\epsilon, p)$, but also $\underline{\underline{\sigma}}^\epsilon(p)$ minimizes the complementary energy

$$J_c(\underline{\underline{\tau}}) = \frac{1}{2}\int_{P^\epsilon} \underline{\underline{\underline{A}}}\,\underline{\underline{\tau}} \;:\; \underline{\underline{\tau}}\, d\underline{x}^\epsilon$$

over all $\underline{\underline{\tau}} \in S'_g(P^\epsilon, p)$ such that $\underline{\operatorname{div}}\, \underline{\underline{\tau}} = -\underline{\pi}_{\underline{V}'} \underline{f}^\epsilon$.

We summarize next, for $p = 1$, some of the HR′ models. To derive the equations of a particular model, we proceed as in Section 6.3. See also [**2**], where the equations for $\mathrm{HR}'_4(1)$ are found explicitly. As the $\mathrm{HR}'_1(1)$, and $\mathrm{HR}'_3(1)$ models are not consistent we do not show them here.

6.4.1. The $\mathrm{HR}'_2(1)$ Model. Assume that the displacement

$$\underline{u}(\underline{x}^\epsilon) = \begin{pmatrix} \underset{\sim}{\eta}(\underset{\sim}{x}^\epsilon) \\ 0 \end{pmatrix} + \begin{pmatrix} -\underset{\sim}{\phi}(\underset{\sim}{x}^\epsilon)x_3^\epsilon \\ \omega(\underset{\sim}{x}^\epsilon) \end{pmatrix},$$

and the stress

$$\underline{\underline{\sigma}}(\underline{x}^\epsilon) = \begin{pmatrix} \underset{\approx}{\sigma}^0(\underset{\sim}{x}^\epsilon) & \varepsilon^{-1}\underset{\sim}{g}^0 x_3^\epsilon \\ \varepsilon^{-1}(\underset{\sim}{g}^0)^T x_3^\epsilon & g_3^1 \end{pmatrix} + \begin{pmatrix} \underset{\approx}{\sigma}^1(\underset{\sim}{x}^\epsilon)x_3^\epsilon & \underset{\sim}{\sigma}^0(\underset{\sim}{x}^\epsilon)[1-\varepsilon^{-2}Q_2(x_3^\epsilon)] + \underset{\sim}{g}^1 \\ (\underset{\sim}{\sigma}^0)^T(\underset{\sim}{x}^\epsilon)[1-\varepsilon^{-2}Q_2(x_3^\epsilon)] + (\underset{\sim}{g}^1)^T & \varepsilon^{-1}g_3^0 x_3^\epsilon \end{pmatrix}. \tag{6.4.1}$$

Then, the equations defining the first components of the displacement for the stretching part are

$$-\varepsilon \,\underset{\sim}{\operatorname{div}}\, \underset{\approx}{A}^{-1}\, \underset{\approx}{e}(\underset{\sim}{\eta}) = \frac{\varepsilon}{2}\underset{\sim}{f}^0 + \underset{\sim}{g}^0 + \varepsilon\frac{\nu}{1-\nu}\,\underset{\sim}{\nabla}\, g_3^1 \qquad \text{in } \Omega, \tag{6.4.2}$$

$$\underset{\sim}{\eta} = 0 \text{ on } \partial\Omega. \tag{6.4.3}$$

Note that here we have basically the same equations as in $\mathrm{HR}_1(1)$ plus a term taking into account the contributions of g_3^1. It is easy to compute the in-plane stress components by substituting

$$\underset{\approx}{\sigma}^0 = \underset{\approx}{A}^{-1}\, \underset{\approx}{e}(\underset{\sim}{\eta}) + \frac{\nu}{1-\nu} g_3^1 \underset{\approx}{\delta}.$$

For the bending part we have that

$$-\frac{\varepsilon^3}{3}\,\underset{\sim}{\operatorname{div}}\, \underset{\approx}{A}^{-1}\, \underset{\approx}{e}(\underset{\sim}{\phi}) + \frac{5}{6}\varepsilon\lambda(\underset{\sim}{\phi} - \underset{\sim}{\nabla}\,\omega) = -\varepsilon(\frac{1}{2}\underset{\sim}{f}^1 + \frac{5}{6}\underset{\sim}{g}^1) - \frac{\nu}{3(1-\nu)}\varepsilon^2\,\underset{\sim}{\nabla}\, g_3^0 \qquad \text{in } \Omega, \tag{6.4.4}$$

$$\frac{5}{6}\varepsilon\lambda\operatorname{div}(\underset{\sim}{\phi} - \underset{\sim}{\nabla}\,\omega) = \frac{\varepsilon}{2}f_3^0 + g_3^0 + \frac{\varepsilon}{6}\operatorname{div}\underset{\sim}{g}^1 \qquad \text{in } \Omega, \tag{6.4.5}$$

$$\underset{\sim}{\phi} = 0, \quad \omega = 0 \qquad \text{on } \partial\Omega. \tag{6.4.6}$$

This time we find the Reissner–Mindlin model with shear correction factor 5/6.

The stress components can be found by substituting

$$\underset{\approx}{\sigma}^1 = -\underset{\approx}{A}^{-1}\, \underset{\approx}{e}(\underset{\sim}{\phi}) + \frac{\nu}{1-\nu}\varepsilon^{-1}g_3^0\underset{\approx}{\delta}, \quad \underset{\sim}{\sigma}^0 = \frac{5}{6}[\lambda(-\underset{\sim}{\phi} + \underset{\sim}{\nabla}\,\omega) - \underset{\sim}{g}^1].$$

6.4.2. The $\mathrm{HR}'_4(1)$ Model. We look for displacement solution in the form:

$$\underline{u}(\underline{x}^\epsilon) = \begin{pmatrix} \underset{\sim}{\eta}(\underset{\sim}{x}^\epsilon) \\ \omega(\underset{\sim}{x}^\epsilon)x_3^\epsilon \end{pmatrix} + \begin{pmatrix} -\underset{\sim}{\phi}(\underset{\sim}{x}^\epsilon)x_3^\epsilon \\ \omega^0(\underset{\sim}{x}^\epsilon) + \omega^2(\underset{\sim}{x}^\epsilon)[Q_2(\underset{\sim}{x}^\epsilon) + \frac{\varepsilon^2}{5}] \end{pmatrix},$$

and the stress

$$\underline{\underline{\sigma}}(\underline{x}^\epsilon) = \begin{pmatrix} \underset{\approx}{\sigma}^0(\underset{\sim}{x}^\epsilon) & \varepsilon^{-1}\underset{\sim}{g}^0 x_3^\epsilon \\ \varepsilon^{-1}(\underset{\sim}{g}^0)^T x_3^\epsilon & (1-\varepsilon^{-2}Q_2(x_3^\epsilon))\sigma_{33}^0(\underset{\sim}{x}^\epsilon) + g_3^1 \end{pmatrix}$$
$$+ \begin{pmatrix} \underset{\approx}{\sigma}^1(\underset{\sim}{x}^\epsilon)x_3^\epsilon & \underset{\sim}{\sigma}^0(\underset{\sim}{x}^\epsilon)(1-\varepsilon^{-2}Q_2(x_3^\epsilon)) + \underset{\sim}{g}^1 \\ (\underset{\sim}{\sigma}^0)^T(\underset{\sim}{x}^\epsilon)(1-\varepsilon^{-2}Q_2(x_3^\epsilon)) + (\underset{\sim}{g}^1)^T & \sigma_{33}^1(\underset{\sim}{x}^\epsilon)(x_3^\epsilon - \varepsilon^{-2}Q_3(x_3^\epsilon)) + \varepsilon^{-1}g_3^0 x_3^\epsilon \end{pmatrix}. \tag{6.4.7}$$

The equations defining the first two displacement components for the pure stretching case are

$$-\varepsilon \underset{\sim}{\operatorname{div}} \underset{\approxeq}{A}^{-1} \underset{\approx}{e}(\underset{\sim}{\eta}) = \frac{\varepsilon}{2}\underset{\sim}{f}^0 + \underset{\sim}{g}^0 + \varepsilon\frac{\nu}{1-\nu}\underset{\sim}{\nabla}(\frac{\varepsilon}{3}\operatorname{div}\underset{\sim}{g}^0 + \frac{1}{2}f_3^1 + g_3^1) \qquad \text{in } \Omega,$$
$$\underset{\sim}{\eta} = 0 \qquad \text{on } \partial\Omega.$$

Next we can compute the other unknowns by substitution.

$$\underset{\approx}{\sigma}^0 = \underset{\approxeq}{A}^{-1}\underset{\approx}{e}(\underset{\sim}{\eta}) + \frac{\nu}{1-\nu}\left(\frac{\varepsilon}{3}\operatorname{div}\underset{\sim}{g}^0 + \frac{1}{2}f_3^1 + g_3^1\right)\underset{\approx}{\delta},$$
$$\sigma_{33}^0 = \frac{\varepsilon}{3}\operatorname{div}\underset{\sim}{g}^0 + \frac{1}{2}f_3^1,$$
$$\omega = \frac{1}{E}[-\nu\operatorname{tr}(\underset{\approx}{\sigma}^0) + \frac{6}{5}\sigma_{33}^0 + g_3^1].$$

For pure bending,

$$-\frac{\varepsilon^3}{3}\underset{\sim}{\operatorname{div}}\underset{\approxeq}{A}^{-1}\underset{\approx}{e}(\underset{\sim}{\phi}) + \frac{5}{6}\varepsilon\lambda(\underset{\sim}{\phi} - \underset{\sim}{\nabla}\omega) = -\varepsilon(\frac{1}{2}\underset{\sim}{f}^1 + \frac{5}{6}\underset{\sim}{g}^1)$$
$$-\frac{\nu}{15(1-\nu)}\varepsilon^3\underset{\sim}{\nabla}(\operatorname{div}\underset{\sim}{g}^1 + 6\varepsilon^{-1}g_3^0 + \frac{1}{2}f_3^0 + \frac{5}{2}\varepsilon^{-2}f_3^2) \qquad \text{in } \Omega,$$
$$\frac{5}{6}\varepsilon\lambda\operatorname{div}(\underset{\sim}{\phi} - \underset{\sim}{\nabla}\omega) = \frac{\varepsilon}{2}f_3^0 + g_3^0 + \frac{\varepsilon}{6}\operatorname{div}\underset{\sim}{g}^1 \qquad \text{in } \Omega,$$
$$\underset{\sim}{\phi} = 0\, quad \omega = 0 \qquad \text{on } \partial\Omega.$$

This is again a Reissner–Mindlin model, with shear correction factor of 5/6. Additional moments of the load are taken into account. Compare with the $\mathrm{HR}_1(1)$ and $\mathrm{HR}_2'(1)$ models.

For the other unknowns,

$$\underset{\approx}{\sigma}^1 = -A^{-1}\underset{\approx}{e}(\underset{\sim}{\phi}) + \frac{\nu}{5(1-\nu)}(\operatorname{div}\underset{\sim}{g}^1 + 6\varepsilon^{-1}g_3^0 + \frac{1}{2}f_3^0 + \frac{5}{2}\varepsilon^{-2}f_3^2)\underset{\approx}{\delta}, \tag{6.4.8}$$

$$\underset{\sim}{\sigma}^0 = \frac{5}{6}\lambda(-\underset{\sim}{\phi} + \underset{\sim}{\nabla}\omega) - \frac{5}{6}\underset{\sim}{g}^1, \quad \sigma_{33}^1 = \frac{1}{5}(\operatorname{div}\underset{\sim}{g}^1 + \varepsilon^{-1}g_3^0 + \frac{1}{2}f_3^0 + \frac{5}{2}\varepsilon^{-2}f_3^2), \tag{6.4.9}$$

$$\omega_2 = -\frac{\nu}{3E}\operatorname{tr}(\underset{\approx}{\sigma}^1) + \frac{10}{21E}\sigma_{33}^1 + \frac{\varepsilon^{-1}}{3E}g_3^0. \tag{6.4.10}$$

CHAPTER 7

Asymptotic Expansion for a Reissner–Mindlin Model

We consider here a simple Reissner–Mindlin model for pure bending of elastic plates. We then assume that the volume plate load $\underline{f}^\epsilon$ and the traction load $\underline{g}^\epsilon$ satisfy

$$\underset{\sim}{f}^\epsilon \text{ odd in } x_3, \quad f_3^\epsilon \text{ even in } x_3,$$
$$\underset{\sim}{g}^\epsilon(\underset{\sim}{x}) = \underset{\sim}{g}^{+,\varepsilon}(\underset{\sim}{x}, \varepsilon) = -\underset{\sim}{g}^{-,\varepsilon}(\underset{\sim}{x}, -\varepsilon), \quad g_3^\epsilon(\underset{\sim}{x}) = g_3^{+,\varepsilon}(\underset{\sim}{x}, \varepsilon) = g_3^{-,\varepsilon}(\underset{\sim}{x}, -\varepsilon).$$

In this case, it is easy to check that $\underset{\sim}{u}^\epsilon$ is odd in x_3 and u_3^ϵ is even in x_3.

7.1. Asymptotic Expansions

We follow the paper by D. Arnold and R. Falk [**3**].[1] Consider a Reissner–Mindlin model given by

$$-\underset{\sim}{L}(\underset{\sim}{\phi}) + \frac{1}{\alpha}\varepsilon^{-2}(\underset{\sim}{\phi} - \underset{\sim}{\nabla}\, w) = \underset{\sim}{F} \qquad \text{in } \Omega,$$
$$\frac{1}{\alpha}\varepsilon^{-2} \operatorname{div}(\underset{\sim}{\phi} - \underset{\sim}{\nabla}\, w) = G \qquad \text{in } \Omega,$$
$$\underset{\sim}{\phi} = 0, \quad \omega = 0 \qquad \text{on } \partial\Omega.$$

in Ω, where $\underset{\sim}{L}(\underset{\sim}{\phi}) = \frac{2}{3} \underset{\sim}{\operatorname{div}} \underset{\approx}{C}^* \underset{\approx}{e}(\underset{\sim}{\phi})$, with $\underset{\approx}{C}^* \underset{\approx}{e}(\underset{\sim}{\phi}) = 2\mu\, \underset{\approx}{e}(\underset{\sim}{\phi}) + \lambda^* \operatorname{div} \underset{\sim}{\phi}\, \underset{\approx}{\delta}$, $\lambda^* = 2\mu\lambda/(2\mu + \lambda)$ and $\alpha = 3/(5\mu)$. Here, λ and μ are the Lamé coefficients. Also,

$$\underset{\sim}{F} = -2\varepsilon^{-2}\left(\underset{\sim}{g}^\epsilon + \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \underset{\sim}{f}^\epsilon(\underset{\sim}{x}, x_3)x_3\, dx_3\right), \qquad G = \varepsilon^{-3}\left(2g_3^\epsilon + \int_{-\varepsilon}^{\varepsilon} f_3^\epsilon(\underset{\sim}{x}, x_3)\, dx_3\right).$$

It is reasonable to expect an asymptotic expansion of the solution of the form

$$\underset{\sim}{\phi} \sim \underset{\sim}{\phi}^I + \underset{\sim}{\phi}^B, \quad w \sim w^I + w^B,$$

where $\underset{\sim}{\phi}^I$ is a series of the form $\sum_i \varepsilon^i \underset{\sim}{\phi}_i$, and $\underset{\sim}{\phi}^B$ represents a boundary layer expansion of the for $\sum_i \varepsilon^i \underset{\sim}{\Phi}_i$, where $\underset{\sim}{\Phi}_i$ depends on ε through the variable ρ/ε where ρ is the distance to $\partial\Omega$, and $\underset{\sim}{\Phi}_i$ behaves like a decaying exponential in ρ, and similarly for w^B. Now, using the identity $\operatorname{div} \underset{\sim}{L}(\underset{\sim}{\phi}) = D\,\Delta \operatorname{div} \underset{\sim}{\phi}$, where $D = 8\mu(\mu+\lambda)/[3(2\mu+\lambda)]$, we obtain

$$D\,\Delta \operatorname{div} \underset{\sim}{\phi} = G - \operatorname{div} \underset{\sim}{F}, \tag{7.1.1}$$
$$\underset{\sim}{\phi} = \underset{\sim}{\nabla} w + \alpha\varepsilon^2(\underset{\sim}{L}\underset{\sim}{\phi} + \underset{\sim}{F}), \tag{7.1.2}$$
$$\operatorname{div} \underset{\sim}{\phi} = \Delta\, w + \alpha\varepsilon^2 G, \tag{7.1.3}$$
$$D\,\Delta^2\, w = G - \operatorname{div} \underset{\sim}{F} - \alpha\varepsilon^2 D\,\Delta\, G. \tag{7.1.4}$$

[1]Actually, I believe that great part of the text itself came from a course on plate theory by Arnold.

Thus w satisfies and a biharmonic equation, and so w does not have a boundary layer. Similarly, $\operatorname{div} \underset{\sim}{\phi}$ satisfies a Poisson equation, which would suggest that the divergence annhilates the boundary layer of $\underset{\sim}{\phi}$, and $\underset{\sim}{\phi}^B = \underset{\sim}{\operatorname{curl}}\, p^B$ for some p^B. Recall that

$$\underset{\sim}{\operatorname{curl}}\, v = \begin{pmatrix} -\partial v/\partial x_2 \\ \partial v/\partial x_1 \end{pmatrix}$$

To construct an asymptotic expansion of this form, we will ask that the pair $(\underset{\sim}{\phi}^I, w)$ satisfy the Reissner–Mindlin system and the pair $(\underset{\sim}{\operatorname{curl}}\, p^B, 0)$ satisfy the homogeneous Reissner–Mindlin system, and that their sum satisfy the boundary conditions. From this we shall deduce recipes for the series representing ϕ^I, $w = w^I$, and p^B.

Inserting $w \sim \sum_i \varepsilon^i w_i$ into (7.1.4) gives

$$D\,\Delta^2 w_i = \delta_{i0}(G - \operatorname{div} \underset{\sim}{F}) - \delta_{i2}\alpha D\,\Delta\, G.$$

Inserting $\phi^I \sim \sum_i \varepsilon^i \underset{\sim}{\phi}_i$ into (7.1.2) and using the identity $\underset{\sim}{L}\underset{\sim}{\nabla} v = D\underset{\sim}{\nabla}\,\Delta\, v$ we get

$$\begin{aligned}
\underset{\sim}{\phi}_1 &= \underset{\sim}{\nabla}\, w_1,\\
\underset{\sim}{\phi}_3 &= \underset{\sim}{\nabla}\, w_3 + \alpha\underset{\sim}{L}\underset{\sim}{\phi}_1 = \underset{\sim}{\nabla}(w_3 + \alpha D\,\Delta\, w_1),\\
\underset{\sim}{\phi}_5 &= \underset{\sim}{\nabla}\, w_5 + \alpha\underset{\sim}{L}\underset{\sim}{\phi}_3 = \underset{\sim}{\nabla}(w_5 + \alpha D\,\Delta\, w_3 + \alpha^2 D^2\,\Delta^2\, w_1),\\
\underset{\sim}{\phi}_7 &= \underset{\sim}{\nabla}\, w_7 + \alpha\underset{\sim}{L}\underset{\sim}{\phi}_5 = \underset{\sim}{\nabla}(w_7 + \alpha D\,\Delta\, w_5 + \alpha^2 D^2\,\Delta^2\, w_3 + \alpha^3 D^3\,\Delta^3\, w_1),
\end{aligned}$$

etc. In view of the biharmonic equations satisfied by the w_i, these simplify to

$$\begin{aligned}
\underset{\sim}{\phi}_1 &= \underset{\sim}{\nabla}\, w_1,\\
\underset{\sim}{\phi}_{2i+1} &= \underset{\sim}{\nabla}(w_{2i+1} + \alpha D\,\Delta\, w_{2i-1}), \quad i = 1, 2, \ldots.
\end{aligned}$$

For the even indexed terms we get

$$\begin{aligned}
\underset{\sim}{\phi}_0 &= \underset{\sim}{\nabla}\, w_0,\\
\underset{\sim}{\phi}_{2i} &= \underset{\sim}{\nabla}(w_{2i} + \alpha D\,\Delta\, w_{2i-2} + \cdots + \alpha^i D^i\,\Delta^i\, w_0) + \alpha^i \underset{\sim}{L}^{i-1}\underset{\sim}{F}, \quad i = 1, 2, \ldots.
\end{aligned}$$

Using the biharmonic equations to simplify this gives

$$\begin{aligned}
\underset{\sim}{\phi}_0 &= \underset{\sim}{\nabla}\, w_0,\\
\underset{\sim}{\phi}_2 &= \underset{\sim}{\nabla}(w_2 + \alpha D\,\Delta\, w_0) + \alpha\underset{\sim}{F},\\
\underset{\sim}{\phi}_4 &= \underset{\sim}{\nabla}[w_4 + \alpha D\,\Delta\, w_2 + \alpha^2 D(G - \operatorname{div}\underset{\sim}{F})] + \alpha^2\underset{\sim}{L}\underset{\sim}{F},\\
\underset{\sim}{\phi}_{2i} &= \underset{\sim}{\nabla}(w_{2i} + \alpha D\,\Delta\, w_{2i-2} - \alpha^i D^{i-1}\,\Delta^{i-1} \operatorname{div}\underset{\sim}{F}) + \alpha^i\underset{\sim}{L}^{i-1}\underset{\sim}{F}, \quad i = 3, 4, \ldots.
\end{aligned}$$

We have thus derived PDE's for the w_i and formulas for the $\underset{\sim}{\phi}_i$ in terms of the w_i. Next we determine differential equations for the boundary layer functions. When we put $\underset{\sim}{\phi} = \underset{\sim}{\operatorname{curl}}\, p^B$, $w = 0$ into the homogeneous Reissner–Mindlin equations, and use the identity $\underset{\sim}{L}\,\underset{\sim}{\operatorname{curl}}\, p = (3\alpha)^{-1}\,\underset{\sim}{\operatorname{curl}}\,\Delta\, p$ we get

$$\underset{\sim}{\operatorname{curl}}(-\frac{1}{3}\varepsilon^2\,\Delta\, p^B + p^B) = 0,$$

or

$$-\frac{1}{3}\varepsilon^2\,\Delta\, p^B + p^B = \text{constant}.$$

Since we want p^B to decay exponentially, the constant must be zero.

Inserting $p^B \sim \sum \varepsilon^i P_i$ for P and gathering like powers of ε, using boundary fitted coordinate (ρ, θ) and $\hat{\rho} = \varepsilon^{-1}\rho$ gives

$$\Delta\, p^B = \sum_{i=0}^{\infty} \varepsilon^{i-2} \left[\frac{\partial^2 P_i}{\partial \hat{\rho}^2} + \sum_{j=0}^{\infty} \hat{\rho}^j \left(a_1^j \frac{\partial P_{i-j-1}}{\partial \hat{\rho}} + a_2^j \frac{\partial^2 P_{i-j-2}}{\partial \theta^2} + a_3^j \frac{\partial P_{i-j-2}}{\partial \theta} \right) \right],$$

where $P_k = 0$ for $k < 0$. Thus the differential equation for p^B leads to

$$-\frac{1}{3} \frac{\partial^2 P_i}{\partial \hat{\rho}^2} + P_i = F_i(\hat{\rho}, \theta),$$

$$F_i(\hat{\rho}, \theta) = \frac{1}{3} \sum_{j=0}^{i-1} \rho^j \left(a_1^j \frac{\partial P_{i-j-1}}{\partial \hat{\rho}} + a_2^j \frac{\partial^2 P_{i-j-2}}{\partial \theta^2} + a_3^j \frac{\partial P_{i-j-2}}{\partial \theta} \right).$$

Note that these are simple ODEs for the P_i. They admit exponentially increasing and decreasing solutions. We shall only allow the latter, i.e., we insist that $\lim_{\hat{\rho}\to\infty} P_i(\hat{\rho}, \theta) = 0$.

Once the P_i are known, we get

$$\underset{\sim}{\operatorname{curl}}\, P_i = \frac{\partial P_i}{\partial \hat{\rho}} \underset{\sim}{\operatorname{curl}}\, \hat{\rho} + \frac{\partial P_i}{\partial \theta} \underset{\sim}{\operatorname{curl}}\, \theta.$$

But

$$\underset{\sim}{\operatorname{curl}}\, \hat{\rho} = \varepsilon^{-1} \underset{\sim}{\operatorname{curl}}\, \rho = -\varepsilon^{-1} \underset{\sim}{s},$$

and

$$\underset{\sim}{\operatorname{curl}}\, \theta = -\sigma(\varepsilon \hat{\rho}, \theta) \underset{\sim}{n} = -\sum_j \varepsilon^j [\hat{\rho}\kappa(\theta)]^j \underset{\sim}{n}.$$

Thus

$$\underset{\sim}{\phi}^B = \underset{\sim}{\operatorname{curl}}\, p^B = -\sum_i \varepsilon^i \left(\frac{\partial P_{i+1}}{\partial \hat{\rho}} \underset{\sim}{s} + \sum_{j=0}^{i} [\hat{\rho}\kappa(\theta)]^j \frac{\partial P_{i-j}}{\partial \theta} \underset{\sim}{n} \right),$$

so $\underset{\sim}{\phi}^B = \sum_i \varepsilon^i \underset{\sim}{\Phi}_i$, with

$$\underset{\sim}{\Phi}_i = -\left(\frac{\partial P_{i+1}}{\partial \hat{\rho}} \underset{\sim}{s} + \sum_{j=0}^{i} [\hat{\rho}\kappa(\theta)]^j \frac{\partial P_{i-j}}{\partial \theta} \underset{\sim}{n} \right).$$

7.1.1. Boundary conditions. Now we turn to the boundary conditions. Since $w \sim \sum \varepsilon^i w_i$, we impose

(7.1.5) $$w_i = 0 \text{ on } \partial\Omega.$$

Now

$$\underset{\sim}{\phi} = \sum \varepsilon^i \underset{\sim}{\phi}_i - \sum_i \varepsilon^i \left[\frac{\partial P_{i+1}}{\partial \hat{\rho}} \underset{\sim}{s} + \sum_{j=0}^{i} [\hat{\rho}\kappa(\theta)]^j \frac{\partial P_{i-j}}{\partial \theta} \underset{\sim}{n} \right],$$

so,

(7.1.6) $$\underset{\sim}{\phi}_i = \frac{\partial P_{i+1}}{\partial \hat{\rho}} \underset{\sim}{s} + \frac{\partial P_i}{\partial \theta} \underset{\sim}{n} \qquad \text{on } \partial\Omega.$$

Now if we take the tangential component of (7.1.5) for $i = -1$ we see that

$$\frac{\partial P_0}{\partial \hat\rho} = 0 \text{ at } \hat\rho = 0.$$

Together with differential equation

$$-\frac{2}{5}\frac{\partial^2 P_0}{\partial \hat\rho^2} + P_0 = 0,$$

and the condition $\lim_{\hat\rho\to\infty} P_i = 0$, we get $P_0 \equiv 0$. If we then take the normal component on (7.1.6) for $i = 0$ we get

$$\underset{\sim}{\phi}_0 \cdot \underset{\sim}{n} = 0,$$

or

$$\frac{\partial w_0}{\partial n} = 0.$$

We thus see that w_0 satisfies the biharmonic equation

$$D\,\Delta^2\, w_0 = G - \operatorname{div} \underset{\sim}{F}$$

together with homogeneous Dirichlet boundary conditions. So, ω_0 and $\phi_0 = \underset{\sim}{\nabla} w_0$ are determined completely. Actually, from (6.2.1), we conclude that w_0 solves the biharmonic model. Hence the Reissner–Mindlin model is consistent. We also have $\underset{\sim}{\phi}_0 \cdot \underset{\sim}{s} = 0$ on $\partial\Omega$, so the tangential component on (7.1.6), case $i = 0$, gives

$$\frac{\partial P_1}{\partial \hat\rho} = 0 \text{ at } \hat\rho = 0.$$

This leads again to $P_1 \equiv 0$. Then we find that w_1 satisfies a homogeneous biharmonic problem with homogeneous Dirichlet boundary conditions, so w_1 and $\underset{\sim}{\phi}_1$ vanish identically. This in turn implies that P_2 satisfies homogeneous Neumann boundary conditions, so it too vanishes. Considering next the normal component of (7.1.6) for $i = 2$ gives the vanishing of $\underset{\sim}{\phi}_2 \cdot n$, or, in view of the relation of $\underset{\sim}{\phi}_2$ and w_2,

$$\frac{\partial w_2}{\partial n} = -\alpha D \frac{\partial\,\Delta\, w_0}{\partial n} - \alpha \underset{\sim}{F} \cdot n.$$

Together with the inhomogeneous biharmonic equation for w_2 and the boundary condition (7.1.5), we can then determine w_2 and ϕ_2.

This time $\underset{\sim}{\phi}_2 \cdot \underset{\sim}{s}$ does not vanish, and so P_3 will not, in general vanish:

$$P_3(\hat\rho, \theta) = c(\theta) e^{-\sqrt{1/3}\hat\rho},$$

where $c(\theta)$ is the trace of $-\sqrt{3}\underset{\sim}{\phi}_2 \cdot \underset{\sim}{s}$ on the boundary. Note that in general, we have $\underset{\sim}{\phi}_i = 0$ for $i = 0$ and 1, but not for $i = 2$.

We continue as follows: knowing P_3, econdition (7.1.6) gives us an inhomogeneous boundary condition for $\underset{\sim}{\phi}_3 \cdot \underset{\sim}{n}$, and hence for $\partial w_3/\partial n$. Then we can determine w_3 and $\underset{\sim}{\phi}_3$. This gives us the Neumann boundary data for P_4, which can then be determined. Etc.

In this way all the functions w_i, $\underset{\sim}{\phi}_i$, P_i, and $\underset{\sim}{\phi}_i$ can be computed. The functions w_1, ϕ_1, $\underset{\sim}{\phi}_0$, and $\underset{\sim}{\phi}_1$ vanish. From the formula for it, we also see that $\underset{\sim}{\phi}_2 \cdot \underset{\sim}{n}$ vanishes. No other terms of the expansion vanish in general.

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