# A MODEL FOR AN ARMA PROCESS SPLIT IN SUB-BANDS

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**ABSTRACT**

This work proposes a model, originated from the power spectral density concept, for the sub-band processes obtained by analysis of an ARMA process by a decimating filter bank. As an example, the model for an AR process analyzed in octaves by a tree-structured FIR filter bank is derived in a recursive way, and some interpretations are given.

## 1. INTRODUCTION

Multi-rate digital processing [1] is today a well-established topic, extensively applied in communications, image and audio industry and other areas, for signal coding, adaptive or statistical processing etc.

A special class of discrete random processes [2] are those obtained by passing white-noise through a linear digital filter—called Moving-Average (MA) for an Finite-Impulse-Response (FIR) filter, Autoregressive (AR) for an all-pole filter or Autoregressive-Moving-Average (ARMA) in the general case.

Many applications in speech and audio employ these processes as models. This is particularly true for audio restoration [3], where ARMA and AR processes play an important role in description of signals. In this context, model-based processing in sub-bands would profit from a similar description of the sub-band signals, if it was possible. This is the motivation behind this work.

This work uses polyphase decomposition of systems [1] and power spectral density of random signals [2] to get a simple model that describes sub-band signals resulting from the analysis of an ARMA process by a decimating filter bank. The special case of an AR process passing through an octave-band filter bank is also detailed, leading to some useful interpretations.

Applications of these results to audio modeling and restoration will appear in future works [4]. Practical simulations yield good results.

The paper is organized as follows. Section 2 covers the general case of the ARMA process divided in $N$ bands, Section 3 derives and discusses the case of the AR process divided in octaves and Section 4 summarizes the results.

## 2. GENERAL MODEL: ARMA PROCESS IN $N$ BANDS

The purpose of this Section is to derive a simple low-rate model for the signals that results from the analysis of an ARMA process in sub-bands.

First, consider the $N$-band multi-rate system shown in Fig. 1, which analyzes the ARMA process $s(k)$ defined by

$$s(k) = \sum_{i=1}^{I} a(i)s(k-i) + \sum_{j=0}^{J} b(j)e(k-j).$$

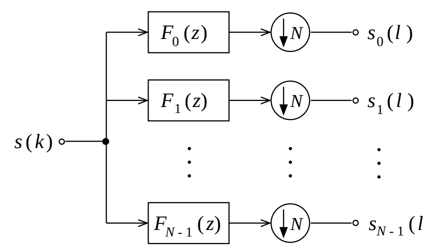


**Figure 1:** ARMA Process analyzed by an $N$-band decimating filter bank.

To obtain the model, we can follow the steps depicted in Fig. 2.

(a) Consider the subsystem corresponding to a given sub-band $n$.

(b) Represent $s(k)$ explicitly as the adequate IIR filter fed by a white-noise process $e(k)$. Group this filter and the analysis filter $F_n(z)$ into $\hat{F}_n(z)$.

(c) Obtain a polyphase representation for $\hat{F}_n(z)$.

(d) Place the decimator before the adder.

(e) For each polyphase $p$, place the decimator before the filter $\hat{F}_{np}(z)$, and represent the respective component of $e(k)$ explicitly as $e_p(l)$.

(f) Being $\sigma_e^2$ the variance of $e(k)$, it is easy to conclude that $e_p(l)$, for $p = 0, 1, \ldots, N-1$, are mutually uncorrelated white-noise processes with variances

$$E[e_p^2(l)] = E[e^2(Nl+p)] = E[e^2(k)] = \sigma_e^2.$$

Since the signals $s_{np}(l)$, for $p = 0, 1, \ldots, N-1$, are colored versions of $e_p(l)$, respectively, they are also mutually uncorrelated. Then, the autocorrelation of $s_n(l)$ can be written without cross-terms as

$$R_{s_n s_n}(\Delta) = \sum_{p=0}^{N-1} R_{s_{np} s_{np}}(\Delta),$$

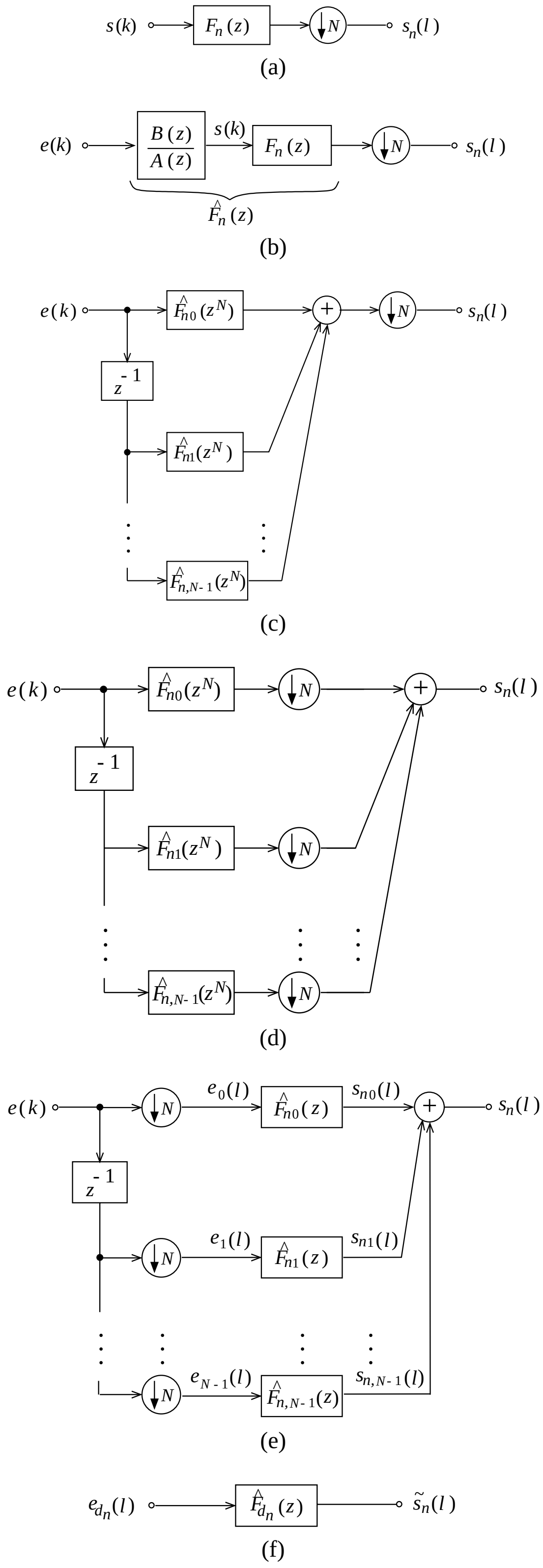


**Figure 2:** Generation of $s_n(l)$: from the multi-rate representation to a low-rate model.

leading to the power spectral density

$$\mathcal{S}_{s_n s_n}(z) = \sum_{p=0}^{N-1} \mathcal{S}_{s_{np} s_{np}}(z) =$$

$$= \sum_{p=0}^{N-1} \hat{F}_{np}(z) \hat{F}_{np}(z^{-1}) \sigma_e^2 = \mathcal{S}_{s_n s_n}(-z).$$

By applying factorization to the expression above,

$$\mathcal{S}_{s_n s_n}(z) \triangleq \hat{F}_{d_n}(z) \hat{F}_{d_n}(z^{-1}) \sigma_e^2,$$

we can get a representation of $\tilde{s}_n(l)$—equivalent to $s_n(l)$ in terms of its power spectral density—as an ARMA process defined by the minimum-phase filter $\hat{F}_{d_n}(z)$ fed by a white-noise process with variance $\sigma_e^2$, denoted as $e_{d_n}(l)$.

Now, a complete equivalent model would consist of $N$ similar sub-systems in parallel. However, this model does not express the mutual correlation between different sub-bands. In the following, we obtain such a description.

Consider 2 distinct sub-bands $n_1$ and $n_2$, each one described as in Fig. 2(e). The cross-correlation between their signals is given by

$$R_{s_{n_1} s_{n_2}}(\Delta) = \sum_{p=0}^{N-1} E[s_{n_1 p}(l) s_{n_2 p}(l+\Delta)]$$

$$= \sum_{p=0}^{N-1} E[(\hat{f}_{n_1 p} * e_p)(l)(\hat{f}_{n_2 p} * e_p)(l+\Delta)],$$

resulting in the cross-power spectral density

$$\mathcal{S}_{s_{n_1} s_{n_2}}(z) = \left[ \sum_{p=0}^{N-1} \hat{F}_{n_1 p}(z) \hat{F}_{n_2 p}(z^{-1}) \right] \sigma_e^2.$$

## 3. MODEL FOR AN AR PROCESS IN OCTAVES

In this section, we apply the results from Section 2 to a usual situation in audio applications: an AR process split in octaves. This special case leads to some useful interpretations.

### 3.1. Model derivation

Start from the order-$I$ autoregressive process

$$s(k) = \sum_{i=1}^{I} a(i) s(k-i) + e(k),$$

where $e(k)$ is white-noise with variance $\sigma_e^2$.

Consider the binary-tree structured FIR filter bank depicted in Fig. 3, used to analyze $s(k)$ in octaves.

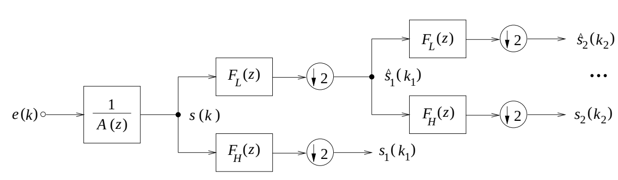


**Figure 3:** Analysis of an AR process by an octave-band decimating filter bank.

In each stage, the analysis filters are the high-pass $F_H(z)$ and the low-pass $F_L(z)$, both of order $Q$. After $m$ stages, there will be $m+1$ sub-bands, from $s_1(k_1)$ to $s_m(k_m)$ besides $\hat{s}_m(k_m)$.

Now, we obtain the polyphase representation for each sub-band. Concerning the first stage, we have

$$\frac{F_H(z)}{A(z)} = \frac{F_H(z)A(-z)}{A(z)A(-z)} \triangleq \frac{B_H(z)}{A_1(z^2)} \triangleq \frac{B_{10}(z^2) + z^{-1}B_{11}(z^2)}{A_1(z^2)}$$

and

$$\frac{F_L(z)}{A(z)} = \frac{F_L(z)}{A(z)}\frac{A(-z)}{A(-z)} \triangleq \frac{B_L(z)}{A_1(z^2)} \triangleq \frac{\hat{B}_{10}(z^2) + z^{-1}\hat{B}_{11}(z^2)}{A_1(z^2)}.$$

These expressions, along with the transformations shown in Fig. 4, describe the polyphase model that leads to $s_1(k_1)$ and $\hat{s}_1(k_1)$. Note that the generation of $s_1(k_1)$ has been omitted from the figure for reasons of space.

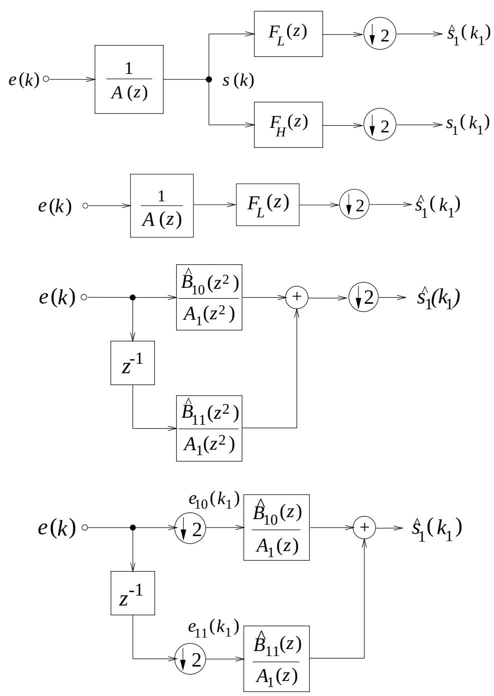


**Figure 4:** Generation of $\hat{s}_1(k_1)$—polyphase representation.

Starting with $\hat{s}_1(k_1)$, we proceed to the second stage,

$$\begin{aligned}
F_H(z)\frac{\hat{B}_{10}(z)}{A_1(z)} &= F_H(z)\frac{\hat{B}_{10}(z)}{A_1(z)}\frac{A_1(-z)}{A_1(-z)} \triangleq \\
&\triangleq \frac{B_{10H}(z)}{A_2(z^2)} \triangleq \frac{B_{20}(z^2)+z^{-1}B_{22}(z^2)}{A_2(z^2)}, \\
F_H(z)\frac{\hat{B}_{11}(z)}{A_1(z)} &= F_H(z)\frac{\hat{B}_{11}(z)}{A_1(z)}\frac{A_1(-z)}{A_1(-z)} \triangleq \\
&\triangleq \frac{B_{11H}(z)}{A_2(z^2)} \triangleq \frac{B_{21}(z^2)+z^{-1}B_{23}(z^2)}{A_2(z^2)}, \\
F_L(z)\frac{\hat{B}_{10}(z)}{A_1(z)} &= F_L(z)\frac{\hat{B}_{10}(z)}{A_1(z)}\frac{A_1(-z)}{A_1(-z)} \triangleq \\
&\triangleq \frac{B_{10L}(z)}{A_2(z^2)} \triangleq \frac{\hat{B}_{20}(z^2)+z^{-1}\hat{B}_{22}(z^2)}{A_2(z^2)}
\end{aligned}$$

and

$$\begin{aligned}
F_L(z)\frac{\hat{B}_{11}(z)}{A_1(z)} &= F_L(z)\frac{\hat{B}_{11}(z)}{A_1(z)}\frac{A_1(-z)}{A_1(-z)} \triangleq \\
&\triangleq \frac{B_{11L}(z)}{A_2(z^2)} \triangleq \frac{\hat{B}_{21}(z^2)+z^{-1}\hat{B}_{23}(z^2)}{A_2(z^2)}.
\end{aligned}$$

These expressions, along with the transformations shown in Fig. 5, describe the polyphase model that leads to $s_2(k_2)$ and $\hat{s}_2(k_2)$. Note that the generation of $s_2(k_2)$ has been omitted from the figure for reasons of space.

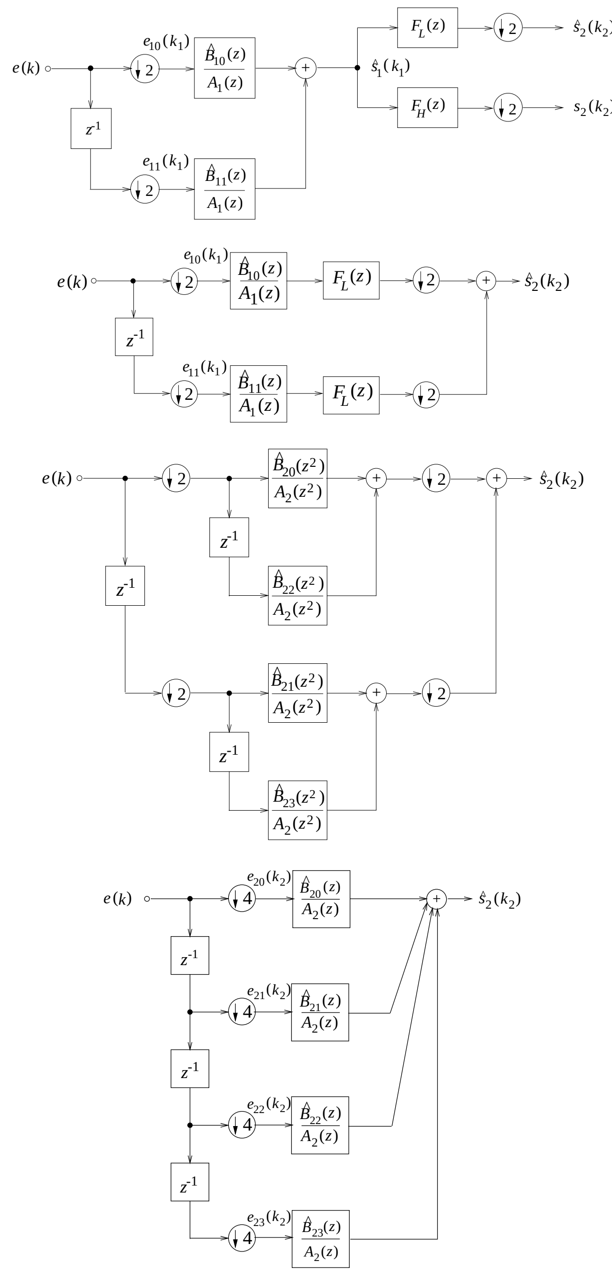


**Figure 5:** Generation of $\hat{s}_2(k_2)$—polyphase representation.

After a general $m$-th stage, one can obtain the polyphase representation of the system that generates signals $s_m(k_m)$ and $\hat{s}_m(k_m)$, shown in Fig. 6.

Referring to the general sub-band signal $s_m$, the same reasoning

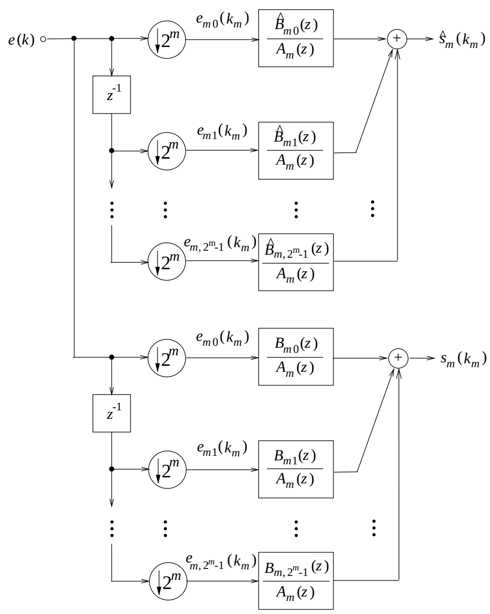


**Figure 6:** Generation of $s_m(k_m)$ and $\hat{s}_m(k_m)$—polyphase representation.

applied in Section 2 leads to the power spectral density

$$\mathcal{S}_{s_m s_m}(z) = \frac{\sum_{p=0}^{2^m-1} B_{mp}(z) B_{mp}(z^{-1})}{A_m(z) A_m(z^{-1})} \sigma_e^2 \triangleq$$

$$\triangleq \frac{B_m(z)}{A_m(z)} \frac{B_m(z^{-1})}{A_m(z^{-1})} \sigma_e^2.$$

In conclusion, as illustrated in Fig. 7, we described $\tilde{s}_m(k_m)$—equivalent to $s_m(k_m)$ in terms of its power spectral density—as an ARMA process defined by the minimum-phase filter

$$\mathcal{M}_m(z) \triangleq \frac{B_m(z)}{A_m(z)},$$

fed by a white-noise process, denoted as $e_{d_m}(k_m)$, with variance $\sigma_e^2$.

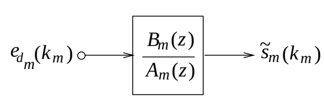


**Figure 7:** Model for the signal in sub-band $m$.

Completing the model, the cross-power spectral density between same rate sub-bands can also be written as

$$\mathcal{S}_{\hat{s}_m s_m}(z) = \frac{\left[\sum_{p=0}^{2^m-1} \hat{B}_{mp}(z) B_{mp}(z^{-1})\right]}{A_m(z) A_m(z^{-1})} \sigma_e^2.$$

### 3.2. Comments

We can draw some conclusions and speculations after the model filters of individual sub-bands.

First, it is possible to deduce that the numerator and denominator orders of $\mathcal{M}_m(z)$ are $\begin{cases} O_{\text{num}}(m+1) = \lceil \frac{O_{\text{num}}(m)+I}{2} \rceil, \ m \geq 1 \\ O_{\text{num}}(1) = \lceil \frac{Q+I}{2} \rceil \end{cases}$ and $O_{\text{den}}(m) = I$, respectively, except for possible canceled terms.

Furthermore, the location of the model filter poles after each stage can be exactly predicted. From stage $m$ to stage $m+1$, the denominator of the model filter transfer function is calculated by

$$A_{m+1}(z^2) = A_m(z) A_m(-z).$$

If we suppose, e.g.,

$$A_m(z) = 1 - 2r\cos\theta z^{-1} + r^2 z^{-2},$$

it can be shown that

$$A_{m+1}(z) = 1 - 2r^2 \cos(2\theta) z^{-1} + (r^2)^2 z^{-2}.$$

This can be generalized to any number of poles and means that each stage squares the value of the model filter poles; so, each complex natural mode has its frequency doubled and its associated selectivity reduced.

Mapping the model filter zeros is a more difficult task, as it results from successive compositions of modified versions of the original model denominator $A(z)$ with the analysis filters. What can be said is that the increasing selectivity resulting from the successive filtering stages should gradually "eliminate" the effect of some modes; another way to visualize this fact is consider that there are zeros canceling poles in the model transfer function. As a consequence, one expects the model orders to be reduced along the stages. This is an interesting issue, because it allows the use of lower-order approximate models for sub-bands. Simulations have confirmed this expectations [4].

## 4. SUMMARY

A low-rate model for the sub-band signals that result from passing an ARMA process through a multi-rate system was proposed. The model, equivalent in terms of the power spectral density, consists of one ARMA process per sub-band. After examination of the correspondent results for an AR process analyzed by an octave-band filter bank, some useful interpretations were drawn, which suggest the use of reduced-order sub-band models. Future works will present applications of these results in audio processing.

## 5. REFERENCES

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