Laboratório Nacional de Computação Científica
Programa de Pós-Graduação em Modelagem Computacional

The Multiscale Hybrid-Hybrid-Mixed method $\mathsf{MH^2M}$

Franklin da Conceição Barros

Petrópolis, RJ - Brasil
March of 2022

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$\mathrm{MH^2M}$

Dissertation submitted to the examining committee in partial fulfillment of the requirements for the degree of Master of Sciences in Computational Modeling.

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Advisor(s): Alexandre Loureiro Madureira and Frédéric Gerard Christian Valentin

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FRANKLIN DA CONCEIÇÃO DE BARROS

THE MULTISCALE HYBRID-HYBRID-MIXED METHOD

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Dedication

This work is dedicated to my family
Luciana Teixeira, Maria Alice, Marcos Vinícius
and Flávio Barros.

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"A mathematician is a person who can find analogies between theorems, a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories; and one can imagine that the ultimate mathematician is one who can see analogies between analogies."

(Stefan Banach)

Abstract

Many problems of practical interest in science and engineering are of the multiscale type, and many of those can be described by partial differential equations (PDEs) with oscillatory coefficients. The corresponding numerical solutions using classical methods are extremely expensive in terms of memory and CPU. Multiscale schemes such as the MsFEM and MHM methods have been developed to solve such problems, based on two-level ideas. This work proposes a numerical method, the Multiscale Hybrid-Hybrid-Mixed method (MH^2M). The starting point is a hybrid formulation of three fields: the solution in each element interior, its flux at the boundary of each element, and its trace on the mesh skeleton. Continuity of traces and fluxes are weakly imposed. Multiscale effects are incorporated into basis functions through localized Neuman problems. A series of static condensations transforms the saddle point problem into an elliptic one, posed at the interfaces. At the discrete level, this drastically reduces the size of the global system. The matrix of the associated linear system is symmetric and positive definite, and can be solved by classical iterative schemes. We prove the well-posedness of the method and establish error estimates. We also perform numerical tests to confirm the theoretical predictions and compare the method with the FEM, MsFEM and MHM schemes.

Keywords: MHM. Numerical methods. Multiscale hybrid method.

Resumo

Muitos problemas de interesse prático em ciência e engenharia são do tipo multiescala, e muitos deles são descritos por equações diferenciais parciais (EDPs) com coeficientes oscilatórios. As soluções numéricas correspondentes usando métodos clássicos são extremamente caras em termos de memória e CPU. Esquemas multiescala como os métodos MsFEM e MHM foram desenvolvidos para resolver tais problemas, baseados em ideias de dois níveis. Este trabalho propõe um método numérico, o método Multiscale Hybrid-Hybrid-Mixed (MH^2M). O ponto de partida é uma formulação híbrida de três campos: a solução no interior de cada elemento, seu traço e fluxo no esqueleto da malha. A continuidade de traços e fluxos é fracamente imposta. Efeitos multiescala são incorporados em funções de base por meio de problemas de Neuman localizados. Uma série de condensações estáticas transforma o problema do ponto de sela em um problema elíptico, colocado nas interfaces. No nível discreto, isso reduz drasticamente o tamanho do sistema global. A matriz do sistema linear associado é simétrica e definida positiva, e pode ser resolvida por esquemas iterativos clássicos. Provamos a boa colocação do método e estabelecemos estimativas de erro. Também realizamos testes numéricos para confirmar as previsões teóricas e comparar o método com os esquemas FEM, MsFEM e MHM.

Palavras-chave: MHM. Métodos numéricos. Método híbrido multiescala.

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List of abbreviations and acronyms

FEM	Finite Element Method
MsFEM	Multiscale Finite Element Method
MHM	Multiscale Hybrid-Mixed
MHHM	Multiscale Hybrid-Hybrid-Mixed
MH^2M	Multiscale Hybrid-Hybrid-Mixed
DC	Discontinuous Galerkin
LDGC	Locally Discontinuous but Globally Continuous
CPU	Central Processing Unit
DoF	Degrees of freedom

List of symbols

$\partial\Omega$	Border of the space Ω 18
$\Gamma_{\mathcal{H}_\Gamma}$	Finite element space 34
$\Lambda_{\mathcal{H}_\Lambda}$	Finite element space 34
Λ	Dual functions space 20
Ω	Open bounded domain 18
Γ	Polyhedral boundary of Ω 20
$\Vert \cdot \Vert_{m,\Omega}$	Norm 18
$\vert \cdot \vert_{m,\Omega}$	Semi-norm 18
∇	Gradient operator
$\nabla \cdot$	Divergent operator
$\mathbb{R}^d$	d-dimensional Euclidean space 18
$\mathcal{C}^n$	Space of functions n times continuously differentiable
$\mathcal{H}$	Refinement level of the mesh 20
$\mathcal{H}^*(\Omega)$	Space of the harmonic functions 31
$\mathcal{T}_\mathcal{H}$	Regular mesh 20
$\mathcal{E}_\mathcal{H}$	Mesh skeleton 20
$\mathbb{P}_n$	Space of the polynomial of degree n 20, 46
$H^1(\mathcal{T}_\mathcal{H})$	Functional space over $\mathcal{T}_\mathcal{H}$ 20
$H_0^{1/2}(\mathcal{E}_\mathcal{H})$	Functional space over $\mathcal{E}_\mathcal{H}$ 20
Λ_*	Dual space for MHM method 20
$\tilde{H}^1(\mathcal{T}_\mathcal{H})$	Space of the functionals with "zero mean" 23
Λ^0	Piecewise constant functional space 24
$\tilde{H}^{-1/2}(\partial K)$	Zero mean functional space over ∂K 24

$\tilde{\Lambda}$	Zero mean functional space over $\mathcal{E}_{\mathcal{H}}$ 24
$T\tilde{\mu}$	Harmonic extension of a function $\tilde{\mu} \in \tilde{\Lambda}$ 25
$\tilde{T}f$	Harmonic extension of a function $f \in L^2(\Omega)$ 25
G	Dirichlet to Neumann operator 26
$\lvert \cdot \rvert_{1,\mathcal{A},K}$	Semi-norm for the space $H^1(K)$ 28
$\lvert \cdot \rvert_{\frac{1}{2},\mathcal{E}_{\mathcal{H}}}$	Semi-norm for the space $H^{1/2}(\mathcal{E}_{\mathcal{H}})$ 28
$V_\xi(K)$	Space of functions 28
$V_\rho(\Omega)$	Space of functions 28
$\tilde{H}^{1/2}(\partial K)$	Zero mean functional space 30
$\lvert \cdot \rvert_\Lambda$	Semi norm for the Λ space 30
$\mathcal{H}_\Gamma$	Mesh size for the finite dimensional space $\Gamma_{\mathcal{H}_\Gamma}$ 34
$\mathcal{H}_\Lambda$	Mesh size for the finite dimensional space $\Lambda_{\mathcal{H}_\Lambda}$ 34
V'	Dual space of the space V 74
$\mathcal{R}(T)$	Range of an operator T 35
$\#(F)$	Cardinality of a set F 55

Contents

1 Introduction

Many problems of practical importance in science and engineering have multiscale characteristics. We mention the modeling of composite materials and flows in porous media as common examples. A large class of multiscale problems are described by partial differential equations with oscillatory coefficients. Coefficients can characterize the heterogeneity of a medium, as in porous media flows, or they can represent the random velocity field in a turbulent transport problem. The direct numerical solution of this class of problems, using, for example, the classical Finite Element Method (FEM), becomes difficult and with a expensive computational cost since the number of degrees of freedom of the resulting discrete system is extremely large, due to the resolution required to achieve meaningful results; moreover, it requires a lot of memory and CPU time, even with supercomputers. On the other hand, the large-scale features of the solution and the averaged effect of small scales on large scales are of primary interest.

Multiscale methods are designed to deal with problems with oscillatory coefficients. Inittialy I. Babuska and J. E. Osborn [5] introduced numerical scheme to deal with these class of problems in one dimension, called Generalized Finite Element Method. Then, this concept was expanded to higher dimension by T. Hou, X. Wu and Z. Cai [20]. Moreover, the theory of the Multiscale Finite Element method was presented by Y. Efendiev and T. Hou [12]. This method is designed to capture accurately the averaged effect of differential operators with oscillatory coefficients on the large scale solutions. Its general idea is to build finite element base functions that capture the small scale information of the leading order differential operator. Parallel implementation is possible due to independence of local problems.

Recently a new family of multiscale finite element method, called the Multiscale Hybrid-Mixed (MHM) method, was introduced by C. Harder, D. Paredes and F. Valentin to solve the Darcy equation with heterogeneous coefficients [2, 23, 6]. This method has a suitable framework to solve linear elasticity problems and diffusion-advection-reaction problems [18, 17]. Moreover, the continuity of the numerical solution is relaxed at the expense of introducing a Lagrange multiplier, while assuring the strong continuity of the dual unknow, which makes dealing with flow simulation in porous media possible.

We introduce a numerical method, called Multiscale Hybrid-Hybrid-Mixed method (MH^2M), derived from a mixed-hybrid formulation of the Three-Field Domain Decomposition method, proposed by Brezzi and Marini [9] to solve second-order elliptical problems. The MH^2M method is defined on traces space, what makes it cheaper than the MHM method, whose associated global system matrix depends on the dimension of the fluxes and piecewise constants spaces. Also, the constat term globally computed in the MHM method is now determined locally. The associated global matrix is simmetric and positive definite, wich enable us to solve the global system using the iterative method called Conjugate Gradient method [21]. The MH^{2}M provide us with a non-conformal primal variable, while conformity of the trace approximation is weakly imposed. Here, the flux is relaxed on the boundary of each element, allowing distinct refinements for each element whitout loss of mass conservation. Error estimates were presented for the Three-Field formulation by F. Brezzi and D. Marini [14], where the bubble functions ensure stability of this formulation on the interfaces. To circumvent the inf-sup condition necessary for the stability of boundary Lagrange multipliers, the Locally Discontinuous but Globally Continuous Galerkin (LDGD) method [3] is based on a stabilized hybrid formulation. The variational formulation of the LDGC method has a penalty function $\beta > 0$ from the Discontinuous Galerkin (DG) method and consists of finding a pair of solutions living in the functions and traces spaces used in the MH^{2}M method. For β sufficiently small, we can approximate the LDGC and MH^{2}M methods by taking in the MH^{2}M method the test space of fluxes composed of normal derivatives of test functions. The stability of MH^{2}M holds by taking function spaces previously satisfying compatibility conditions introduced in this work.

This work is presented as follows: in Chapter 2 we establish some notations and preliminary definitions. Then we present the MH^{2}M method in infinity dimension broken function spaces and demonstrate the well-posedness of its global elliptical problem, among other functional analysis results. In Chapter 3 we present the MH^{2}M and show that the well posedness of its discrete global problem is valid, since the discrete inf-sup condition holds. Error estimates are also presented. We study a simple example assuming consistency at the sencod level. We end this chapter by presenting some relationships between the MH^{2}M and MsFEM methods. Finally, Chapter 4 presents numerical results in order to confirm the theoretical estimates. Furthermore, it evaluates the robustness of the MH^{2}M method compared to the MsFEM and the MHM method.

2 Setting and preliminary results

In this section we set the model problem after briefly introducing some function spaces. Starting from a three-field formulation, we obtain, by means of space decompositions techniques, an elliptical problem, which characterizes the MH^{2}M method in finite-dimensional spaces. Several properties are proved in the last subsection.

We suppose that our problem is posed in an open domain $\Omega \subset \mathbb{R}^n$, with a polyhedral boundary $\partial\Omega$. In practice $n = 2$ or 3, but we present the problem in a two-dimensional setting for the sake of simplicity. In the sequel we introduce some notations about the Sobolev spaces. They are based on

$$L^2(\Omega) := \left\{ v;\ \int_\Omega |v|^2\, dx = \|v\|^2_{L^2(\Omega)} < +\infty \right\} \tag{2.1}$$

the space of square integrable functions on Ω. We then define, for an integer $m \geq 0$,

$$H^m(\Omega) := \left\{ v;\ D^\alpha v \in L^2(\Omega),\ |\alpha| \leq m \right\}, \tag{2.2}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ and

$$D^\alpha v := \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n, \tag{2.3}$$

these derivatives being taken in the sense of distributions. On this space, we shall use the semi-norms

$$|v|^2_{k,\Omega} := \sum_{|\alpha|=k} |D^\alpha v|^2_{L^2(\Omega)}, \quad k \in \{0, 1, \cdots, n\} \tag{2.4}$$

and the norm

$$\|v\|^2_{m,\Omega} := \sum_{k \leq m} |v|^2_{k,\Omega}. \tag{2.5}$$

The space $L^2(\Omega)$ is then $H^0(\Omega)$ and we shall usually write $\|v\|_{0,\Omega}$ to denote its norm $\|v\|_{L^2(\Omega)}$. We refer [11, 1, 15, 7] to the other Sobolev spaces that appear throughout this presentation.

2.1 The model

The boundary value problem considered in this work consists of finding a solution $u : \Omega \to \mathbb{R}$ to

$$\begin{aligned} -\nabla \cdot (\mathcal{A}\nabla u) = f \quad & \text{in } \Omega; \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{2.6}$$

where f is a given regular function and the diffusion coefficient $\mathcal{A} = \{\mathcal{A}_{ij}\}$ is a simmetric tensor in $[L^\infty(\Omega)]^{d\times d}$ wich is assumed to be uniformly elliptic, that is, there are positive constants $a_{\min}$ and $a_{\max}$ so that

$$a_{\min}|\mathbf{v}|^2 \leq \mathcal{A}(\mathbf{x})\mathbf{v}\cdot\mathbf{v} \leq a_{\max}|\mathbf{v}|^2, \quad \forall\, \mathbf{v} \in \mathbb{R}^d, \tag{2.7}$$

where $|\cdot|$ is the Euclidean norm. The classical solution to this problem is a function $u \in \mathcal{C}^2(\Omega)\cap\mathcal{C}(\overline{\Omega})$ satisfying the equation (2.6) everywhere in Ω and fulfilling the boundary condition at every $x \in \partial\Omega$. The associated standard variational formulation for (2.6) requires $f \in L^2(\Omega)$ and looks for a weak solution $u \in H^1_0(\Omega)$ such that

$$a(u,v) = F(v), \quad \forall\, v \in H^1_0(\Omega), \tag{2.8}$$

where the bilinear form $a : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ and the linear form $F : H^1_0(\Omega) \to \mathbb{R}$ are defined as:

$$a(v,w) := \int_\Omega \mathcal{A}\nabla v \cdot \nabla w \, dx \quad \text{and} \quad F(v) := \int_\Omega f v \, dx, \tag{2.9}$$

for all $v, w \in H^1_0(\Omega)$. The variational formulation reduces the regularity constraint over solution u and over data f. To prove the existence and uniqueness of this solution it is enough to verify the continuity and the coerciveness of the bilinear form $a(\cdot,\cdot)$ and the continuity of the linear form $F(\cdot)$; thus the result follows from the Lax-Milgram Lemma A.0.1. Indeed, we get from the Cauchy-Schwarz inequality that

$$|a(v,w)| \leq a_{\max}\int_\Omega |\nabla v \cdot \nabla w| \, dx \leq a_{\max}\|\nabla v\|_{0,\Omega}\|\nabla w\|_{0,\Omega} \leq a_{\max}\|v\|_{1,\Omega}\|w\|_{1,\Omega}; \tag{2.10}$$

$$|F(v)| \leq \int_\Omega |fv| \, dx \leq \|f\|_{0,\Omega}\|v\|_{1,\Omega}, \tag{2.11}$$

for all $v, w \in H_0^1(\Omega)$, that establishes the continuitity for $a(\cdot,\cdot)$ and $F(\cdot)$. Moreover, the coercivity for $a(\cdot,\cdot)$ holds, since:

$$a(v,v) \geq a_{\min}|v|_{1,\Omega}^2 \geq \alpha\|v\|_{1,\Omega}^2, \tag{2.12}$$

where the Poincaré inequality Theorem A.0.3 was used at the last inequality. The Lax-Milgran Lemma provide us the following stability result:

$$\|u\|_{1,\Omega} \leq \frac{1}{\alpha}\|f\|_{0,\Omega}. \tag{2.13}$$

2.2 Infinite dimensional problem using hybrid formulation

We introduce the mixed-hybrid formulation to the problem (2.6). Let $\mathcal{H} \in]0,1[$ be a parameter, $\mathcal{T}_\mathcal{H}$ be a regular mesh of the domain Ω composed of elements $K \in \mathcal{T}_\mathcal{H}$ and let $\mathcal{E}_\mathcal{H}$ the mesh skeleton composed of faces elements. For a fixed element $K \in \mathcal{T}_\mathcal{H}$, ∂K denotes its boundary, and $\mathbf{n}^K$ the unit size normal vector that points outward K. We denote by $\mathbf{n}$ the outward normal vector on $\partial\Omega$. Consider the following broken spaces:

$$\begin{aligned} H^1(\mathcal{T}_\mathcal{H}) &:= \{v \in L^2(\Omega);\ v|_K \in H^1(K),\ K \in \mathcal{T}_\mathcal{H}\}; \\ \Lambda &:= \left\{\mu \in H^{-1/2}(\partial K);\ \exists\, \sigma \in H(\mathrm{div}, K);\ \mu = \sigma \cdot \mathbf{n}^K|_{\partial K},\ K \in \mathcal{T}_\mathcal{H}\right\} \\ H_0^{1/2}(\mathcal{E}_\mathcal{H}) &:= \{v|_{\mathcal{E}_\mathcal{H}};\ v \in H_0^1(\Omega)\}. \end{aligned} \tag{2.14}$$

We can identify Λ with $\prod_{K\in\mathcal{T}_\mathcal{H}} H^{-1/2}(\partial K)$. Let

$$\Lambda_* := \{\mu \in H^{-1/2}(\mathcal{E}_\mathcal{H});\ \exists\, \sigma \in H(\mathrm{div};\Omega);\ \mu|_F = \sigma \cdot \mathbf{n}^K|_F,\ \forall\, F \in \mathcal{E}_\mathcal{H}\}. \tag{2.15}$$

Consider the finite dimensional subspaces $\Lambda^0 \subset \Lambda$ and $\Lambda_*^0 \subset \Lambda_*$ defined as:

$$\Lambda^0 := \prod_{K\in\mathcal{T}_\mathcal{H}} \{\mu \in \mathbb{P}_0(F),\ F \in \partial K\}; \tag{2.16}$$

$$\Lambda_*^0 := \{\mu \in \mathbb{P}_0(F),\ F \in \mathcal{E}_\mathcal{H}\}, \tag{2.17}$$

where $\mathbb{P}_0(F)$ stands for the space of constants functions over $F \in \mathcal{E}_{\mathcal{H}}$. See in the Figure 1 the illustrative functions for Λ^0 and Λ^0_* spaces.

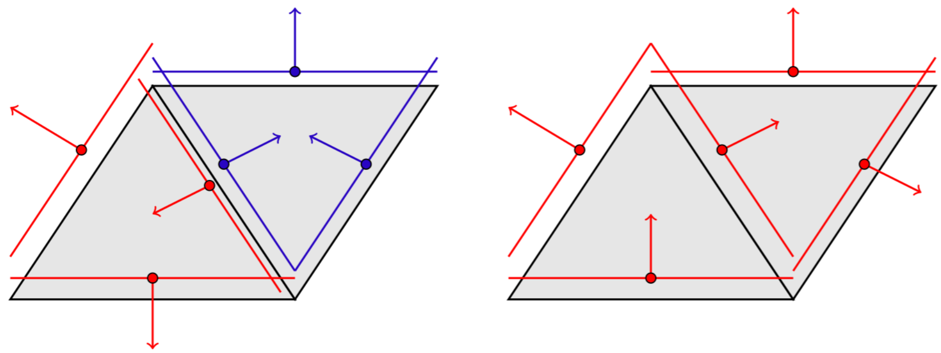

Figure 1 – Function on the space Λ^0 in the MHHM method and a function on the space Λ^0_* in the MHM method.

For $w, v \in L^2(\Omega)$, $\rho \in H_0^{1/2}(\mathcal{E}_{\mathcal{H}})$ and $\mu \in \Lambda$, define:

$$(w, v)_{\mathcal{T}_{\mathcal{H}}} := \sum_{K \in \mathcal{T}_{\mathcal{H}}} \int_K wv \, dx, \quad \langle \mu, \rho \rangle_{\mathcal{E}_{\mathcal{H}}} := \sum_{K \in \mathcal{T}_{\mathcal{H}}} \langle \mu, \rho \rangle_{\partial K}, \tag{2.18}$$

where $\langle \cdot, \cdot \rangle_{\partial K}$ is the dual product involving $H^{-1/2}(\partial K)$ and $H^{1/2}(\partial K)$, and we are using the same notation for a function in $H_0^{1/2}(\mathcal{E}_{\mathcal{H}})$ and its restriction to an element boundary ∂K. Therefore, we get from Corollary A.0.1 that:

$$\langle \mu, \rho \rangle_{\partial K} := \int_K v \nabla \cdot \sigma \, dx + \int_K \sigma \cdot \nabla v \, dx, \tag{2.19}$$

for all $\sigma \in H(\text{div}; K)$ such that $\sigma \cdot \mathbf{n}^K = \mu$, and for all $v \in H^1(K)$ such that $v|_{\partial K} = \rho$. We also denote:

$$\langle \mu, v \rangle_{\mathcal{E}_{\mathcal{H}}} := \sum_{K \in \mathcal{T}_{\mathcal{H}}} \langle \mu, v \rangle_{\partial K}, \tag{2.20}$$

whenever $v \in H^1(\mathcal{T}_{\mathcal{H}})$ and $\mu \in \Lambda$. We write, abusing notation, $\langle \mu, v \rangle_{\partial K} = \langle \mu, v|_{\partial K} \rangle_{\partial K}$, for all $\mu \in H^{-1/2}(\partial K)$ and all $v \in H^1(\mathcal{T}_{\mathcal{H}})$.

Then, the mixed-hybrid formulation to the boundary value problem (2.6) consists of finding $u \in H^1(\mathcal{T}_\mathcal{H})$, $\rho \in H_0^{1/2}(\mathcal{E}_\mathcal{H})$ and $\lambda \in \Lambda$ such that:

$$\begin{aligned}
&(\mathcal{A}\nabla u, \nabla v)_{\mathcal{T}_\mathcal{H}} \quad -\langle \lambda, v\rangle_{\mathcal{E}_\mathcal{H}} && = (f,v)_{\mathcal{T}_\mathcal{H}}, && \forall\, v \in H^1(\mathcal{T}_\mathcal{H});\\
&-\langle \mu, u\rangle_{\mathcal{E}_\mathcal{H}} \qquad\qquad\qquad +\langle \mu, \rho\rangle_{\mathcal{E}_\mathcal{H}} && = 0, && \forall\, \mu \in \Lambda;\\
&\qquad\qquad\quad \langle \lambda, \xi\rangle_{\mathcal{E}_\mathcal{H}} && = 0, && \forall\, \xi \in H_0^{1/2}(\mathcal{E}_\mathcal{H}).
\end{aligned} \tag{2.21}$$

This is the Three-field domain decomposition method proposed in [9], and it is equivalent to (2.6) as shown below.

Theorem 2.2.1. *Let u be the weak solution of (2.6), in the sense that u satisfies the variational formulation (2.8). Then (2.21) holds with $\rho = u|_{\mathcal{E}_\mathcal{H}}$ and $\lambda = \mathcal{A}\nabla u \cdot \boldsymbol{n}^K \in \Lambda$. Conversely, if $u \in H^1(\mathcal{T}_\mathcal{H})$, $\rho \in H_0^{1/2}(\mathcal{E}_\mathcal{H})$ and $\lambda \in \Lambda$ are solution of (2.21), then u is the weak solution of the boundary value problem (2.6).* —

Proof. Let u be the weak solution of (2.6). Then (2.8) holds in each element $K \in \mathcal{T}_\mathcal{H}$. For $\lambda = \mathcal{A}\nabla u \cdot \mathbf{n}^K$ we get

$$\begin{aligned}
\langle \lambda, \xi\rangle_{\mathcal{E}_\mathcal{H}} &= \sum_{K\in\mathcal{T}_\mathcal{H}} \langle \lambda, \xi\rangle_{\partial K}\\
&= \sum_{K\in\mathcal{T}_\mathcal{H}} \left\{\int_K \nabla\cdot(\mathcal{A}\nabla u) v\, dx + \int_K \mathcal{A}\nabla u \cdot \nabla v\, dx\right\}\\
&= \sum_{K\in\mathcal{T}_\mathcal{H}} \left\{-\int_K f v\, dx + \int_K f v\, dx\right\} = 0,
\end{aligned} \tag{2.22}$$

for all $\xi \in H_0^{1/2}(\mathcal{E}_\mathcal{H})$, where $v \in H_0^1(\Omega)$ is such that $v|_{\mathcal{E}_\mathcal{H}} = \xi$. Moreover, the first equation in (2.21) becomes:

$$\begin{aligned}
(\mathcal{A}\nabla u, \nabla v)_{\mathcal{T}_\mathcal{H}} - \langle \lambda, v\rangle_{\mathcal{E}_\mathcal{H}} &= \sum_{K\in\mathcal{T}_\mathcal{H}} \left[\int_K \mathcal{A}\nabla u\cdot\nabla v\, dx - \langle \mathcal{A}\nabla u \cdot \mathbf{n}^K, v\rangle_{\partial K}\right]\\
&= \sum_{K\in\mathcal{T}_\mathcal{H}} \int_K -\nabla\cdot(\mathcal{A}\nabla u) v\, dx\\
&= \sum_{K\in\mathcal{T}_\mathcal{H}} \int_K f v\, dx = (f,v)_{\mathcal{T}_\mathcal{H}},
\end{aligned} \tag{2.23}$$

where the second identity holds since $\lambda \in \Lambda$ is a trace of a function belonging to the space $H(\mathrm{div}; K)$, for all $K \in \mathcal{T}_\mathcal{H}$. The second equation of (2.21) is immediate for $\rho = u|_{\mathcal{E}_\mathcal{H}}$.

Conversely, suppose $u \in H^1(\mathcal{T}_\mathcal{H})$, $\rho \in H_0^{1/2}(\mathcal{E}_\mathcal{H})$ and $\lambda \in \Lambda$ solve (2.21). We gather from the third equation in (2.21) and Lemma A.0.2 that there exists $\sigma \in H(\mathrm{div};\Omega)$ such that $\lambda|_{\partial K} = \sigma \cdot \mathbf{n}^K$, for all $K \in \mathcal{T}_\mathcal{H}$. Since $H_0^1(\Omega) \subset H^1(\mathcal{T}_\mathcal{H})$, we get from Lemma A.0.2

that the first equation becomes:

$$(\mathcal{A}\nabla u, \nabla v)_{\mathcal{T}_\mathcal{H}} = (f, v)_\Omega, \quad \forall\, v \in H_0^1(\Omega). \tag{2.24}$$

it remains to show the continuity of u over $\mathcal{E}_\mathcal{H}$. Let $K \in \mathcal{T}_\mathcal{H}$. Then, we obtain from the second equation that

$$\langle \mu, u - \rho \rangle_{\partial K} = 0, \quad \forall\, \mu \in \Lambda|_{\partial K}, \tag{2.25}$$

so that $u = \rho$ over ∂K. Then, for each face $F \in \mathcal{E}_\mathcal{H}$ shared by two elements $K_i, K_j \in \mathcal{T}_\mathcal{H}$ we have, denoting $u|_{F_i}$ for the restriction $(u|_{\partial K_i})|_F$, that $u|_{F_i} = u|_{F_j} = \rho|_F$. Therefore, follows directly from Theorem A.0.2, that $u \in H_0^1(\Omega)$. So, we can rewrite (2.24) as:

$$(\mathcal{A}\nabla u, \nabla v)_\Omega = (f, v)_\Omega, \quad \forall\, v \in H_0^1(\Omega), \tag{2.26}$$

which concludes the proof. □

The equivalence given by the previous theorem implies the well posedness of the hybrid-mixed formulation (2.21). The next step is to decompose both $H^1(\mathcal{T}_\mathcal{H})$ and Λ spaces in the form "constant" plus "zero average". Let

$$H^1(\mathcal{T}_\mathcal{H}) = \mathbb{P}_0(\mathcal{T}_\mathcal{H}) \oplus \tilde{H}^1(\mathcal{T}_\mathcal{H}), \tag{2.27}$$

where $\mathbb{P}_0(\mathcal{T}_\mathcal{H})$ is the space of piecewise constants in each element and $\tilde{H}^1(\mathcal{T}_\mathcal{H})$ is the space of functions that have zero-average within each element border:

$$\tilde{H}^1(\mathcal{T}_\mathcal{H}) := \left\{ v \in H^1(\mathcal{T}_\mathcal{H});\ \int_{\partial K} v\, ds = 0,\ \forall\, K \in \mathcal{T}_\mathcal{H} \right\}. \tag{2.28}$$

Then, for $v \in H^1(\mathcal{T}_\mathcal{H})$, we write $v = v^0 + \tilde{v}$, where $v^0 \in \mathbb{P}_0(\mathcal{T}_\mathcal{H})$ and $\tilde{v} \in \tilde{H}^1(\mathcal{T}_\mathcal{H})$. This space differs from the space $H_0^\perp(\mathcal{T}_\mathcal{H})$ introduced in the MHM method, which works with zero-average functionals inside each element, that is,

$$H_0^\perp(\mathcal{T}_\mathcal{H}) := \left\{ v \in H^1(\mathcal{T}_\mathcal{H});\ \int_K v\, dx = 0,\ \forall\, K \in \mathcal{T}_\mathcal{H} \right\}. \tag{2.29}$$

Taking a further step, we decompose Λ into a space of "constats" plus "zero-average" functionals over the border of the elements of $\mathcal{T}_{\mathcal{H}}$. For each $K_i \in \mathcal{T}_{\mathcal{H}}$, let $\lambda_i^0 \in \Lambda$ such that:

$$\langle \lambda_i^0, v \rangle_{\mathcal{E}_{\mathcal{H}}} = \int_{\partial K_i} v \, ds, \qquad \forall \, v \in H^1(\mathcal{T}_{\mathcal{H}}). \tag{2.30}$$

Let N be the number of elements of $\mathcal{T}_{\mathcal{H}}$ and define the following spaces:

$$\begin{aligned} \Lambda^0 &:= \operatorname{span}\left\{ \lambda_i^0, \, \forall \, i = 1, \cdots, N \right\}; \\ \tilde{H}^{-1/2}(\partial K) &:= \left\{ \tilde{\mu} \in H^{-1/2}(\partial K); \, \langle \tilde{\mu}, 1 \rangle_{\partial K} = 0 \right\}; \\ \tilde{\Lambda} := \mathbb{P}_0(\mathcal{T}_{\mathcal{H}})^{\perp} &:= \left\{ \tilde{\mu} \in \Lambda; \, \langle \tilde{\mu}, v^0 \rangle_{\mathcal{E}_{\mathcal{H}}} = 0, \, \forall \, v_0 \in \mathbb{P}_0(\mathcal{T}_{\mathcal{H}}) \right\} = \prod_{K \in \mathcal{T}_{\mathcal{H}}} \tilde{H}^{-1/2}(\partial K). \end{aligned} \tag{2.31}$$

We can now decompose $\Lambda = \Lambda^0 \oplus \tilde{\Lambda}$ as follows. Given $\mu \in \Lambda$, let $\mu^0 \in \Lambda^0$ and $\tilde{\mu} \in \tilde{\Lambda}$ such that

$$\begin{aligned} \langle \mu^0, v^0 \rangle_{\mathcal{E}_{\mathcal{H}}} &= \langle \mu, v^0 \rangle_{\mathcal{E}_{\mathcal{H}}}, & \forall \, v^0 \in \mathbb{P}_0(\mathcal{T}_{\mathcal{H}}); \\ \langle \tilde{\mu}, v \rangle_{\mathcal{E}_{\mathcal{H}}} &= \langle \mu, v \rangle_{\mathcal{E}_{\mathcal{H}}} - \langle \mu^0, v \rangle_{\mathcal{E}_{\mathcal{H}}}, & \forall \, v \in H^1(\mathcal{T}_{\mathcal{H}}). \end{aligned} \tag{2.32}$$

Note that $\tilde{\mu} \in \tilde{\Lambda}$ since

$$\langle \tilde{\mu}, v^0 \rangle_{\mathcal{E}_{\mathcal{H}}} = \langle \mu, v^0 \rangle_{\mathcal{E}_{\mathcal{H}}} - \langle \mu^0, v^0 \rangle_{\mathcal{E}_{\mathcal{H}}} = 0, \tag{2.33}$$

and $\mu = \mu^0 + \tilde{\mu}$. According to the above definitions, we write $u = u^0 + \tilde{u}$, where $u^0 \in \mathbb{P}_0(\mathcal{T}_{\mathcal{H}})$ and $\tilde{u} \in \tilde{H}^1(\mathcal{T}_{\mathcal{H}})$, and also $\lambda = \lambda^0 + \tilde{\lambda}$, for $\lambda^0 \in \Lambda^0$ and $\tilde{\lambda} \in \tilde{\Lambda}$.

We get from (2.21) that $\lambda^0 \in \Lambda^0$ and $u^0 \in \mathbb{P}_0(\mathcal{T}_{\mathcal{H}})$ are solution of the following problems:

$$\begin{aligned} \langle \lambda^0, v^0 \rangle_{\mathcal{E}_{\mathcal{H}}} &= -(f, v^0)_{\mathcal{T}_{\mathcal{H}}}, & \forall \, v^0 \in \mathbb{P}_0(\mathcal{T}_{\mathcal{H}}); \\ \langle \mu^0, u^0 \rangle_{\mathcal{E}_{\mathcal{H}}} &= \langle \mu^0, \rho \rangle_{\mathcal{E}_{\mathcal{H}}}, & \forall \, \mu^0 \in \Lambda^0. \end{aligned} \tag{2.34}$$

Note that λ^0 is defined by the first equation of (2.34). The piecewise constant u^0 is obtained from the second equation after the computation of ρ. We also obtain from (2.21) that $(\tilde{u}, \tilde{\lambda}, \rho) \in \tilde{H}^1(\mathcal{T}_{\mathcal{H}}) \times \tilde{\Lambda} \times H_0^{1/2}(\mathcal{E}_{\mathcal{H}})$ solves:

$$\begin{array}{lllll} (\mathcal{A}\nabla \tilde{u}, \nabla \tilde{v})_{\mathcal{T}_{\mathcal{H}}} & -\langle \tilde{\lambda}, \tilde{v} \rangle_{\mathcal{E}_{\mathcal{H}}} & & = (f, \tilde{v})_{\mathcal{T}_{\mathcal{H}}}, & \forall \, \tilde{v} \in \tilde{H}^1(\mathcal{T}_{\mathcal{H}}), \\ -\langle \tilde{\mu}, \tilde{u} \rangle_{\mathcal{E}_{\mathcal{H}}} & & +\langle \tilde{\mu}, \rho \rangle_{\mathcal{E}_{\mathcal{H}}} & = 0 & \forall \, \tilde{\mu} \in \tilde{\Lambda}, \\ & \langle \tilde{\lambda}, \xi \rangle_{\mathcal{E}_{\mathcal{H}}} & & = -\langle \lambda^0, \xi \rangle_{\mathcal{E}_{\mathcal{H}}}, & \forall \, \xi \in H_0^{1/2}(\mathcal{E}_{\mathcal{H}}). \end{array} \tag{2.35}$$

The first equation of (2.35) is local and allows the introduction of local solvers. Let $K \in \mathcal{T}_{\mathcal{H}}$. For $\tilde{\mu} \in \tilde{\Lambda}$ and $f \in L^2(\Omega)$, let $T : \tilde{\Lambda} \to \tilde{H}^1(\mathcal{T}_{\mathcal{H}})$ and $\tilde{T} : L^2(\Omega) \to \tilde{H}^1(\mathcal{T}_{\mathcal{H}})$ be such that

$$\int_K \mathcal{A}\nabla(T\tilde{\mu}) \cdot \nabla\tilde{v}\, dx = \langle \tilde{\mu}, \tilde{v}\rangle_{\partial K}, \quad \forall\, \tilde{v} \in \tilde{H}^1(K), \tag{2.36}$$

and

$$\int_K \mathcal{A}\nabla(\tilde{T}f) \cdot \nabla\tilde{v}\, dx = \int_K f\tilde{v}\, dx, \quad \forall\, \tilde{v} \in \tilde{H}^1(K). \tag{2.37}$$

The local problems (2.36) and (2.37) are well-posed. Indeed, it is simple to show that their bilinear and linear forms are continuous. The coercivity for the bilinear forms holds from the generalized Poincaré inequality introduced in Theorem A.0.4, which establishes that there exists a positive constant $C(K)$ such that:

$$\|v\|_{0,K} \leq C(K)\left(\left|\int_{\partial K} v\, ds\right| + |v|_{1,K}\right), \tag{2.38}$$

for all $v \in H^1(K)$. According to the Lemma A.0.3 and the Definition A.0.1, the constant C depends on the domain geometry. The T operator is self-adjoint. In fact, since $\mathcal{A}$ is a symmetric tensor, we have:

$$\langle \tilde{\lambda}, T\tilde{\mu}\rangle_{\partial K} = \int_K \mathcal{A}\nabla T\tilde{\lambda} \cdot \nabla T\tilde{\mu}\, dx = \int_K \mathcal{A}\nabla T\tilde{\mu} \cdot \nabla T\tilde{\lambda}\, dx = \langle \tilde{\mu}, T\tilde{\lambda}\rangle_{\partial K}, \tag{2.39}$$

for all $\tilde{\mu},\ \tilde{\lambda} \in \tilde{\Lambda}$, and $K \in \mathcal{T}_{\mathcal{H}}$. Moreover, for $K \in \mathcal{T}_{\mathcal{H}}$, follows from (2.36) that a function $\tilde{\mu} \in \tilde{H}^{-1/2}(\partial K)$ is the trace of $\mathcal{A}\nabla T\tilde{\mu} \in H(\text{div}; K)$. Then, we get from the boundeness of the trace operator in the space $H(\text{div}; K)$ [15, Theorem 1.7] and from (2.7) that

$$\begin{aligned} |\tilde{\mu}|^2_{-\frac{1}{2},\partial K} &\leq \|\mathcal{A}\nabla T\tilde{\mu}\|^2_{\text{div},K} = (\|\mathcal{A}\nabla T\tilde{\mu}\|^2_{0,K} + \|\nabla \cdot (\mathcal{A}\nabla T\tilde{\mu})\|^2_{0,K}) \\ &= \|\mathcal{A}\nabla T\tilde{\mu}\|^2_{0,K} = \int_K \mathcal{A}\nabla T\tilde{\mu} \cdot \nabla T\tilde{\mu}\, dx = \langle \tilde{\mu}, T\tilde{\mu}\rangle_{\partial K} \\ &\leq |\tilde{\mu}|_{-\frac{1}{2},\partial K}|T\tilde{\mu}|_{1,K}, \end{aligned} \tag{2.40}$$

where the last inequality follows from Trace Inequality in $H^1(K)$ [15, Theorem 1.4] and Generalized Poincaré Inequality Theorem A.0.4. Then,

$$|T\tilde{\mu}|_{1,K} \geq C|\tilde{\mu}|_{-\frac{1}{2},\partial K}, \tag{2.41}$$

for all $\tilde{\mu} \in \tilde{\Lambda}$.

Therefore, we can write $\tilde{u} = T\tilde{\lambda} + \tilde{T}f$. Applying a static condensation in (2.35), we have the saddle point problem of finding $(\tilde{\lambda}, \rho) \in \tilde{\Lambda} \times H_0^{1/2}(\mathcal{E}_{\mathcal{H}})$ such that:

$$\begin{array}{lcll} -\langle \tilde{\mu}, T\tilde{\lambda}\rangle_{\mathcal{E}_{\mathcal{H}}} + \langle \tilde{\mu}, \rho\rangle_{\mathcal{E}_{\mathcal{H}}} & = & \langle \tilde{\mu}, \tilde{T}f\rangle_{\mathcal{E}_{\mathcal{H}}}, & \forall\, \tilde{\mu} \in \tilde{\Lambda}; \\ \langle \tilde{\lambda}, \xi\rangle_{\mathcal{E}_{\mathcal{H}}} & = & -\langle \lambda^0, \xi\rangle_{\mathcal{E}_{\mathcal{H}}}, & \forall\, \xi \in H_0^{1/2}(\mathcal{E}_{\mathcal{H}}). \end{array} \tag{2.42}$$

The coercivity of the bilinear form $\langle \cdot, T\cdot\rangle$ on $\tilde{\Lambda}$ space allows us to project a function $\phi \in H_0^{1/2}(\mathcal{E}_{\mathcal{H}})$ on the space $\tilde{\Lambda}$. Then, problem (2.42) can be simplified by applying static condensation again. In fact, for all $\mu \in \tilde{\Lambda}$, we have:

$$\langle \tilde{\mu}, T\tilde{\mu}\rangle_{\mathcal{E}_{\mathcal{H}}} = \sum_{K \in \mathcal{T}_{\mathcal{H}}} \langle \tilde{\mu}, T\tilde{\mu}\rangle_{\partial K}, \tag{2.43}$$

and, for a fixed $K \in \mathcal{T}_{\mathcal{H}}$,

$$\begin{aligned} \langle \tilde{\mu}, T\tilde{\mu}\rangle_{\partial K} &= \int_K \mathcal{A}\nabla(T\tilde{\mu}) \cdot \nabla(T\tilde{\mu})\, dx \\ &\geq a_{\min} \|\nabla T\tilde{\mu}\|_{0,K}^2 \geq C\|T\tilde{\mu}\|_{1,K}^2 \\ &\geq C|\tilde{\mu}|_{-\frac{1}{2},\partial K}^2, \end{aligned} \tag{2.44}$$

where we have used in the last two inequalities the Genaralized Poincaré inequality (2.38) and the injectivity of the operator T (2.41). We define:

$$G : \prod_{K \in \mathcal{T}_{\mathcal{H}}} H^{1/2}(\partial K) \longrightarrow \tilde{\Lambda}, \tag{2.45}$$

such that, for a fixed $K \in \mathcal{T}_{\mathcal{H}}$ and given $\phi \in H^{1/2}(\partial K)$, if $\tilde{\lambda}_\phi|_{\partial K} = G\phi|_{\partial K} \in \tilde{H}^{-1/2}(\partial K)$, then,

$$\int_K \mathcal{A}\nabla(T\tilde{\lambda}_\phi) \cdot \nabla T\tilde{\mu}\, dx = \langle \tilde{\mu}, T\tilde{\lambda}_\phi\rangle_{\partial K} = \langle \tilde{\mu}, \phi\rangle_{\partial K}, \quad \forall\, \tilde{\mu} \in \tilde{\Lambda}, \tag{2.46}$$

where the first identity follows from (2.36). We have from (2.42) that $\tilde{\lambda} = G(\rho - \tilde{T}f)$. From the second equation of (2.42), we have the global problem of finding $\rho \in H_0^{1/2}(\mathcal{E}_{\mathcal{H}})$ such that

$$\langle G\rho, \xi\rangle_{\mathcal{E}_{\mathcal{H}}} = -\langle \lambda^0, \xi\rangle_{\mathcal{E}_{\mathcal{H}}} + \langle G\tilde{T}f, \xi\rangle_{\mathcal{E}_{\mathcal{H}}}, \quad \forall\, \xi \in H_0^{1/2}(\mathcal{E}_{\mathcal{H}}). \tag{2.47}$$

The following description of the exact solution holds:

$$u = u^0 + TG\rho + (I - TG)\tilde{T}f. \tag{2.48}$$

Moreover,

$$\lambda = \lambda^0 + G(\rho - \tilde{T}f). \tag{2.49}$$

Note that both T and G can be solved locally. The matrix to compute λ^0 and u^0 from (2.34) is diagonal and there is a finite number of unknows. Thus, (2.47) is only global infinite dimensional system of equations, depending on $\mathcal{A}$ throught the G operator. Regarding solvability of (2.35), from u, λ and ρ solving (2.21) the decompositions $u = u^0 + \tilde{u}$ and $\lambda = \lambda^0 + \tilde{\lambda}$ yield solutions for (2.35). To check the uniqueness, if $f = 0$, the first equation in (2.35) implies $\lambda^0 = 0$. By the injectivity of T and $\tilde{T}$ operators we have $\tilde{u} = T\tilde{\lambda} + \tilde{T}f = 0$. Since $G \circ \tilde{T}$ is injective, we have from the global problem (2.47) that $\rho = 0$. Then (2.42) implies $\tilde{\lambda} = 0$. Finally, we obtain $u^0 = 0$ in (2.34).

The left hand side of (2.47) induces a symmetric, continuous and $H_0^{1/2}(\mathcal{E}_{\mathcal{H}})$-elliptic bilinear form as we will see next. Define the bilinear forms $g_K : H^{1/2}(\partial K) \times H^{1/2}(\partial K) \to \mathbb{R}$ and $g : H_0^{1/2}(\mathcal{E}_{\mathcal{H}}) \times H_0^{1/2}(\mathcal{E}_{\mathcal{H}}) \to \mathbb{R}$ be such that:

$$\begin{aligned} g_K(\xi, \phi) &:= \langle G\xi, \phi\rangle_{\partial K}, \quad \forall\, \xi, \phi \in H^{1/2}(\partial K),\ \forall\, K \in \mathcal{T}_{\mathcal{H}}; \\ g(\xi, \phi) &:= \sum_{K \in \mathcal{T}_{\mathcal{H}}} g_K(\xi, \phi), \quad \forall\, \xi, \phi \in H_0^{1/2}(\mathcal{E}_{\mathcal{H}}). \end{aligned} \tag{2.50}$$

Note that $g_K(\cdot, \cdot)$ is symmetric since, for $\xi, \phi \in H^{1/2}(\partial K)$, we obtain by definition of G and symmetry of T that:

$$\langle \tilde{\lambda}_\xi, \phi\rangle_{\partial K} = \langle \tilde{\lambda}_\xi, T\tilde{\lambda}_\phi\rangle_{\partial K} = \langle \tilde{\lambda}_\phi, T\tilde{\lambda}_\xi\rangle_{\partial K} = \langle \tilde{\lambda}_\phi, \xi\rangle_{\partial K}.$$

Remark 2.2.1. Another way, not explored in this work, to solve the three-field formulation (2.21) consists to replace local Neumann problems by local sadle-point problems. Then, we can apply an static condensation to get the trace ρ in terms of the solution u and flux λ, since the associated block matrix is non-singular. —

2.3 Functional analysis results

We establish here the well posedness of the global problem (2.47), through the Lax-Milgran Lemma, namely: continuity and coercivity of the bilinear form $g(\cdot,\cdot)$ defined in (2.50). We first introduce some definitions and technical results.

Definition 2.3.1. Consider the following semi-norms:

$$\begin{aligned} |v|_{1,\mathcal{A},K} &:= \|\mathcal{A}^{1/2}\nabla v\|_{0,K} = \left(\int_K \mathcal{A}^{1/2}\nabla v \cdot \mathcal{A}^{1/2}\nabla v \, dx\right)^{1/2}; \\ |\xi|_{\frac{1}{2},\partial K} &:= \inf_{\phi \in V_\xi(K)} |\phi|_{1,\mathcal{A},K}; \\ |\rho|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} &:= \inf_{\phi \in V_\rho(\Omega)} |\phi|_{1,\mathcal{A},\Omega}; \end{aligned} \tag{2.51}$$

where, for $\xi \in H^{1/2}(\partial K)$ and $\rho \in H_0^{1/2}(\mathcal{E}_\mathcal{H})$, we define:

$$V_\xi(K) := \{v \in H^1(K);\ v|_{\partial K} = \xi\} \quad \text{and} \quad V_\rho(\Omega) := \{v \in H_0^1(\Omega);\ v|_{\mathcal{E}_\mathcal{H}} = \rho\}. \tag{2.52}$$

Futhermore, from the Spectral Theorem in finite dimensional spaces [22, page 160], the positive matrix $\mathcal{A}^{1/2}$ is the unique square root of the symmetric positive definite matrix $\mathcal{A}$. —

Lemma 2.3.1. *Let $K \in \mathcal{T}_\mathcal{H}$ and $\xi \in H^{1/2}(\partial K)$. Then*

$$|\xi|_{\frac{1}{2},\partial K} = |\phi_\xi|_{1,\mathcal{A},K}, \tag{2.53}$$

where ϕ_ξ is the weak solution of

$$\begin{aligned} -\nabla \cdot (\mathcal{A}\nabla\phi_\xi) &= 0, \quad \text{in } K; \\ \phi_\xi &= \xi, \quad \text{on } \partial K. \end{aligned} \tag{2.54}$$

Moreover, if $\xi \in H_0^{1/2}(\mathcal{E}_h)$ and (2.54) holds for all $K \in \mathcal{T}_h$, then $\phi_\xi \in H_0^1(\Omega)$ and

$$|\xi|^2_{\frac{1}{2},\mathcal{E}_\mathcal{H}} = |\phi_\xi|^2_{1,\mathcal{A},\Omega} = \sum_{K\in\mathcal{T}_\mathcal{H}} |\xi|^2_{\frac{1}{2},\partial K}. \tag{2.55}$$

—

Proof. For a fixed $K \in \mathcal{T}_{\mathcal{H}}$, define the energy functional $J : H^1(K) \to \mathbb{R}$ such that:

$$J(v) := \int_K \mathcal{A}\nabla v \cdot \nabla v \, dx, \quad \forall\, v \in H^1(K), \tag{2.56}$$

and consider the minimization problem in $V_\xi(K)$:

$$\inf_{v \in V_\xi(K)} J(v) = |\xi|_{\frac{1}{2},\partial K}. \tag{2.57}$$

The associated variational formulation for (2.57) satisfies, for $\phi_\xi \in V_\xi$:

$$0 = \langle J'(\phi_\xi), v\rangle = 2\int_K \mathcal{A}\nabla\phi_\xi \cdot \nabla v \, dx, \tag{2.58}$$

for all $v \in H_0^1(K)$. Since (2.58) is the weak formulation for (2.54), we conclude that ϕ_ξ minimizes the functional (2.56), that is,

$$|\xi|_{\frac{1}{2},\partial K} = \inf_{v \in V_\xi} J(v) = J(\phi_\xi) = |\phi_\xi|_{\mathcal{A},1,K}. \tag{2.59}$$

Furthermore, let $\xi \in H_0^{1/2}(\mathcal{E}_{\mathcal{H}})$ and assume that (2.54) holds in each element $K \in \mathcal{T}_{\mathcal{H}}$. Then, we get from the first part of the proof that $\phi_\xi|_{\partial K} \in H^1(K)$, $\forall\, K \in \mathcal{T}_{\mathcal{H}}$. In addition, if $\mathcal{I}_{\mathcal{H}} := \mathcal{E}_{\mathcal{H}} \setminus \partial\Omega$ is the interior faces set, we have:

$$\sum_{K \in \mathcal{E}_{\mathcal{H}}} \langle \tau \cdot \mathbf{n}^K, \phi_\xi\rangle_{\partial K} = \sum_{F \in \mathcal{I}_{\mathcal{H}}} \int_F \left(\phi_\xi|_{K_i,F} - \phi_\xi|_{K_j,F}\right) \tau \cdot \mathbf{n}^F \, ds \tag{2.60}$$

$$= \sum_{F \in \mathcal{I}_{\mathcal{H}}} \int_F (\xi|_F - \xi|_F)\, \tau \cdot \mathbf{n}^F \, ds = 0, \tag{2.61}$$

for all $\tau \in [\mathcal{C}_0^\infty(\Omega)]^d$, where, in (2.60), $K_i, K_j \in \mathcal{T}_{\mathcal{H}}$ are adjacent elements that share face $F \in \mathcal{E}_{\mathcal{H}}$. Therefore, we get from the characterazion of the $H^1(\Omega)$ space, Lemma A.0.1, and the fact that $\xi = 0$ over $\partial\Omega$ that $\phi_\xi \in H_0^1(\Omega)$. Finally:

$$\begin{aligned}\sum_{K \in \mathcal{T}_{\mathcal{H}}} |\xi|^2_{\frac{1}{2},\partial K} &= \sum_{K \in \mathcal{T}_{\mathcal{H}}} \inf_{v \in V_\xi(K)} |v|^2_{1,\mathcal{A},K} = \sum_{K \in \mathcal{T}_{\mathcal{H}}} |\phi_\xi|^2_{1,\mathcal{A},K} \\ &= |\phi_\xi|^2_{1,\mathcal{A},\Omega} = \inf_{v \in V_\xi(\Omega)} |v|^2_{1,\mathcal{A},\Omega} = |\xi|^2_{\frac{1}{2},\mathcal{E}_{\mathcal{H}}},\end{aligned} \tag{2.62}$$

wich proves (2.55). □

Definition 2.3.2.

1. For a fixed element $K \in \mathcal{T}_{\mathcal{H}}$, let the space of zero average functionals over the boundary ∂K:

$$\tilde{H}^{1/2}(\partial K) := \left\{ \xi \in H^{1/2}(\partial K);\ \int_{\partial K} \xi \, ds = 0 \right\}. \tag{2.63}$$

2. We define the following semi-norms on the space Λ:

$$|\mu|_{-\frac{1}{2},\partial K} := \sup_{\tilde{\phi} \in \tilde{H}^{\frac{1}{2}}(\partial K)} \frac{\langle \mu, \tilde{\phi} \rangle_{\partial K}}{|\tilde{\phi}|_{\frac{1}{2},\partial K}}, \quad |\mu|_{\Lambda}^2 := \sum_{K \in \mathcal{T}_h} |\mu|_{-\frac{1}{2}\partial K}^2, \tag{2.64}$$

for all $\mu \in \Lambda$.

—

Remark 2.3.1. About previous definitions, for $K \in \mathcal{T}_{\mathcal{H}}$, we mention:

1. The semi-norm $|\cdot|_{\frac{1}{2},\partial K}$ on the space $\tilde{H}^{1/2}(\partial K)$ is a norm. To verify that, it is enough to show that the semi-norm $|\cdot|_{1,\mathcal{A},K}$ on the space $\tilde{H}^1(K)$, evaluated on each element $K \in \mathcal{T}_{\mathcal{H}}$, is a norm. Indeed, for $\tilde{v} \in \tilde{H}^1(K)$, we have:

$$|\tilde{v}|_{1,\mathcal{A},K} = 0 \quad \Leftrightarrow \quad \nabla \tilde{v} = 0 \quad \Leftrightarrow \quad \tilde{v} = \text{constant}, \tag{2.65}$$

so that, the zero mean property implies $\tilde{v} = 0$.

2. Follows from (2.46) that, for all $\xi \in H^{1/2}(\partial K)$,

$$(TG\xi)|_{\partial K} = \xi + c_\xi \in \tilde{H}^{1/2}(\partial K), \quad \text{where } c_\xi := -\frac{1}{|\partial K|} \int_{\partial K} \xi \, ds. \tag{2.66}$$

We stress that such identity holds only over the boundary of each element.

3. The semi-norm $|\cdot|_{-\frac{1}{2},\partial K}$ on the zero mean functions space $\tilde{H}^{-\frac{1}{2}}(\partial K)$ becomes a norm. However, the composition $|\cdot|_{-\frac{1}{2},\partial K} \circ G : H^{\frac{1}{2}}(\partial K) \to \mathbb{R}$ is still a semi-norm.

—

In the following lemma we collect some useful technical results.

Lemma 2.3.2. *For a fixed $K \in \mathcal{T}_{\mathcal{H}}$ the following result holds:*

(i) $|T\tilde{\mu}|_{\frac{1}{2},\partial K} = |T\tilde{\mu}|_{1,\mathcal{A},K} = |\tilde{\mu}|_{-\frac{1}{2},\partial K}, \quad \forall\, \tilde{\mu} \in \tilde{\Lambda}$;

(ii) $|G\xi|_{-\frac{1}{2},\partial K} = |\xi|_{\frac{1}{2},\partial K}, \quad \forall\, \xi \in H^{1/2}(\partial K)$;

(iii) $\langle \tilde{\mu}, \xi \rangle_{\partial K} \leq |\xi|_{\frac{1}{2},\partial K} |\tilde{\mu}|_{-\frac{1}{2},\partial K}, \quad \forall\, \tilde{\mu} \in \tilde{\Lambda},\ \forall\, \xi \in H^{1/2}(\partial K)$,

Also, for all $\xi \in H^{1/2}(\partial K)$,

$$\sup_{\tilde{\mu} \in \tilde{H}^{-\frac{1}{2}}(\partial K)} \frac{\langle \tilde{\mu}, \xi \rangle_{\partial K}}{|\tilde{\mu}|_{-\frac{1}{2},\partial K}} = |\xi|_{\frac{1}{2},\partial K}. \tag{2.67}$$

Proof. For a fixed $K \in \mathcal{T}_{\mathcal{H}}$ and given $\tilde{\mu} \in \tilde{\Lambda}$, follows from (2.36) that $T\tilde{\mu} \in \tilde{H}^1(K)$ is harmonic and the first identity in (i) holds from Lemma 2.3.1.

For the second identity, let $\mathcal{H}^*(K)$ be the space of harmonic functions in $\tilde{H}^1(K)$. Futhermore, from Lemma 2.3.1, functionals of the space $\mathcal{H}^*(K)$ minimizes the semi-norm $|\cdot|_{\frac{1}{2},\partial K}$. Then, we get from (2.64) that:

$$|\tilde{\mu}|_{-\frac{1}{2},\partial K} = \sup_{\psi \in \tilde{H}^1(K)} \frac{\langle \tilde{\mu}, \psi \rangle_{\partial K}}{|\psi|_{\frac{1}{2},\partial K}} = \sup_{\psi \in \mathcal{H}^*(K)} \frac{\langle \tilde{\mu}, \psi \rangle_{\partial K}}{|\psi|_{\frac{1}{2},\partial K}}. \tag{2.68}$$

Since, for $\phi \in \mathcal{H}^*$, we get $\tilde{\mu}_\phi := \mathcal{A}\nabla\phi \cdot \mathbf{n}^K \in \tilde{H}^{-1/2}(\partial K)$ is such that $T\tilde{\mu}_\phi = \phi$, then $T : H^{-1/2}(\partial K) \to \mathcal{H}^*$ is surjective. It follows from the previous arguments and the first identity in (i) that:

$$|\tilde{\mu}|_{-\frac{1}{2},\partial K} = \sup_{\lambda \in \tilde{\Lambda}} \frac{\langle \tilde{\mu}, T\lambda \rangle_{\partial K}}{|T\lambda|_{\frac{1}{2},\partial K}} = \sup_{\lambda \in \tilde{\Lambda}} \frac{\int_K \mathcal{A}\nabla T\tilde{\mu} \cdot \nabla T\lambda \, dx}{|T\lambda|_{1,\mathcal{A},K}} = |T\tilde{\mu}|_{1,\mathcal{A},K}. \tag{2.69}$$

Next, (ii) follows from (i) with $\tilde{\mu} = G\xi$ and (2.66). In fact, since $\xi = TG\xi - c_\xi$, we get from (i) that:

$$|\xi|_{\frac{1}{2},\partial K} = |TG\xi - c_\xi|_{\frac{1}{2},\partial K} = |TG\xi - c_\xi|_{\mathcal{A},1,K} = |TG\xi|_{\mathcal{A},1,K} = |G\xi|_{-\frac{1}{2},\partial K}. \tag{2.70}$$

Finally, to show (iii), first denote ψ_ξ the $\mathcal{A}$-harmonic extension of ξ, that is, the solution of (2.54). Follows from (i) and the Cauchy-Schwarz inequality that:

$$\frac{\langle \tilde{\mu}, \xi \rangle_{\partial K}}{|\tilde{\mu}|_{-\frac{1}{2},\partial K}} = \frac{\int_K \mathcal{A}\nabla T\tilde{\mu} \cdot \nabla \psi_\xi \, dx}{|T\tilde{\mu}|_{1,\mathcal{A},K}} \leq |\psi_\xi|_{1,\mathcal{A},K} = |\xi|_{\frac{1}{2},\partial K}, \tag{2.71}$$

where we used Lemma (2.3.1) at last step. Finally, for the semi-norm (2.67), we have:

$$\sup_{\tilde{\mu}\in\tilde{H}^{-\frac{1}{2}}(\partial K)} \frac{\langle\tilde{\mu},\xi\rangle_{\partial K}}{|\tilde{\mu}|_{-\frac{1}{2},\partial K}} = \sup_{\tilde{\mu}\in\tilde{H}^{-\frac{1}{2}}(\partial K)} \frac{\int_K \mathcal{A}\nabla T\tilde{\mu}\cdot\nabla(TG\xi)\,dx}{|T\tilde{\mu}|_{1,\mathcal{A},K}} = |TG\xi|_{1,\mathcal{A},K} = |\xi|_{\frac{1}{2},\partial K}, \quad (2.72)$$

where (ii) was used at the last identity. □

Proposition 2.3.1. *Let the bilinear forms g_K and g as defined in (2.50). The following coercivity results hold:*

$$g_K(\tilde{\xi},\tilde{\xi}) = |\tilde{\xi}|^2_{\frac{1}{2},\partial K}, \quad \forall\,\tilde{\xi}\in\tilde{H}^{1/2}(\partial K); \quad (2.73)$$

$$g(\xi,\xi) = |\xi|^2_{\frac{1}{2},\mathcal{E}_\mathcal{H}}, \quad \forall\,\xi\in H^{1/2}_0(\mathcal{E}_\mathcal{H}). \quad (2.74)$$

Moreover,

$$g(\xi,\rho) \le |\xi|_{\frac{1}{2},\mathcal{E}_\mathcal{H}}|\rho|_{\frac{1}{2},\mathcal{E}_\mathcal{H}}, \quad \forall\,\xi,\rho\in H^{1/2}_0(\mathcal{E}_\mathcal{H}). \quad (2.75)$$

Proof. For $\tilde{\xi}\in\tilde{H}^{1/2}(\partial K)$, let $\tilde{\lambda}_{\tilde{\xi}} = G\tilde{\xi}$. The local coercivity holds since, from the definition of the T operator (2.36) and the identity (2.66),

$$|\tilde{\xi}|^2_{\frac{1}{2},\partial K} = \inf_{\phi\in V_{\tilde{\xi}}(K)} |\phi|^2_{1,\mathcal{A},K} = |T\tilde{\lambda}_{\tilde{\xi}}|^2_{1,\mathcal{A},K} = \int_K \mathcal{A}\nabla T\tilde{\lambda}_{\tilde{\xi}}\cdot\nabla T\tilde{\lambda}_{\tilde{\xi}}\,dx = \langle\tilde{\lambda}_\xi,\tilde{\xi}\rangle_{\partial K} = g_K(\tilde{\xi},\tilde{\xi}), \quad (2.76)$$

and (2.73) follows. To show (2.74) assume that $\xi\in H^{1/2}_0(\mathcal{E}_h)$. Then:

$$g(\xi,\xi) = \sum_{K\in\mathcal{T}_h} g_K(\tilde{\xi},\tilde{\xi}) = \sum_{K\in\mathcal{T}_h} |\tilde{\xi}|^2_{\frac{1}{2},\partial K} = |\xi|^2_{1,\mathcal{E}_\mathcal{H}}, \quad (2.77)$$

from Lemma 2.3.1. Then (2.74) follows. To show (2.75), we obtain from (2.66):

$$g(\xi,\rho) = \sum_{K\in\mathcal{T}_\mathcal{H}} g_K(\xi,\rho) = \sum_{K\in\mathcal{T}_\mathcal{H}} \langle\tilde{\lambda}_\xi,\rho\rangle_{\partial K} = \sum_{K\in\mathcal{T}_\mathcal{H}} \int_K \mathcal{A}\nabla T\tilde{\lambda}_\xi\cdot\nabla T\tilde{\lambda}_\rho\,dx, \quad (2.78)$$

for all $\xi,\rho\in H^{1/2}_0(\mathcal{E}_h)$. Then, it follows from the Cauchy-Schwarz inequality and Lemma

2.3.1 that:

$$g(\xi,\rho) \leq \sum_{K\in\mathcal{T}_{\mathcal{H}}} |T\tilde{\lambda}_{\xi}|_{1,\mathcal{A},K}|T\tilde{\lambda}_{\rho}|_{1,\mathcal{A},K} = \sum_{K\in\mathcal{T}_{h}} |\xi|_{\frac{1}{2},\partial K}|\rho|_{\frac{1}{2},\partial K} \leq |\xi|_{\frac{1}{2},\mathcal{E}_{\mathcal{H}}}|\rho|_{\frac{1}{2},\mathcal{E}_{\mathcal{H}}}, \tag{2.79}$$

where we used in the last inequality the Holder's inequality, Proposition A.0.1. □

We get from the last Proposition 2.3.1 the following theorem.

Theorem 2.3.1. *The elliptical global problem* (2.47) *is well-posed.* —

3 The MH^{2}M method

3.1 Galerkin scheme

We now introduce the MH^{2}M method, which corresponds to the discrete version of the elliptical problem (2.47). Let $\mathcal{H}_\Gamma$, $\mathcal{H}_\Lambda$ and h be parameters. Assume that $\Gamma_{\mathcal{H}_\Gamma} \subset H_0^{1/2}(\mathcal{E}_\mathcal{H})$ and $\tilde{\Lambda}_{\mathcal{H}_\Lambda} \subset \tilde{\Lambda}$ are finite dimensional spaces. Let $\tilde{\Gamma}_{\mathcal{H}_\Gamma} := \Gamma_{\mathcal{H}_\Gamma} \cap \tilde{H}^{1/2}(\mathcal{E}_\mathcal{H})$ be the space of functions in $\Gamma_{\mathcal{H}_\Gamma}$ with zero average in each element boundary. For a finite dimensional space $U_\mathcal{H}$ over $\mathcal{T}_\mathcal{H}$ (or $\mathcal{E}_\mathcal{H}$) we use notation $U_{\mathcal{H},K} := U_\mathcal{H}|_K$.

The finite dimensional spaces defined above cannot be chosen arbitrarily, since it is necessary to ensure the injectivity of the operators T and G defined in (2.36) and (2.46). In other words, the following compatibility conditions must be satisfied. Let the finite dimensional space $\tilde{V}_h \subset \tilde{H}^1(\mathcal{T}_\mathcal{H})$ be such that,

$$\tilde{\mu}_{\mathcal{H}_\Lambda} \in \tilde{\Lambda}_{\mathcal{H}_\Lambda} \quad \text{and} \quad \langle \tilde{\mu}_{\mathcal{H}_\Lambda}, \tilde{v}_h \rangle_{\partial K} = 0,\ \forall\, \tilde{v}_h \in \tilde{V}_{h,K},\ \forall\, K \in \mathcal{T}_\mathcal{H} \quad \Longrightarrow \quad \tilde{\mu}_{\mathcal{H}_\Lambda} = 0. \tag{3.1}$$

We define $T_h : \tilde{\Lambda} \to \tilde{V}_h$ be such that, for a given $\tilde{\mu} \in \tilde{\Lambda}$,

$$\int_K \mathcal{A}\nabla(T_h\tilde{\mu}) \cdot \nabla\tilde{v}_h\, dx = \langle \tilde{\mu}, \tilde{v}_h \rangle_{\partial K}, \quad \forall\, \tilde{v}_h \in \tilde{V}_{h,K}. \tag{3.2}$$

Moreover, we have the discrete operator $\tilde{T}_h : L^2(\Omega) \to \tilde{V}_h$ be such that, for a given $f \in L^2(\Omega)$,

$$\int_K \mathcal{A}\nabla\tilde{T}_h f \cdot \nabla\tilde{v}_h\, dx = \int_k f\tilde{v}_h\, dx, \quad \forall\, \tilde{v}_h \in \tilde{V}_{h,K}. \tag{3.3}$$

for all $K \in \mathcal{T}_\mathcal{H}$. Now, a second conditions is that there exists a space $\tilde{\Lambda}_{\mathcal{H}_0} \subset \tilde{\Lambda}$ such that,

$$\tilde{\xi}_{\mathcal{H}_\Gamma} \in \tilde{\Gamma}_{\mathcal{H}_\Gamma} \quad \text{and} \quad \langle \tilde{\mu}_{\mathcal{H}_\Lambda}, \tilde{\xi}_{\mathcal{H}_\Gamma} \rangle_{\partial K} = 0,\ \forall\, \tilde{\mu}_{\mathcal{H}_\Lambda} \in \tilde{\Lambda}_{\mathcal{H}_0,K},\ \forall\, K \in \mathcal{T}_\mathcal{H} \quad \Longrightarrow \quad \tilde{\xi}_{\mathcal{H}_\Gamma} = 0. \tag{3.4}$$

Remark 3.1.1. An important characteristic of the three-field formulation is that it allows different discretizations for $\lambda_{\mathcal{H}_\Lambda}$ on each opposite sides of the edges. —

We assume henceforward that $\tilde{\Lambda}_{\mathcal{H}_\Lambda}$ is a finite dimensional space such that $\tilde{\Lambda}_{\mathcal{H}_0} \subset \tilde{\Lambda}_{\mathcal{H}_\Lambda} \subset \tilde{\Lambda}$. Let the operator $G_h : H_0^{1/2}(\mathcal{E}_\mathcal{H}) \to \tilde{\Lambda}_{\mathcal{H}_\Lambda}$, the discrete equivalent to G defined as follows: for $\phi \in H_0^{1/2}(\mathcal{E}_\mathcal{H})$, define $\tilde{\lambda}_\phi = G_h\phi$ such that, for $K \in \mathcal{T}_\mathcal{H}$,

$$\int_K \mathcal{A}\nabla(T_h\tilde{\lambda}_\phi) \cdot \nabla T_h\tilde{\mu}_{\mathcal{H}_\Lambda}\, dx = \langle \tilde{\mu}_{\mathcal{H}_\Lambda}, T_h\tilde{\lambda}_\phi\rangle_{\partial K} = \langle \tilde{\mu}_{\mathcal{H}_\Lambda}, \phi\rangle_{\partial K}, \quad \forall\, \tilde{\mu}_{\mathcal{H}_\Lambda} \in \tilde{\Lambda}_{\mathcal{H}_\Lambda}. \tag{3.5}$$

Figure 2 illustrates the relationships between the finite dimensional subspaces defined above, as well as the action of operators G_h and T_h as shown in equation (2.66). Follows from the second identity of the Lemma 2.3.2 (i) that the range $\mathcal{R}(T_h)$ consists of the discrete harmonic extensions of the functions $\tilde{\lambda}_{\mathcal{H}_\Lambda} \in \tilde{\Lambda}_{\mathcal{H}_\Lambda}$ through the T_h operator. The dashed lines connect spaces with the same dimension, accordingly the compatibility conditions (3.1) and (3.4).

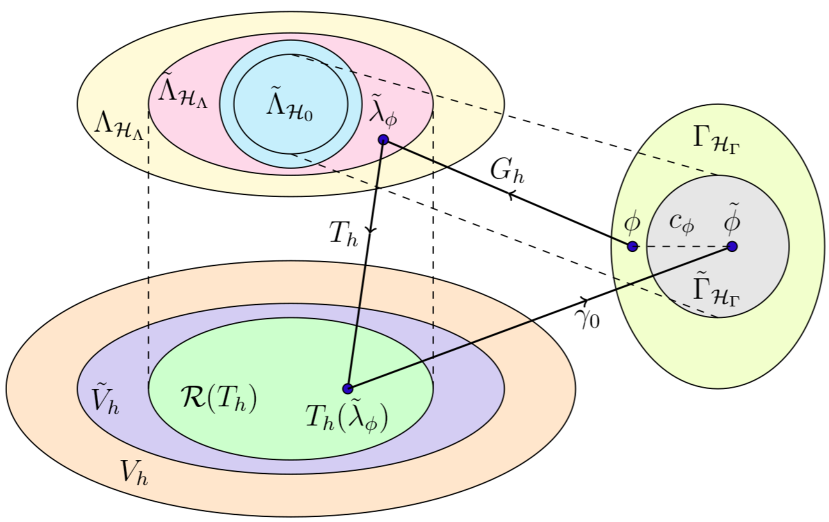

Figure 2 – Illustrative relationship between finite dimensional subspaces.

Here, $\gamma_0 : H^1(K) \to H^{1/2}(\partial K)$ is the trace operator defined as $\gamma_0(v) := v|_{\partial K}$. The Galerkin scheme related to the continous problem (2.47) is to find $\rho_{\mathcal{H}_\Gamma} \in \Gamma_{\mathcal{H}_\Gamma}$ such that:

$$\langle G_h\rho_{\mathcal{H}_\Gamma}, \xi_{\mathcal{H}_\Gamma}\rangle_{\mathcal{E}_\mathcal{H}} = -\langle \lambda^0, \xi_{\mathcal{H}_\Gamma}\rangle_{\mathcal{E}_\mathcal{H}} + \langle G_h\tilde{T}_hf, \xi_{\mathcal{H}_\Gamma}\rangle_{\mathcal{E}_\mathcal{H}}, \quad \forall\, \xi_{\mathcal{H}_\Gamma} \in \Gamma_{\mathcal{H}_\Gamma}. \tag{3.6}$$

Then, $\lambda_{\mathcal{H}_\Lambda}$ and u_h are given by:

$$\lambda_{\mathcal{H}_\Lambda} = \lambda^0 + G_h(\rho_{\mathcal{H}_\Gamma} - \tilde{T}_hf), \tag{3.7}$$
$$u_h = u^0 + T_hG_h\rho_{\mathcal{H}_\Gamma} + (I - T_hG_h)\tilde{T}_hf. \tag{3.8}$$

Remark 3.1.2. It is important to note that mass conservation to the approximated flux $\lambda_{\mathcal{H}_\Lambda} \in \Lambda_{\mathcal{H}_\Lambda}$ holds, since, for $K \in \mathcal{T}_\mathcal{H}$, applying Green Identity and taking $\lambda_{\mathcal{H}_\Lambda}|_{\partial K} = \mathcal{A}\nabla u_h \cdot \mathbf{n}^K$, we have:

$$-\int_K f\, dx = \int_K \nabla \cdot (\mathcal{A}\nabla u_h)\, dx = \langle \mathcal{A}\nabla u_h \cdot \mathbf{n}^K, 1\rangle_{\partial K} \tag{3.9}$$
$$= \langle \lambda_{\mathcal{H}_\Lambda}, 1\rangle_{\partial K} = \langle \lambda^0 + \tilde{\lambda}_{\mathcal{H}_\Lambda}, 1\rangle_{\partial K} = \langle \lambda^0, 1\rangle_{\partial K}.$$

Then, we obtain the first equation of (2.34) with $v_0 = 1$. —

We introduce a discretization of the bilinear forms $g(\cdot,\cdot)$ and $g_K(\cdot,\cdot)$ as in (2.50).

Definition 3.1.1. Let the bilinear forms $g_{h,K} : H^{1/2}(\partial K) \times H^{1/2}(\partial K) \to \mathbb{R}$, for $K \in \mathcal{T}_\mathcal{H}$, and $g_h : H_0^{1/2}(\mathcal{E}_\mathcal{H}) \times H_0^{1/2}(\mathcal{E}_\mathcal{H}) \to \mathbb{R}$ such that:

$$g_{h,K}(\xi,\phi) := \langle G_h\xi, \phi\rangle_{\partial K}, \quad \text{and} \quad g_h(\xi,\phi) := \sum_{K\in\mathcal{T}_\mathcal{H}} g_{h,K}(\xi,\phi), \tag{3.10}$$

for all $\xi, \phi \in H_0^{1/2}(\mathcal{E}_\mathcal{H})$. —

The next result shows that a discrete inf-sup condition follows from the compatibility condition (3.4).

Proposition 3.1.1. *For a fixed element $K \in \mathcal{T}_\mathcal{H}$ let the finite dimensional space $\tilde{\Lambda}_{\mathcal{H}_0} \subset \tilde{\Lambda}$ introduced in (3.4). Then, there exists a positive constant γ_K independent of $\mathcal{H}$ sucht that:*

$$\sup_{\tilde{\mu}\in\tilde{\Lambda}} \frac{\langle \tilde{\mu}, \xi_{\mathcal{H}_\Gamma}\rangle_{\partial K}}{|\tilde{\mu}|_{-\frac{1}{2},\partial K}} \le \gamma_K \sup_{\tilde{\mu}_{\mathcal{H}_\Lambda}\in\tilde{\Lambda}_{\mathcal{H}_0}} \frac{\langle \tilde{\mu}_{\mathcal{H}_\Lambda}, \xi_{\mathcal{H}_\Gamma}\rangle_{\partial K}}{|\tilde{\mu}_{\mathcal{H}_\Lambda}|_{-\frac{1}{2},\partial K}}, \tag{3.11}$$

for all $\xi_{\mathcal{H}_\Gamma} \in \Gamma_{\mathcal{H}_\Gamma}$. —

Proof. For a fixed $K \in \mathcal{T}_\mathcal{H}$, let $\hat{K} \subset \mathbb{R}^n$ a simplicial element and $T_K : \mathbb{R}^n \to \mathbb{R}^n$ an affine mapping such that $T_K\hat{x} := B_K\hat{x} + b_K$ for all $\hat{x} \in \mathbb{R}^n$, where $B_K \in \mathbb{R}^{n\times n}$ is a non singular matrix and $b_K \in \mathbb{R}^n$. Thus $K = T_K(\hat{K})$. Moreover we denote $v_{\hat{h}} := v_h \circ T_K$, where $\hat{h}$ is the diameter of $\hat{K}$ and $v_h : K \to \mathbb{R}$ is a functional. Here we use the same notation T_K for the mappping $T_K|_{\partial K}$. We gather from the equivalence of norms on finite dimensional spaces, the compatibility condition (3.4) and Corollary 5.1.1 [7, page 272] that there exists a positive constant $\gamma_{\hat{K}}$ such that:

$$|\xi_{\hat{\mathcal{H}}_\Gamma}|_{\Gamma_{\hat{\mathcal{H}}_\Gamma}} := \sup_{\tilde{\mu}_{\hat{\mathcal{H}}_\Lambda} \in \tilde{\Lambda}_{\hat{\mathcal{H}}_0}} \frac{\langle \tilde{\mu}_{\hat{\mathcal{H}}_\Lambda}, \xi_{\hat{\mathcal{H}}_\Gamma} \rangle_{\partial \hat{K}}}{|\tilde{\mu}_{\hat{\mathcal{H}}_\Lambda}|_{\Lambda_{\hat{\mathcal{H}}}}} \geq \gamma_{\hat{K}}^{-1} |\xi_{\hat{\mathcal{H}}_\Gamma}|_{\frac{1}{2},\partial \hat{K}}. \tag{3.12}$$

Thus, for $K \in \mathcal{T}_{\mathcal{H}}$,

$$\begin{aligned}
|\xi_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\partial K} &\leq C \|B_K^{-1}\| |\det\, B_K|^{\frac{1}{2}} |\xi_{\hat{\mathcal{H}}_\Gamma}|_{\frac{1}{2},\partial \hat{K}} && \text{(Lemma A.0.4)} \\
&\leq C \|B_K^{-1}\| |\det\, B_K|^{\frac{1}{2}} \gamma_{\hat{K}} |\xi_{\hat{\mathcal{H}}_\Gamma}|_{\Gamma_{\hat{\mathcal{H}}_\Gamma}} && (3.12) \\
&\leq C \gamma_{\hat{K}} \|B_K^{-1}\| |\det\, B_K|^{\frac{1}{2}} \|B_K\| |\det\, B_K|^{-\frac{1}{2}} |\xi_{\mathcal{H}_\Gamma}|_{\Gamma_{\mathcal{H}_\Gamma}} && \text{(Lemma A.0.4)} \\
&\leq C \gamma_{\hat{K}} \frac{\hat{\mathcal{H}}}{\rho_K} \frac{\mathcal{H}_K}{\hat{\rho}} |\xi_{\mathcal{H}_\Gamma}|_{\Gamma_{\mathcal{H}_\Gamma}} && \text{(Lemma A.0.5)} \\
&\leq \gamma_K \sup_{\tilde{\mu}_{\mathcal{H}_\Lambda} \in \tilde{\Lambda}_{\mathcal{H}_0}} \frac{\langle \tilde{\mu}_{\mathcal{H}_\Lambda}, \xi_{\mathcal{H}_\Gamma} \rangle_{\partial K}}{|\tilde{\mu}_{\mathcal{H}_\Lambda}|_{-\frac{1}{2},\partial K}}.
\end{aligned}$$

Where we have used the mesh regularity assumption on the last inequality. □

3.2 Main results

We start this section proving the coercicity and boundedness of the discrete bilinear forms (3.10).

Proposition 3.2.1. *Let g_K and g as introduced in (3.10) and assume that (3.4) holds. Then we get the following coercivity results:*

$$g_{h,K}(\tilde{\xi}_{\mathcal{H}_\Gamma}, \tilde{\xi}_{\mathcal{H}_\Gamma}) \geq \gamma_K^{-2} |\tilde{\xi}_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\partial K}, \quad \forall\, \tilde{\xi}_{\mathcal{H}_\Gamma} \in \tilde{\Gamma}_{\mathcal{H}_\Gamma,K}; \tag{3.13}$$

$$g_h(\xi_{\mathcal{H}_\Gamma}, \xi_{\mathcal{H}_\Gamma}) \geq \gamma^{-2} |\xi_{\mathcal{H}_\Gamma}|^2_{\frac{1}{2},\mathcal{E}_{\mathcal{H}}}, \quad \forall\, \xi_{\mathcal{H}_\Gamma} \in \Gamma_{\mathcal{H}_\Gamma}, \tag{3.14}$$

where the constant γ_K is the same as the inequality (3.11) and $\gamma = \max\{\gamma_K;\ K \in \mathcal{T}_{\mathcal{H}}\}$. Moreover, assume (3.1) holds. If T_h is injective with constant C_{T_K} on $K \in \mathcal{T}_{\mathcal{H}}$, that is,

$$|T_h \tilde{\mu}_{\mathcal{H}_\Lambda}|_{1,\mathcal{A},K} \geq C_{T_K} |\tilde{\mu}_{\mathcal{H}_\Lambda}|_{-\frac{1}{2},\partial K}, \quad \forall\, \tilde{\mu}_{\mathcal{H}_\Lambda} \in \tilde{\Lambda}_{\mathcal{H}_\Lambda}, \tag{3.15}$$

then

$$g_{h,K}(\xi_{\mathcal{H}_\Gamma},\rho_{\mathcal{H}_\Gamma}) \leq C_{T_K}^{-1}|\xi_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\partial K}|\rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\partial K}, \quad \forall\, \xi_{\mathcal{H}_\Gamma},\rho_{\mathcal{H}_\Gamma} \in \Gamma_{\mathcal{H}_\Gamma,K}; \tag{3.16}$$

$$g_{h}(\xi_{\mathcal{H}_\Gamma},\rho_{\mathcal{H}_\Gamma}) \leq C_{T_\mathcal{H}}^{-1}|\xi_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}}|\rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}}, \quad \forall\, \xi_{\mathcal{H}_\Gamma},\rho_{\mathcal{H}_\Gamma} \in \Gamma_{\mathcal{H}_\Gamma}, \tag{3.17}$$

where $C_{T_\mathcal{H}} := \inf\{C_{T_K},\ K \in \mathcal{T}_\mathcal{H}\}$. —

Proof. For $\tilde{\xi}_{\mathcal{H}_\Gamma} \in \tilde{\Gamma}_{\mathcal{H}_\Gamma,K}$, denote $\tilde{\lambda}_{\tilde{\xi}} := G_h\tilde{\xi}_{\mathcal{H}_\Gamma}$. The local coercivity (3.13) holds since, from discrete inf-sup condition (3.11), we have:

$$|\tilde{\xi}_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\partial K} = \sup_{\tilde{\mu}\in\tilde{\Lambda}} \frac{\langle \tilde{\mu}, \tilde{\xi}_{\mathcal{H}_\Gamma}\rangle_{\partial K}}{|\tilde{\mu}|_{-\frac{1}{2},\partial K}} \leq \gamma_K \sup_{\tilde{\mu}_{\mathcal{H}_\Lambda}\in\tilde{\Lambda}_{\mathcal{H}_\Lambda}} \frac{\langle \tilde{\mu}_{\mathcal{H}_\Lambda}, \tilde{\xi}_{\mathcal{H}_\Gamma}\rangle_{\partial K}}{|\tilde{\mu}_{\mathcal{H}_\Lambda}|_{-\frac{1}{2},\partial K}}. \tag{3.18}$$

From (3.5), the Cauchy-Schwarz inequality and Lemma 2.3.2(i), we get:

$$|\tilde{\xi}_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\partial K} \leq \gamma_K \sup_{\tilde{\mu}_{\mathcal{H}_\Lambda}\in\tilde{\Lambda}_{\mathcal{H}_\Lambda}} \frac{\langle \tilde{\mu}_{\mathcal{H}_\Lambda}, T_h\tilde{\lambda}_\xi\rangle_{\partial K}}{|\tilde{\mu}_{\mathcal{H}_\Lambda}|_{-\frac{1}{2},\partial K}} \leq \gamma_K |T_h\tilde{\lambda}_\xi|_{\frac{1}{2},\partial K} = \gamma_K|T_h\tilde{\lambda}_\xi|_{1,\mathcal{A},K}. \tag{3.19}$$

By the characterization of the operator T_h (3.2) and again by (2.66), we obtain

$$|\tilde{\xi}_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\partial K} \leq \gamma_K \langle \tilde{\lambda}_\xi, T_h\tilde{\lambda}_\xi\rangle_{\partial K}^{1/2} = \gamma_K \left[g_{h,K}(\tilde{\xi}_{\mathcal{H}_\Gamma},\tilde{\xi}_{\mathcal{H}_\Gamma})\right]^{1/2}, \tag{3.20}$$

and (3.13) holds.

Next, to verify (3.14), assume that $\xi_{\mathcal{H}_\Gamma} \in \Gamma_{\mathcal{H}_\Gamma}$. Then, $\xi_{\mathcal{H}_\Gamma} = \tilde{\xi}_{\mathcal{H}_\Gamma} + \xi^0_{\mathcal{H}_\Gamma}$, where $\tilde{\xi}_{\mathcal{H}_\Gamma}|_{\partial K} \in \tilde{\Gamma}_{\mathcal{H}_\Gamma}$ and $\xi^0_{\mathcal{H}_\Gamma}$ is constant on ∂K, $\forall\, K \in \mathcal{T}_\mathcal{H}$. Then, we gather from definition (3.10), the local coerciveness (3.13) and the identity (2.55), that:

$$g_h(\xi_{\mathcal{H}_\Gamma},\xi_{\mathcal{H}_\Gamma}) = \sum_{K\in\mathcal{T}_\mathcal{H}} g_{h,K}(\xi_{\mathcal{H}_\Gamma},\xi_{\mathcal{H}_\Gamma}) \geq \sum_{K\in\mathcal{T}_\mathcal{H}} \gamma_K^{-2}|\tilde{\xi}_{\mathcal{H}_\Gamma}|^2_{\frac{1}{2},\partial K} \tag{3.21}$$

$$\geq \gamma^{-2}\sum_{K\in\mathcal{T}_\mathcal{H}} |\tilde{\xi}_{\mathcal{H}_\Gamma}|^2_{\frac{1}{2},\partial K} = \gamma^{-2}\sum_{K\in\mathcal{T}_\mathcal{H}} |\xi_{\mathcal{H}_\Gamma}|^2_{\frac{1}{2},\partial K} = \gamma^{-2}|\xi_{\mathcal{H}_\Gamma}|^2_{\frac{1}{2},\mathcal{E}_\mathcal{H}}. \tag{3.22}$$

To prove continuity, let us show the injectivity of T_h. Just as we show inequality (3.11) from (3.4), it is possible to verify using similar arguments that follows from the compatibility condition (3.1) that there exists a constant $C_{T_K} > 0$ dependents on the shape-regularity of $K \in \mathcal{T}_{\mathcal{H}}$ such that:

$$|\tilde{\mu}_{\mathcal{H}_\Lambda}|_{-\frac{1}{2},\partial K} \leq C_{T_K} \sup_{\tilde{v}_h \in \tilde{V}_{h,K}} \frac{\langle \tilde{\mu}_{\mathcal{H}_\Lambda}, \tilde{v}_h \rangle_{\partial K}}{|\tilde{v}_h|_{\frac{1}{2},\partial K}}. \tag{3.23}$$

Then, we get from (3.23) and the Cauchy-Schwarz inequality that:

$$C_{T_K}^{-1} |\tilde{\mu}_{\mathcal{H}_\Lambda}|_{-\frac{1}{2},\partial K} \leq \sup_{\tilde{v}_h \in \tilde{V}_{h,K}} \frac{\langle \tilde{\mu}_{\mathcal{H}_\Lambda}, \tilde{v}_h \rangle_{\partial K}}{|\tilde{v}_h|_{\frac{1}{2},\partial K}} = \sup_{\tilde{v}_h \in \tilde{V}_{h,K}} \frac{\int_K \mathcal{A} \nabla T_h \tilde{\mu}_{\mathcal{H}_\Lambda} \cdot \nabla \tilde{v}_h \, dx}{|\tilde{v}_h|_{1,\mathcal{A},K}} = |T_h \tilde{\mu}_{\mathcal{H}_\Lambda}|_{1,\mathcal{A},K}. \tag{3.24}$$

Then (3.15) holds. We can now deduce the following inequality:

$$C_{T_K} |\tilde{\lambda}_\xi|^2_{-\frac{1}{2},\partial K} \leq |T_h \tilde{\lambda}_\xi|^2_{1,\mathcal{A},K} = \langle \tilde{\lambda}_\xi, T_h \tilde{\lambda}_\xi \rangle_{\partial K}. \tag{3.25}$$

Using (3.25), follows from (3.5) and Lemma 2.3.2 (iii) that,

$$C_{T_K} |\tilde{\lambda}_\xi|^2_{-\frac{1}{2},\partial K} \leq \langle \tilde{\lambda}_\xi, \tilde{\xi}_{\mathcal{H}_\Gamma} \rangle_{\partial K} = \langle \tilde{\lambda}_\xi, \xi_{\mathcal{H}_\Gamma} \rangle_{\partial K} \leq |\tilde{\lambda}_\xi|_{-\frac{1}{2},\partial K} |\xi_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\partial K}. \tag{3.26}$$

Therefore,

$$|\tilde{\lambda}_\xi|_{-\frac{1}{2},\partial K} \leq C_{T_K}^{-1} |\xi_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\partial K}. \tag{3.27}$$

We can now estimate (3.16) and (3.17). From (3.27) we get:

$$g_{h,K}(\xi_{\mathcal{H}_\Gamma}, \rho_{\mathcal{H}_\Gamma}) = \langle \tilde{\lambda}_\xi, \rho_{\mathcal{H}_\Gamma} \rangle_{\partial K} \leq |\tilde{\lambda}_\xi|_{-\frac{1}{2},\partial K} |\rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\partial K} \leq C_{T_K}^{-1} |\xi_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\partial K} |\rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\partial K}. \tag{3.28}$$

Finally, from the Holder's Inequality (Proposition A.0.1), we have

$$g_h(\xi, \rho) \leq \sum_{K \in \mathcal{T}_{\mathcal{H}}} C_{T_K}^{-1} |\xi_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\partial K} |\rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\partial K} \leq C_{T_{\mathcal{H}}}^{-1} |\xi_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_{\mathcal{H}}} |\rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_{\mathcal{H}}}. \tag{3.29}$$

which completes the proof. □

The next Lemma estimates $G - G_h$ based on the First Strang Lemma [13]. We see that convergence depends on the approximability property of the $\tilde{\Lambda}_{\mathcal{H}_\Lambda}$ space and on the consistency of the method at the second level.

Lemma 3.2.1. *Let* $\phi \in H_0^{1/2}(\mathcal{E}_{\mathcal{H}})$ *and assume* (3.15) *holds. Then:*

$$|G\phi - G_h\phi|_\Lambda \le E(\phi), \tag{3.30}$$

where,

$$E(\phi) := \inf_{\tilde{\mu}_{\mathcal{H}_\Lambda} \in \tilde{\Lambda}_{\mathcal{H}_\Lambda}} \left\{ \left(C_T^{-1} + 1 \right) |G\phi - \tilde{\mu}_{\mathcal{H}_\Lambda}|_\Lambda + C_T^{-1} S(\tilde{\mu}_{\mathcal{H}_\Lambda}) \right\}.$$

and

$$S(\tilde{\mu}_{\mathcal{H}_\Lambda}) := \sup_{\tilde{\eta}_{\mathcal{H}_\Lambda} \in \tilde{\Lambda}_{\mathcal{H}_\Lambda}} \frac{\langle \tilde{\mu}_{\mathcal{H}_\Lambda}, T\tilde{\eta}_{\mathcal{H}_\Lambda} \rangle_{\mathcal{E}_{\mathcal{H}}} - \langle \tilde{\mu}_{\mathcal{H}_\Lambda}, T_h\tilde{\eta}_{\mathcal{H}_\Lambda} \rangle_{\mathcal{E}_{\mathcal{H}}}}{|\tilde{\eta}_{\mathcal{H}_\Lambda}|_\Lambda}$$

—

Proof. Consider the bilinear forms $a_T : \tilde{\Lambda} \times \tilde{\Lambda} \to \mathbb{R}$ and $a_{T_h} : \tilde{\Lambda}_{\mathcal{H}_\Lambda} \times \tilde{\Lambda}_{\mathcal{H}_\Lambda} \to \mathbb{R}$ defined as:

$$a_T(\tilde{\lambda}, \tilde{\mu}) := \langle \tilde{\lambda}, T\tilde{\mu} \rangle_{\mathcal{E}_{\mathcal{H}}} \quad \text{and} \quad a_{T_h}(\tilde{\lambda}_{\mathcal{H}_\Lambda}, \tilde{\mu}_{\mathcal{H}_\Lambda}) := \langle \tilde{\lambda}_{\mathcal{H}_\Lambda}, T_h\tilde{\mu}_{\mathcal{H}_\Lambda} \rangle_{\mathcal{E}_{\mathcal{H}}}, \tag{3.31}$$

for all $\tilde{\lambda}, \tilde{\mu} \in \tilde{\Lambda}$ and for all $\tilde{\lambda}_{\mathcal{H}_\Lambda}, \tilde{\mu}_{\mathcal{H}_\Lambda} \in \tilde{\Lambda}_{\mathcal{H}_\Lambda}$. We get from the injectivity of T_h operator (3.15) and its definition (3.2) the following coercivity result:

$$\begin{aligned} C_T |\tilde{\mu}_{\mathcal{H}_\Lambda}|_\Lambda^2 &\le |T_h\tilde{\mu}_{\mathcal{H}_\Lambda}|_{1,\mathcal{A},\Omega}^2 = (\mathcal{A}\nabla T_h\tilde{\mu}_{\mathcal{H}_\Lambda}, \nabla T_h\tilde{\mu}_{\mathcal{H}_\Lambda})_{\mathcal{T}_{\mathcal{H}}} \\ &= \langle \tilde{\mu}_{\mathcal{H}_\Lambda}, T_h\tilde{\mu}_{\mathcal{H}_\Lambda} \rangle_{\mathcal{E}_{\mathcal{H}}} = a_{T_h}(\tilde{\mu}_{\mathcal{H}_\Lambda}, \tilde{\mu}_{\mathcal{H}_\Lambda}), \end{aligned} \tag{3.32}$$

for all $\tilde{\mu}_{\mathcal{H}_\Lambda} \in \tilde{\Lambda}_{\mathcal{H}_\Lambda}$. Moreover, by Lemma 2.3.2 (i) and (iii), and Holder's inequality (Proposition A.0.1) we obtain:

$$\begin{aligned} a_T(\tilde{\mu}, \tilde{\eta}) &= \langle \tilde{\mu}, T\tilde{\eta} \rangle_{\mathcal{E}_{\mathcal{H}}} = \sum_{K \in \mathcal{T}_{\mathcal{H}}} \langle \tilde{\mu}, T\tilde{\eta} \rangle_{\partial K} \\ &\le \sum_{K \in \mathcal{T}_{\mathcal{H}}} |\tilde{\mu}|_{-\frac{1}{2},\partial K} |T\tilde{\eta}|_{\frac{1}{2},\partial K} = \sum_{K \in \mathcal{T}_{\mathcal{H}}} |\tilde{\mu}|_{-\frac{1}{2},\partial K} |\tilde{\eta}|_{-\frac{1}{2},\partial K} \\ &\le |\tilde{\mu}|_\Lambda |\tilde{\eta}|_\Lambda \end{aligned} \tag{3.33}$$

for all $\tilde{\mu}, \tilde{\eta} \in \tilde{\Lambda}$. In what follows we denote $\tilde{\lambda} = G\phi$ and $\tilde{\lambda}_{\mathcal{H}_\Lambda} = G_h\phi$, for $\phi \in H^{1/2}(\mathcal{E}_\mathcal{H})$. Note that, by simmetry of the T and T_h operators, and by definitions of G and G_h, we have:

$$a_T(\tilde{\lambda}, \tilde{\mu}_{\mathcal{H}_\Lambda}) = \langle \tilde{\lambda}, T\tilde{\mu}_{\mathcal{H}_\Lambda}\rangle_{\mathcal{E}_\mathcal{H}} = \langle \tilde{\mu}_{\mathcal{H}_\Lambda}, T\tilde{\lambda}\rangle_{\mathcal{E}_\mathcal{H}} = \langle \tilde{\mu}_{\mathcal{H}_\Lambda}, \phi\rangle_{\mathcal{E}_\mathcal{H}}; \tag{3.34}$$

and,

$$a_{T_h}(\tilde{\lambda}_{\mathcal{H}_\Lambda}, \tilde{\mu}_{\mathcal{H}_\Lambda}) = \langle \tilde{\lambda}_{\mathcal{H}_\Lambda}, T_h\tilde{\mu}_{\mathcal{H}_\Lambda}\rangle_{\mathcal{E}_\mathcal{H}} = \langle \tilde{\mu}_{\mathcal{H}_\Lambda}, T_h\tilde{\lambda}_{\mathcal{H}_\Lambda}\rangle_{\mathcal{E}_\mathcal{H}} = \langle \tilde{\mu}_{\mathcal{H}_\Lambda}, \phi\rangle_{\mathcal{E}_\mathcal{H}}. \tag{3.35}$$

Hence,

$$a_T(\tilde{\lambda}, \tilde{\mu}_{\mathcal{H}_\Lambda}) = a_{T_h}(\tilde{\lambda}_{\mathcal{H}_\Lambda}, \tilde{\mu}_{\mathcal{H}_\Lambda}) = \langle \tilde{\mu}_{\mathcal{H}_\Lambda}, \phi\rangle_{\mathcal{E}_\mathcal{H}}. \tag{3.36}$$

Finally, the estimate (3.30) holds, since, by the coercivity of a_{T_h} (3.32) and using the simmetry of the operators T and T_h, we get:

$$\begin{aligned}
C_T|\tilde{\lambda}_{\mathcal{H}_\Lambda} - \tilde{\mu}_{\mathcal{H}_\Lambda}|_\Lambda^2 &\leq a_{T_h}(\tilde{\lambda}_{\mathcal{H}_\Lambda} - \tilde{\mu}_{\mathcal{H}_\Lambda}, \tilde{\lambda}_{\mathcal{H}_\Lambda} - \tilde{\mu}_{\mathcal{H}_\Lambda}) && \text{by (3.32)}\\
&= a_{T_h}(\tilde{\lambda}_{\mathcal{H}_\Lambda}, \tilde{\lambda}_{\mathcal{H}_\Lambda} - \tilde{\mu}_{\mathcal{H}_\Lambda}) - a_{T_h}(\tilde{\mu}_{\mathcal{H}_\Lambda}, \tilde{\lambda}_{\mathcal{H}_\Lambda} - \tilde{\mu}_{\mathcal{H}_\Lambda})\\
&+ a_T(\tilde{\mu}_{\mathcal{H}_\Lambda}, \tilde{\lambda}_{\mathcal{H}_\Lambda} - \tilde{\mu}_{\mathcal{H}_\Lambda}) - a_T(\tilde{\mu}_{\mathcal{H}_\Lambda}, \tilde{\lambda}_{\mathcal{H}_\Lambda} - \tilde{\mu}_{\mathcal{H}_\Lambda})\\
&= a_T(\tilde{\lambda} - \tilde{\mu}_{\mathcal{H}_\Lambda}, \tilde{\lambda}_{\mathcal{H}_\Lambda} - \tilde{\mu}_{\mathcal{H}_\Lambda}) + a_T(\tilde{\mu}_{\mathcal{H}_\Lambda}, \tilde{\lambda}_{\mathcal{H}_\Lambda} - \tilde{\mu}_{\mathcal{H}_\Lambda})\\
&- a_{T_h}(\tilde{\mu}_{\mathcal{H}_\Lambda}, \tilde{\lambda}_{\mathcal{H}_\Lambda} - \tilde{\mu}_{\mathcal{H}_\Lambda}) && \text{by (3.36)}\\
&\leq |\tilde{\lambda}_{\mathcal{H}_\Lambda} - \tilde{\mu}_{\mathcal{H}_\Lambda}|_\Lambda \left(|\tilde{\lambda} - \tilde{\mu}_{\mathcal{H}_\Lambda}|_\Lambda + \sup_{\tilde{\eta}_{\mathcal{H}_\Lambda} \in \tilde{\Lambda}_{\mathcal{H}_\Lambda}} \frac{a_T(\tilde{\eta}_{\mathcal{H}_\Lambda}, \tilde{\mu}_{\mathcal{H}_\Lambda}) - a_{T_h}(\tilde{\eta}_{\mathcal{H}_\Lambda}, \tilde{\mu}_{\mathcal{H}_\Lambda})}{|\tilde{\eta}_{\mathcal{H}_\Lambda}|_\Lambda} \right).
\end{aligned}$$

Therefore,

$$C_T|\tilde{\lambda}_{\mathcal{H}_\Lambda} - \tilde{\mu}_{\mathcal{H}_\Lambda}|_\Lambda \leq |\tilde{\lambda} - \tilde{\mu}_{\mathcal{H}_\Lambda}|_\Lambda + \sup_{\tilde{\eta}_{\mathcal{H}_\Lambda} \in \tilde{\Lambda}_{\mathcal{H}_\Lambda}} \frac{a_T(\tilde{\eta}_{\mathcal{H}_\Lambda}, \tilde{\mu}_{\mathcal{H}_\Lambda}) - a_{T_h}(\tilde{\eta}_{\mathcal{H}_\Lambda}, \tilde{\mu}_{\mathcal{H}_\Lambda})}{|\tilde{\eta}_{\mathcal{H}_\Lambda}|_\Lambda}. \tag{3.37}$$

Using the triangular inequality,

$$|\tilde{\lambda}_\mathcal{H} - \tilde{\lambda}|_\Lambda \leq |\tilde{\lambda}_\mathcal{H} - \tilde{\mu}_\mathcal{H}|_\Lambda + |\tilde{\lambda} - \tilde{\mu}_\mathcal{H}|_\Lambda, \tag{3.38}$$

we get

$$|\tilde{\lambda}_{\mathcal{H}_\Lambda} - \tilde{\lambda}|_\Lambda \leq \inf_{\tilde{\mu}_{\mathcal{H}_\Lambda} \in \tilde{\Lambda}_{\mathcal{H}_\Lambda}} \left\{ (C_T^{-1} + 1)|\tilde{\lambda} - \tilde{\mu}_{\mathcal{H}_\Lambda}|_\Lambda + C_T^{-1} S(\tilde{\mu}_{\mathcal{H}_\Lambda}) \right\}, \tag{3.39}$$

and

$$S(\tilde{\mu}_{\mathcal{H}_\Lambda}) := \sup_{\tilde{\eta}_{\mathcal{H}_\Lambda} \in \tilde{\Lambda}_{\mathcal{H}_\Lambda}} \frac{a_T(\tilde{\eta}_{\mathcal{H}_\Lambda}, \tilde{\mu}_{\mathcal{H}_\Lambda}) - a_{T_h}(\tilde{\eta}_{\mathcal{H}_\Lambda}, \tilde{\mu}_{\mathcal{H}_\Lambda})}{|\tilde{\eta}_{\mathcal{H}_\Lambda}|_\Lambda}$$

which completes the proof. □

In the next theorem we derive estimates for the approximation errors $\rho - \rho_{\mathcal{H}_\Gamma}$, $\lambda - \lambda_{\mathcal{H}_\Lambda}$ and $u - u_h$.

Theorem 3.2.1. *Let γ and $\gamma_{\mathcal{H}_\Gamma}$ solve (2.47) and (3.6); u and u_h be as presented in (2.48) and (3.8); and λ and $\lambda_{\mathcal{H}_\Lambda}$ be as in (2.49) and (3.7). Then, problem (3.6) is well-posed and:*

$$|\rho - \rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} \leq \inf_{\phi_{\mathcal{H}_\Gamma} \in \Gamma_\mathcal{H}} \left\{ 2|\rho - \phi_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} + E(\phi_{\mathcal{H}_\Gamma}) \right\} + |(\tilde{T} - \tilde{T}_h)f|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} + E(\tilde{T}_h f). \tag{3.40}$$

Moreover,

$$|\lambda - \lambda_{\mathcal{H}_\Lambda}|_\Lambda \leq |\rho - \rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} + E(\rho_{\mathcal{H}_\Gamma}) + |(\tilde{T} - \tilde{T}_h)f|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} + E(\tilde{T}_h f); \tag{3.41}$$

$$|u - u_h|_{1,\mathcal{A},\mathcal{T}_\mathcal{H}} \leq |\lambda - \lambda_{\mathcal{H}_\Lambda}|_\Lambda + \|T - T_h\| |\tilde{\lambda}_{\mathcal{H}_\Lambda}|_\Lambda + |(\tilde{T} - \tilde{T}_h)f|_{\frac{1}{2},\mathcal{E}_\mathcal{H}}. \tag{3.42}$$

Finally, the following weak continuity

$$\langle \mu_{\mathcal{H}_\Lambda}, u_h - \rho_{\mathcal{H}_\Gamma} \rangle_{\mathcal{E}_\mathcal{H}} = 0, \quad \forall\, \mu_{\mathcal{H}_\Lambda} \in \Lambda_{\mathcal{H}_\Lambda}, \tag{3.43}$$

holds. —

Proof. It follows from Proposition 3.2.1 that the bilinear form g_h is bounded and coercive on the space $\Gamma_{\mathcal{H}_\Gamma}$. Then, the Lax-Milgran Lemma ensures the well-posedness of (3.6). Let us derive (3.40). From the triangular inequality,

$$|\rho - \rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} \leq |\rho - \phi_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} + |\phi_{\mathcal{H}_\Gamma} - \rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}}. \tag{3.44}$$

Since $\Gamma_{\mathcal{H}_\Gamma} \subset H_0^{1/2}(\mathcal{E}_{\mathcal{H}})$, we gather from the continous global problem (2.47) and its discrete version (3.6) that

$$\langle G_h \rho_{\mathcal{H}_\Gamma} - G\rho, \xi_{\mathcal{H}_\Gamma} \rangle_{\mathcal{E}_{\mathcal{H}}} = \langle (G_h \tilde{T}_h - G\tilde{T}) f, \xi_{\mathcal{H}_\Gamma} \rangle_{\mathcal{E}_{\mathcal{H}}}, \quad \forall\, \xi_{\mathcal{H}_\Gamma} \in \Gamma_{\mathcal{H}_\Gamma}. \tag{3.45}$$

By the coercivity of the bilinear form g_h (3.14) and taking $\xi_{\mathcal{H}_\Gamma} = \phi_{\mathcal{H}_\Gamma} - \rho_{\mathcal{H}_\Gamma}$ in (3.45), we have:

$$\begin{aligned}
&\gamma^{-2} |\phi_{\mathcal{H}_\Gamma} - \rho_{\mathcal{H}_\Gamma}|^2_{\frac{1}{2},\mathcal{E}_{\mathcal{H}}} \leq g_{\mathcal{H}}(\phi_{\mathcal{H}_\Gamma} - \rho_{\mathcal{H}_\Gamma}, \phi_{\mathcal{H}_\Gamma} - \rho_{\mathcal{H}_\Gamma}) \\
&= \langle G_h \phi_{\mathcal{H}_\Gamma} - G_h \rho_{\mathcal{H}_\Gamma} + (G_h \rho_{\mathcal{H}_\Gamma} - G\rho), \phi_{\mathcal{H}_\Gamma} - \rho_{\mathcal{H}_\Gamma} \rangle_{\mathcal{E}_{\mathcal{H}}} + \langle (G\tilde{T} - G_h \tilde{T}_h) f, \phi_{\mathcal{H}_\Gamma} - \rho_{\mathcal{H}_\Gamma} \rangle_{\mathcal{E}_{\mathcal{H}}} \\
&= \langle G_h \phi_{\mathcal{H}_\Gamma} - G\rho, \phi_{\mathcal{H}_\Gamma} - \rho_{\mathcal{H}_\Gamma} \rangle_{\mathcal{E}_{\mathcal{H}}} + \langle (G\tilde{T} - G_h \tilde{T}_h) f, \phi_{\mathcal{H}_\Gamma} - \rho_{\mathcal{H}_\Gamma} \rangle_{\mathcal{E}_{\mathcal{H}}} \\
&= \langle (G_h - G)\phi_{\mathcal{H}_\Gamma}, \phi_{\mathcal{H}_\Gamma} - \rho_{\mathcal{H}_\Gamma} \rangle_{\mathcal{E}_{\mathcal{H}}} + \langle G(\phi_{\mathcal{H}_\Gamma} - \rho), \phi_{\mathcal{H}_\Gamma} - \rho_{\mathcal{H}_\Gamma} \rangle_{\mathcal{E}_{\mathcal{H}}} \\
&+ \langle (G\tilde{T} - G_h \tilde{T}_h) f, \phi_{\mathcal{H}_\Gamma} - \rho_{\mathcal{H}_\Gamma} \rangle_{\mathcal{E}_{\mathcal{H}}} \\
&\leq \left(E(\phi_{\mathcal{H}_\Gamma}) + |\phi_{\mathcal{H}_\Gamma} - \rho|_{\frac{1}{2},\mathcal{E}_{\mathcal{H}}} + |(G\tilde{T} - G_h \tilde{T}_h) f|_\Lambda \right) |\phi_{\mathcal{H}_\Gamma} - \rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_{\mathcal{H}}},
\end{aligned} \tag{3.46}$$

were we have used the estimate (3.30) and Lemma 2.3.2-*ii* in the last inequality. It follows from the triangular inequality, Lemma 2.3.2-*i* and estimate (3.30) that

$$\begin{aligned}
|(G\tilde{T} - G_h \tilde{T}_h) f|_\Lambda &\leq |(G\tilde{T} - G\tilde{T}_h) f|_\Lambda + |(G\tilde{T}_h - G_h \tilde{T}_h) f|_\Lambda \\
&\leq |(\tilde{T} - \tilde{T}_h) f|_{\frac{1}{2},\mathcal{E}_{\mathcal{H}}} + E(\tilde{T}_h f).
\end{aligned} \tag{3.47}$$

Therefore, (3.40) follows from (3.44), (3.46) and (3.47).

Let us deduce (3.41). We gather from the continuous global problem (2.47) and its discrete version (3.6) that:

$$\lambda - \lambda_{\mathcal{H}_\Lambda} = \tilde{\lambda} - \tilde{\lambda}_{\mathcal{H}_\Lambda} = (G\rho - G_h \rho_{\mathcal{H}_\Gamma}) + (G_h \tilde{T}_h f - G\tilde{T} f). \tag{3.48}$$

To estimate the first term on the right hand side of (3.48) we proceed as follows:

$$\begin{aligned}
\langle G\rho - G_h \rho_{\mathcal{H}_\Gamma}, \tilde{\phi} \rangle_{\mathcal{E}_{\mathcal{H}}} &= \langle G\rho - G\rho_{\mathcal{H}_\Gamma}, \tilde{\phi} \rangle_{\mathcal{E}_{\mathcal{H}}} + \langle G\rho_{\mathcal{H}_\Gamma} - G_h \rho_{\mathcal{H}_\Gamma}, \tilde{\phi} \rangle_{\mathcal{E}_{\mathcal{H}}} \\
&\leq \left(|\rho - \rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_{\mathcal{H}}} + |G\rho_{\mathcal{H}_\Gamma} - G_h \rho_{\mathcal{H}_\Gamma}|_\Lambda \right) |\tilde{\phi}|_{\frac{1}{2},\mathcal{E}_{\mathcal{H}}} \\
&\leq \left(|\rho - \rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_{\mathcal{H}}} + E(\rho_{\mathcal{H}_\Gamma}) \right) |\tilde{\phi}|_{\frac{1}{2},\mathcal{E}_{\mathcal{H}}}.
\end{aligned} \tag{3.49}$$

Therefore,

$$|G\rho - G_h\rho_{\mathcal{H}_\Gamma}|_\Lambda = \sup_{\tilde{\phi}\in\tilde{H}_0^{1/2}(\mathcal{E}_\mathcal{H})} \frac{\langle G\rho - G_h\rho_{\mathcal{H}_\Gamma}, \tilde{\phi}\rangle_{\mathcal{E}_\mathcal{H}}}{|\tilde{\phi}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}}} \leq |\rho - \rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} + E(\rho_{\mathcal{H}_\Gamma}). \tag{3.50}$$

(3.51)

Thus, the estimate (3.41) holds from the identity (3.48), (3.47) and (3.50).

In the sequel, since:

$$\tilde{u} - \tilde{u}_h = T\tilde{\lambda} - T_h\tilde{\lambda}_{\mathcal{H}_\Lambda} + \tilde{T}f - \tilde{T}_hf \tag{3.52}$$

$$= T\tilde{\lambda} - (T\tilde{\lambda}_{\mathcal{H}_\Lambda} + T\tilde{\lambda}_{\mathcal{H}_\Lambda}) - T_h\tilde{\lambda}_{\mathcal{H}_\Lambda} + \tilde{T}f - \tilde{T}_hf \tag{3.53}$$

$$= T(\tilde{\lambda} - \tilde{\lambda}_{\mathcal{H}_\Lambda}) + (T - T_h)\tilde{\lambda}_{\mathcal{H}_\Lambda} + (\tilde{T} - \tilde{T}_h)f, \tag{3.54}$$

we obtain by Lemma 2.3.2 (i) that:

$$|\tilde{u} - \tilde{u}_h|_{1,\mathcal{A},\mathcal{T}_\mathcal{H}} \leq |\tilde{\lambda} - \tilde{\lambda}_{\mathcal{H}_\Lambda}|_\Lambda + \|T - T_h\||\tilde{\lambda}_{\mathcal{H}_\Lambda}|_\Lambda + |(\tilde{T} - \tilde{T}_h)f|_{\frac{1}{2},\mathcal{E}_\mathcal{H}}. \tag{3.55}$$

Since u^0 and u_h^0 are piecewise constants, then $|u^0 - u_h^0|_{1,\mathcal{A},\mathcal{T}_\mathcal{H}} = 0$, so that (3.42) holds from (3.48). We get from the second equation in (2.34) that (3.43) holds for $\mu^0 \in \Lambda^0$. For $\tilde{\mu}_{\mathcal{H}_\Lambda} \in \Lambda_{\mathcal{H}_\Lambda}$ we have

$$\langle\tilde{\mu}_{\mathcal{H}_\Lambda}, u_h\rangle_{\mathcal{E}_\mathcal{H}} = \langle\tilde{\mu}_{\mathcal{H}_\Lambda}, \tilde{u}_h\rangle_{\mathcal{E}_\mathcal{H}} = \langle\tilde{\mu}_{\mathcal{H}_\Lambda}, T_hG_h\rho_{\mathcal{H}_\Gamma} + (I - T_hG_h)\tilde{T}_hf\rangle_{\mathcal{E}_\mathcal{H}} = \langle\tilde{\mu}_{\mathcal{H}_\Lambda}, \rho_{\mathcal{H}_\Gamma}\rangle_{\mathcal{E}_\mathcal{H}}, \tag{3.56}$$

since, by (3.5), $T_hG_h\tilde{\xi}_{\mathcal{H}_\Gamma}|_{\mathcal{E}_\mathcal{H}} = \tilde{\xi}_{\mathcal{H}_\Gamma}$, $\forall\, \tilde{\xi}_{\mathcal{H}_\Gamma} \in \Gamma_{\mathcal{H}_\Gamma}$ weakly, which concludes the proof. □

Therefore, according to Theorem 3.2.1, the error estimates $u - u_h$, $\lambda - \lambda_{\mathcal{H}_\lambda}$ and $\rho - \rho_{\mathcal{H}_\Gamma}$ depend on the approximation of the both finite dimensional spaces, traces $\Gamma_{\mathcal{H}_\Gamma}$ and fluxes $\Lambda_{\mathcal{H}_\Lambda}$. Thus, it is necessary to enrich them, respecting the compatibility condition (3.1) and (3.4), to ensure convergence of the approximation error to zero. In addition, the consistency of the method at the second level affects the error estimates. In fact, local problems need to be well resolved to ensure an approximation in the space of flows. Next, we assume that local problems admit exact solutions and focus our attention on first-level problems. Next we will consider

3.3 $V_h = H^1(\mathcal{T}_{\mathcal{H}})$ assumption

In this section we assume that local problems have exact solutions, so that $V_h = H^1(\mathcal{T}_{\mathcal{H}})$ and, consequently, $T_h = T$ and $\tilde{T}_h = \tilde{T}$. We review the main results under this hypothesis and then present an example of the MHHM method with error estimates. We start by noting that the compatibility condition (3.1) becomes, for $K \in \mathcal{T}_{\mathcal{H}}$:

$$\tilde{\mu}_{\mathcal{H}_\Lambda} \in \tilde{\Lambda}_{\mathcal{H}_\Lambda} \quad \text{and} \quad \langle \tilde{\mu}_{\mathcal{H}_\Lambda}, \tilde{v} \rangle_{\partial K} = 0, \, \forall \, \tilde{v} \in \tilde{H}^1(K) \quad \Longrightarrow \quad \tilde{\mu}_{\mathcal{H}_\Lambda} = 0. \tag{3.57}$$

Since the trace of functions in $\tilde{H}^1(K)$ is $\tilde{H}^{1/2}(\partial K)$, condition (3.57) is satisfied for all choice of $\tilde{\Lambda}_{\mathcal{H}_\Lambda}$. On the other hand, the compatibility condition (3.4) between spaces $\tilde{\Gamma}_{\mathcal{H}_\Gamma}$ and $\tilde{\Lambda}_{\mathcal{H}_0}$ remains the same, that is, for each $K \in \mathcal{T}_{\mathcal{H}}$:

$$\tilde{\xi}_{\mathcal{H}_\Gamma} \in \tilde{\Gamma}_{\mathcal{H}_\Gamma, K} \quad \text{and} \quad \langle \tilde{\mu}_{\mathcal{H}_\Lambda}, \tilde{\xi}_{\mathcal{H}_\Gamma} \rangle_{\partial K} = 0, \, \forall \, \tilde{\mu}_{\mathcal{H}_\Lambda} \in \tilde{\Lambda}_{\mathcal{H}_0, K} \quad \Longrightarrow \quad \tilde{\xi}_{\mathcal{H}_\Gamma} = 0. \tag{3.58}$$

Follows from (3.58) that the discrete inf-sup is valid, as presented in Proposition 3.1.1. The assumption $T_h = T$ simplifies the redefinition of the discrete operator G_h as follows: for $\phi \in H_0^{1/2}(\mathcal{E}_{\mathcal{H}})$, let $\tilde{\lambda}_\phi = G_h \phi$ such that, for $K \in \mathcal{T}_{\mathcal{H}}$,

$$\int_K \mathcal{A} \nabla (T \tilde{\lambda}_\phi) \cdot \nabla T \tilde{\mu}_{\mathcal{H}_\Lambda} \, dx = \langle \tilde{\mu}_{\mathcal{H}_\Lambda}, T \tilde{\lambda}_\phi \rangle_{\partial K} = \langle \tilde{\mu}_{\mathcal{H}_\Lambda}, \phi \rangle_{\partial K}, \quad \forall \, \tilde{\mu}_{\mathcal{H}_\Lambda} \in \tilde{\Lambda}_{\mathcal{H}_\Lambda}. \tag{3.59}$$

In this case, the harmonic extension $T G_h \phi$ belongs to the infinite dimensional space $\tilde{H}^1(\mathcal{T}_{\mathcal{H}})$, instead of $\tilde{V}_h$ as in the T_h operator case. Next, we revisit the main approximation results in terms of spaces $\Lambda_{\mathcal{H}_\Lambda}$ and $\Gamma_{\mathcal{H}_\Gamma}$, provided the compatibility condition (3.58) is satisfied.

Lemma 3.3.1. *Let* $\phi \in H^{1/2}(\mathcal{E}_{\mathcal{H}})$. *Then,*

$$|G\phi - G_h \phi|_\Lambda \leq E(\phi), \tag{3.60}$$

where,

$$E(\phi) := \left(C_T^{-1} + 1 \right) \inf_{\tilde{\mu}_{\mathcal{H}_\Lambda} \in \tilde{\Lambda}_{\mathcal{H}_\Lambda}} |G\phi - \tilde{\mu}_{\mathcal{H}_\Lambda}|_\Lambda. \tag{3.61}$$

Theorem 3.3.1. *With the assumptions of the Theorem 3.2.1, the following error estimates hold:*

$$|\rho - \rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} \leq \inf_{\phi_{\mathcal{H}_\Gamma} \in \Gamma_{\mathcal{H}_\Gamma}} \left\{ 2|\rho - \phi_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} + E(\phi_{\mathcal{H}_\Gamma}) \right\} + E(\tilde{T}f); \tag{3.62}$$

$$|\lambda - \lambda_{\mathcal{H}_\Lambda}|_\Lambda \leq |\rho - \rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} + E(\rho_{\mathcal{H}_\Gamma}) + E(\tilde{T}f). \tag{3.63}$$

$$|u - u_h|_{1,\mathcal{A},\mathcal{T}_\mathcal{H}} \leq |\lambda - \lambda_{\mathcal{H}_\Lambda}|_\Lambda. \tag{3.64}$$

—

Therefore, the estimate $G - G_h$ depends only on the approximation property of the $\tilde{\Lambda}_{\mathcal{H}_\Lambda}$-space as (3.61) shows. In all cases of the Theorem 3.3.1 the estimates depends on approximation properties in both $\tilde{\Lambda}_{\mathcal{H}_\Lambda}$ and $\Gamma_{\mathcal{H}_\Gamma}$ spaces.

3.4 A simple case

In this section we present an example of a pair of finite element spaces satisfying the compatibility conditions (3.1) and (3.4). Then, we estimate the approximation rates for this case.

3.4.1 Compatibility issues

Assuming $V_h = H^1(\mathcal{T}_\mathcal{H})$, let the following finite element spaces:

$$\Gamma^1_{\mathcal{H}_\Gamma} := \left\{ \xi_{\mathcal{H}_\Gamma} \in H_0^{1/2}(\mathcal{E}_\mathcal{H});\ \xi_{\mathcal{H}_\Gamma}|_F \in \mathbb{P}_1(F),\ \forall\, F \in \mathcal{E}_\mathcal{H} \right\}; \tag{3.65}$$

$$\Lambda^0_{\mathcal{H}_\Lambda} := \prod_{K \in \mathcal{T}_\mathcal{H}} \left\{ \mu_{\mathcal{H}_\Lambda} \in L^2(\partial K);\ \mu_{\mathcal{H}_\Lambda}|_F \in \mathbb{P}_0(F),\ \forall\, F \in \partial K \right\}, \tag{3.66}$$

where $\mathbb{P}_1(F)$ is the space of linear functions over the face F and $\mathbb{P}_0(F)$ is the space of constant functions over F. We illustrate in Figure 3 typical functions of these spaces.

Remark 3.4.1. For the spaces (3.65) and (3.66) we have $\mathcal{H} = \mathcal{H}_\Lambda = \mathcal{H}_\Gamma$. —

The next result shows that the zero-mean subspaces arising from (3.65) and (3.66) satisfy the compatibility condition (3.58). Before, it is interesting to note that, in this case, the smallest space of finite-dimensional functions that keep the stability conditions satisfied is formed by linear functions. So, for a fixed $K \in \mathcal{T}_\mathcal{H}$, we have the triple $\mathbb{P}_1(K) - \mathbb{P}_0(\partial K) - \mathbb{P}_1(\partial K)$ (function, flow, trace).

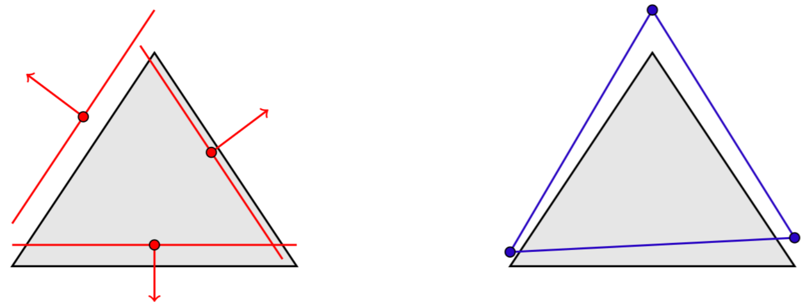

Figure 3 – Representative piecewise constant function of the space $\Lambda^0_{\mathcal{H}_\Lambda,K}$ and a continuous piecewise linear function of $\Gamma^1_{\mathcal{H}_\Gamma,K}$.

Lemma 3.4.1. *The zero-mean finite element spaces $\tilde{\Gamma}^1_{\mathcal{H}_\Gamma} \subset \Gamma^1_{\mathcal{H}_\Gamma}$ and $\tilde{\Lambda}^0_{\mathcal{H}_\Lambda} \subset \Lambda^0_{\mathcal{H}_\Lambda}$, where $\Gamma^1_{\mathcal{H}_\Gamma}$ and $\Lambda^0_{\mathcal{H}_\Lambda}$ are introduced in (3.65) and (3.66) satisfy the compatibility condtion (3.1) and (3.4).*

Proof. Let $K \in \mathcal{T}_\mathcal{H}$ with faces F_1, F_2, F_3. Consider $\{\tilde{\mu}_{\mathcal{H}_\Lambda,1}, \tilde{\mu}_{\mathcal{H}_\Lambda,2}\} \subset \tilde{\Lambda}^0_{\mathcal{H}_\Lambda}(\partial K)$ a basis function such that $\tilde{\mu}_{\mathcal{H}_\Lambda,i}|_{F_j} = \delta_{ij}$, $\forall\, i \in \{1,2\}$ and $\forall\, j \in \{1,2,3\}$. Now, given a function $\tilde{\xi}_{\mathcal{H}_\Gamma} \in \tilde{\Gamma}^1_{\mathcal{H}_\Gamma}(\partial K)$, we have by the hypothesis that:

$$\langle \tilde{\mu}_{\mathcal{H}_\Lambda,i}, \tilde{\xi}_{\mathcal{H}_\Gamma} \rangle_{\partial K} = 0, \quad \forall \in \{1,2\}, \tag{3.67}$$

so that $\tilde{\xi}_{\mathcal{H}_\Gamma} = 0$ at the midpoint of each face F_1 and F_2. Since

$$\langle \mu^0_{\mathcal{H}_\Lambda}, \tilde{\xi}_{\mathcal{H}_\Gamma} \rangle_{\partial K} = 0, \tag{3.68}$$

we get $\tilde{\xi}_{\mathcal{H}_\Gamma} = 0$ at the midpoint of F_3. Therefore, $\tilde{\xi}_{\mathcal{H}_\Gamma} = 0$, since it vanishes at three non-collinear points. □

The discrete inf-sup condition follows from Proposition 3.1.1.

3.4.2 Error estimates

Next, we develop the approximation errors associated with the first level problems. For this, let $\hat{K}$ be a reference element and, for a fixed $K \in \mathcal{T}_\mathcal{H}$, let $T_K : \mathbb{R}^d \to \mathbb{R}^d$, $d \in \{2,3\}$ be an affine mapping such that, for all $\hat{x} \in \mathbb{R}^d$, $T_K\hat{x} = B_K\hat{x} + b_K$, where $B_K \in \mathbb{R}^{d\times d}$ is an invertible matrix and $b_K \in \mathbb{R}^d$ is a vector. Then $T_K(\hat{K}) = K$. For a space function $V(K)$ and $v \in V(K)$, we define $\hat{v} := v \circ T_K \in V(\hat{K})$. Let us start by getting an estimate for

$|\mu - \mu_{\mathcal{H}_\Lambda}|_\Lambda$, where $\mu \in \Lambda \cap L^2(\mathcal{E}_\mathcal{H})$. We have from Definition 2 that:

$$|\lambda - \lambda_{\mathcal{H}_\Lambda}|^2_\Lambda = \sum_{K \in \mathcal{T}_\mathcal{H}} |\lambda - \lambda_{\mathcal{H}_\Lambda}|^2_{-\frac{1}{2},\partial K} = \sum_{K \in \mathcal{T}_\mathcal{H}} \left\{ \sup_{\tilde{\phi} \in \tilde{H}^{\frac{1}{2}}(\partial K)} \frac{|\langle \lambda - \lambda_{\mathcal{H}_\Lambda}, \tilde{\phi} \rangle_{\partial K}|}{|\tilde{\phi}|_{\frac{1}{2},\partial K}} \right\}^2. \tag{3.69}$$

Applying the Cauchy-Schwarz inequality, we obtain:

$$\langle \lambda - \lambda_{\mathcal{H}_\Lambda}, \tilde{\phi} \rangle_{\partial K} \leq |\lambda - \lambda_{\mathcal{H}_\Lambda}|_{0,\partial K} |\tilde{\phi}|_{0,\partial K} = \left[\sum_{F \in \partial K} |\lambda - \lambda_{\mathcal{H}_\Lambda}|^2_{0,F} \right]^{1/2} |\tilde{\phi}|_{0,\partial K}. \tag{3.70}$$

Now, from [15, Lemma 3.18], the folling estimate holds:

$$|\lambda - \lambda_{\mathcal{H}_\Lambda}|_{0,F} \leq C \mathcal{H}_K^{1/2} |\tau_\lambda|_{1,K}, \tag{3.71}$$

where $\tau_\lambda \in [H^1(K)]^d$ is such that $\tau_\lambda \cdot \mathbf{n}^K = \lambda$ over ∂K. On the other hand, we get from Lemma A.0.4 and from the equivalence norm result for fractional Sobolev spaces, in [19, Proposition 2.1]:

$$|\tilde{\phi}|_{0,\partial K} = |\mathrm{det} B_K|^{1/2} |\hat{\phi}|_{0,\partial \hat{K}} \leq C |\mathrm{det} B_K|^{1/2} |\hat{\phi}|_{\frac{1}{2},\partial \hat{K}}. \tag{3.72}$$

Moreover, follows from Lemma 2.9 [19, page 513] that:

$$|\tilde{\phi}|_{0,\partial K} \leq C |\mathrm{det} B_K|^{1/2} \|B_K\|^{1/2} |\mathrm{det} B_K|^{-1/2} |\tilde{\phi}|_{\frac{1}{2},\partial K} \leq C |\mathcal{H}_K|^{1/2} |\tilde{\phi}|_{\frac{1}{2},\partial K}. \tag{3.73}$$

Thus, we obtain:

$$|\tilde{\phi}|_{0,\partial K} \leq C \mathcal{H}_K^{1/2} |\tilde{\phi}|_{\frac{1}{2},\partial K}. \tag{3.74}$$

We gather from (3.69), (3.70), (3.71) and (3.74) that there exists a positive constat C, independ of $\mathcal{H}$, such that:

$$|\lambda - \lambda_{\mathcal{H}_\Lambda}|_\Lambda \leq C \mathcal{H} |\tau_\lambda|_{1,\mathcal{T}_\mathcal{H}}. \tag{3.75}$$

Let $\tau_\lambda = \mathcal{A}\nabla u$. Then $|\mathcal{A}\nabla u|_{1,\mathcal{T}_\mathcal{H}} \leq a_{\max}|u|_{2,\mathcal{T}_\mathcal{H}}$, so that:

$$|\lambda - \lambda_{\mathcal{H}_\Lambda}|_\Lambda \leq C\mathcal{H}|u|_{2,\mathcal{T}_\mathcal{H}}. \tag{3.76}$$

Define the following interpolant operators $\Pi_1^* : H_0^{1/2}(\mathcal{E}_\mathcal{H}) \to \mathbb{P}_1(\mathcal{E}_\mathcal{H})$ and $\Pi_1 : H^1(\mathcal{T}_\mathcal{H}) \to \mathbb{P}_1(\mathcal{T}_\mathcal{H})$, where the spaces $\mathbb{P}_1(\mathcal{E}_\mathcal{H})$ and $\mathbb{P}_1(\mathcal{T}_\mathcal{H})$ are defined as:

$$\mathbb{P}_1(\mathcal{E}_\mathcal{H}) := \{\xi \in H_0^{1/2}(\mathcal{E}_\mathcal{H}); \xi|_F \in \mathbb{P}_1(F),\ \forall\, F \in \mathcal{E}_\mathcal{H}\}; \tag{3.77}$$

$$\mathbb{P}_1(\mathcal{T}_\mathcal{H}) := \{v \in H^1(\mathcal{T}_\mathcal{H});\ v|_K \in \mathbb{P}_1(K);\ \forall\, K \in \mathcal{T}_\mathcal{H}\}. \tag{3.78}$$

Then, we gather from the definition 2.3.1 and the interpolation error estimate introduced in (4.5) [15, page 94] that:

$$|\rho - \Pi_1^*\rho|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} := \inf_{v\in V_\rho(\Omega)} |v - \Pi_1 v|_{1,\mathcal{A},\mathcal{T}_\mathcal{H}} \leq |u - \Pi_1 u|_{1,\mathcal{A},\mathcal{T}_\mathcal{H}} \leq C\mathcal{H}|u|_{2,\mathcal{T}_\mathcal{H}}. \tag{3.79}$$

where the space V_ρ is as defined in (2.52).

Remark 3.4.2. We gather from the definition of $E(\cdot)$ in (3.60) that

$$\inf_{\phi_{\mathcal{H}_\Gamma}\in\Gamma_{\mathcal{H}_\Gamma}} E(\phi_{\mathcal{H}_\Gamma}) = 0; \tag{3.80}$$

—

Now we use the previous results to estimate the error-norms presented in Theorem 3.3.1. Follows from (3.76), (3.79) and Remark 3.4.2(*a*) that

$$\begin{aligned}
|\rho - \rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} &\leq \inf_{\phi_{\mathcal{H}_\Gamma}\in\Gamma_{\mathcal{H}_\Gamma}} \left\{2|\rho - \phi_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} + E(\phi_{\mathcal{H}_\Gamma})\right\} + E(\tilde{T}f) \\
&\leq 2|\rho - \Pi_1^*\rho|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} + E(\Pi_1^*\rho) + E(\tilde{T}f) \\
&= C\mathcal{H}|u|_{2,\mathcal{T}_\mathcal{H}} + C\mathcal{H}|\tilde{T}f|_{2,\mathcal{T}_\mathcal{H}} \\
&\leq C\mathcal{H}|u|_{2,\mathcal{T}_\mathcal{H}} + C\mathcal{H}\|f\|_{0,\Omega} \\
&\leq C\mathcal{H}\|f\|_{0,\Omega}.
\end{aligned} \tag{3.81}$$

where we use the fact that $\Pi_1\gamma \in \mathbb{P}_1(F) \subset \Gamma_{\mathcal{H}_\Gamma}$ and the regularity [16, Theorem 1.10]. From (3.63) and (3.64), we obtain:

$$\begin{aligned} |\lambda - \lambda_{\mathcal{H}_\Lambda}|_\Lambda &\leq |\rho - \rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} + E(\rho_{\mathcal{H}_\Gamma}) + E(\tilde{T}f) \\ &\leq C\mathcal{H}|u|_{2,\mathcal{T}_\mathcal{H}} + C\mathcal{H}\|f\|_{0,\Omega} \\ &\leq C\mathcal{H}\|f\|_{0,\Omega}. \end{aligned} \tag{3.82}$$

These error estimates are summarized in the Theorem below.

Theorem 3.4.1. *Under assumptions of Theorem 3.2.1, let $V_h = H^1(\mathcal{T}_\mathcal{H})$, $T_h = T$ and the finite element spaces $\Gamma^1_{\mathcal{H}_\Gamma}$ and $\Lambda^0_{\mathcal{H}_\Lambda}$ introduced in* (3.65) *and* (3.66)*. Then, there exists constants $C > 0$ such that*

$$\begin{aligned} |\rho - \rho_{\mathcal{H}_\Gamma}|_{\frac{1}{2},\mathcal{E}_\mathcal{H}} &\leq C\mathcal{H}\|f\|_{0,\Omega}; \\ |\lambda - \lambda_{\mathcal{H}_\Lambda}|_\Lambda &\leq C\mathcal{H}\|f\|_{0,\Omega}; \\ |u - u_h|_{1,\mathcal{A},\mathcal{T}_\mathcal{H}} &\leq C\mathcal{H}\|f\|_{0,\Omega}. \end{aligned} \tag{3.83}$$

—

3.5 On the relation between the present method and the MsFEM

Note that equation (2.47) that defines our method has some sort of relation with the definition of the MsFEM. What we show bellow is that a Galerkin discretization of (2.47) yields some nodal values under certain conditions. We first consider a continuous version of MsFEM, seeking $\rho \in H^{1/2}_0(\mathcal{E}_\mathcal{H})$ such that:

$$\int_\Omega \mathcal{A}\nabla\mathcal{E}(\rho)\cdot\nabla\mathcal{E}(\xi)\,dx = \int_\Omega f\mathcal{E}(\xi)\,dx, \quad \forall\,\xi \in H^{1/2}_0(\mathcal{E}_\mathcal{H}) \tag{3.84}$$

where we denote the $\mathcal{A}$-harmonic extension $\mathcal{E} : H^{1/2}_0(\mathcal{E}_\mathcal{H}) \to \mathcal{H}^*(\Omega) \cap H^1_0(\Omega)$ (the space of piecewise harmonic fuctions that are also in $H^1_0(\Omega)$). Formulation (3.84) results from the usual weak $H^1_0(\Omega)$ formulation. A discretization of (3.84) yields the MsFEM.

Let us show that, from (2.66), the identity

$$\mathcal{E}(\xi) = TG\xi - c_\xi = TG\xi + \frac{1}{|\partial K|}\int_{\partial K} \xi\,ds, \tag{3.85}$$

for all $\xi \in H_0^{1/2}(\mathcal{E}_{\mathcal{H}})$, holds. For $K \in \mathcal{T}_{\mathcal{H}}$, let $\xi \in H^{1/2}(\partial K)$ and denote $v_\xi := \mathcal{E}(\xi)$. Then, $\tilde{v}_\xi = v_\xi + c_\xi$ belongs to the harmonic function space $\mathcal{H}^*(K)$. Since map $T : \tilde{\Lambda} \to \mathcal{H}^*(K)$ is surjective, there exists $\tilde{\lambda}_\xi \in \tilde{H}^{-1/2}(\partial K)$ such that $T\tilde{\lambda}_\xi = \tilde{v}_\xi$. From (2.66), v_ξ satisfies:

$$\int_K \mathcal{A}\nabla T\tilde{\mu} \cdot \nabla \tilde{v}_\xi \, dx = \langle \tilde{\mu}, \tilde{v}_\xi \rangle_{\partial K} = \langle \tilde{\mu}, \xi \rangle_{\partial K}, \tag{3.86}$$

for all $\tilde{\mu} \in \tilde{\Lambda}$, so that $\tilde{v}_\xi = TG\xi$.

Then we gather from a discretization of (3.85) and from (3.86) that

$$\begin{aligned}(\mathcal{E}(\rho), \mathcal{E}(\xi))_{\mathcal{T}_{\mathcal{H}}} &= \sum_{K \in \mathcal{T}_{\mathcal{H}}} \int_K \mathcal{A}\nabla TG\rho \cdot \nabla TG\xi \, dx \\ &= \langle G\rho, TG\xi \rangle_{\mathcal{E}_{\mathcal{H}}} = \langle G\rho, \xi \rangle_{\mathcal{E}_{\mathcal{H}}},\end{aligned} \tag{3.87}$$

for all $\rho,\ \xi \in H_0^{1/2}(\mathcal{E}_{\mathcal{H}})$, that is, the left-hand side of the global problem (2.47). From (3.87), we get by joining the right-hand sides of (3.84) and (2.47) the following identity:

$$-\langle \lambda^0, \xi \rangle_{\mathcal{E}_{\mathcal{H}}} + \langle G\tilde{T}f, \xi \rangle_{\mathcal{E}_{\mathcal{H}}} = \int_\Omega f\mathcal{E}(\xi) \, dx, \tag{3.88}$$

for all $\xi \in H_0^{1/2}(\mathcal{E}_{\mathcal{H}})$. We can also deduce (3.88) in the following way: First, note that

$$\langle G\tilde{T}f, \xi \rangle_{\mathcal{E}_{\mathcal{H}}} = \langle G\xi, \tilde{T}f \rangle_{\mathcal{E}_{\mathcal{H}}} = \sum_{K \in \mathcal{T}_{\mathcal{H}}} \int_K \mathcal{A}\nabla\tilde{T}f \cdot \nabla TG\xi \, dx = \sum_{K \in \mathcal{T}_{\mathcal{H}}} \int_K fTG\xi \, dx. \tag{3.89}$$

Thus,

$$\begin{aligned}-\langle \lambda^0, \xi \rangle_{\mathcal{E}_{\mathcal{H}}} + \langle G\tilde{T}f, \xi \rangle_{\mathcal{E}_{\mathcal{H}}} &= \sum_{K \in \mathcal{T}_{\mathcal{H}}} \left(\int_{\partial K} \lambda^0 \xi \, dx + \int_K fTG\xi \, dx \right) \\ &= \sum_{K \in \mathcal{T}_{\mathcal{H}}} \left(\int_{\partial K} \lambda^0 c_\xi \, dx + \int_K fTG\xi \, dx \right) \\ &= \sum_{K \in \mathcal{T}_{\mathcal{H}}} \left[-\int_K (fc_\xi + fTG\xi) \, dx \right] \\ &= \sum_{K \in \mathcal{T}_{\mathcal{H}}} \int_K f(TG\xi - c_\xi) \, dx = \int_\Omega f\mathcal{E}(\xi) \, dx.\end{aligned}$$

Assuming that there are exact solutions for second level problems, follows from (3.85) that solutions ρ from (2.47) and (3.84) coincide pointwise if Λ is an infinite dimensional space. But, for $G = Gh$, solutions match if $\mathcal{R}(\mathcal{E}) \subset \mathcal{R}(TG_h)$. Moreover, for $\mathcal{A} = \mathcal{I}$, then solutions match, even when $T = T_h$.

4 Numerical Results

In this section we investigate the numerical performance of the MH^{2}M method. We briefly present the computational algorithm for the MH^{2}M, and consider two model problem: one that admits an analytical solution and another with heterogeneous coefficients.

4.1 Computational algorithm

Let $N_0, N_1 \in \mathbb{N}$ and $\mathcal{B}_\Gamma := \{\xi_1, \cdots, \xi_{N_0}\}$ be a basis for $\Gamma_{\mathcal{H}_\Gamma}$ and $\mathcal{B}_{\tilde{\Lambda}} := \{\tilde{\mu}_1, \cdots, \tilde{\mu}_{N_1}\}$ be a basis for $\tilde{\Lambda}_{\mathcal{H}_\Lambda}$. The computational algorithm consists of the following steps:

1. compute $\lambda^0 \in \mathbb{P}^0(\mathcal{T}_\mathcal{H})$ from the first equation of (2.34);
2. compute $T_h\tilde{\mu}_j \in \tilde{\Lambda}_\mathcal{H}$ from (3.2), for each $\tilde{\mu}_j \in \mathcal{B}_{\tilde{\Lambda}}$.
3. for each $\xi_i \in \mathcal{B}_\Gamma$, compute $G_h\xi_i \in \tilde{\Lambda}_{\mathcal{H}_\Lambda}$ from (3.5);
4. compute $G_h\tilde{T}_hf \in \tilde{\Lambda}_{\mathcal{H}_\Lambda}$ using (3.5);
5. solve the discrete global problem (3.6) to obtain the coefficients of $\rho_{\mathcal{H}_\Gamma} = \sum_{i=1}^{N} \alpha_i\xi_i \in \Gamma_{\mathcal{H}_\Gamma}$.
6. compute $\tilde{\lambda}_{\mathcal{H}_\Lambda} = G_h(\rho_{\mathcal{H}_\Gamma} - \tilde{T}_hf)$;
7. compute $\tilde{u}_h = T_h\tilde{\lambda}_{\mathcal{H}_\Lambda} + \tilde{T}_hf$;
8. compute u_h^0 from the second equation of (2.34).
9. compute $u_h = u_h^0 + \tilde{u}_h = u_h^0 + T_hG_h\rho_{\mathcal{H}_\Lambda} + (I - T_hG_h)\,\tilde{T}_hf$.

4.2 Numerical validation

In this section we evaluate the performance of the proposed MH^{2}M method through two types of problems and, at the same time, compare it with the FEM, MsFEM and MHM methods. The first case consist to solve the Poisson model ($\mathcal{A} \equiv 1$ in (2.6)) with an smooth analytical solution. The goal is to confirm the theoretical results. In the second case, the tensor $\mathcal{A}$ represents a heterogeneous field varying in multiple scales, thus, problem (2.6) verifies the robustness of the MH^{2}M method. In what follows, the domain is a unit square $\Omega :=]0,1[\times]0,1[$, the mesh $\mathcal{T}_\mathcal{H}$ and sub-mesh $\mathcal{T}_h$ are composed by uniform triangles, on Figure 4. The parameters h and $\mathcal{H}$ indicate the refinement level of the fine scale and the coarse scale. More precisely, they are the number of partition on each face $F \in \partial\Omega$ and the number of faces $F \in \partial\Omega$ of the boundary $\partial\Omega$. The second level problems, defined in each element $K \in \mathcal{T}_{\mathcal{H}}$, are solved by the classical FEM [13]. Here, for each $K \in \mathcal{T}_{\mathcal{H}}$, the Hilbert space $H^1(K)$ is approximated by the space of continuous piecewise linear functions.

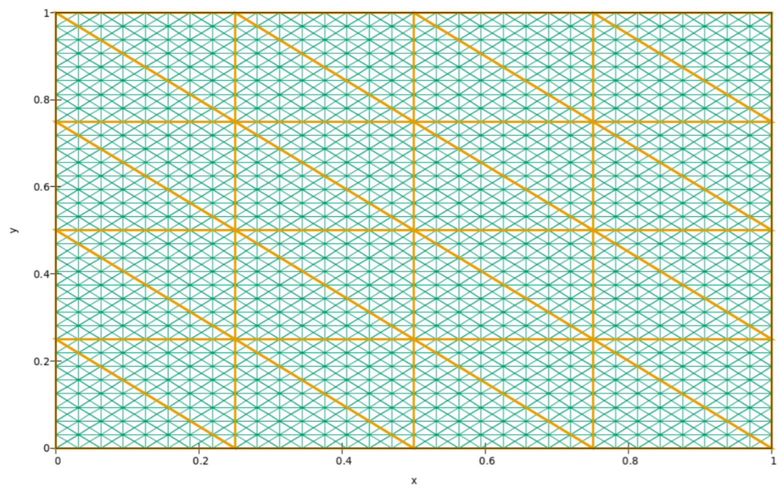

Figure 4 – Representative uniform triangular mesh and sub-mesh in the unit square with $\mathcal{H} = 1/4$ and $h = \mathcal{H}/8$.

4.2.1 Problem with known solution

Consider the boundary value problem (2.6) where the tensor $\mathcal{A}$ is the identity matrix $\mathcal{I}$, and input data $f \in L^2(\Omega)$ is given by:

$$f(x, y) = -2[x(x-1) + y(y-1)], \quad \forall\,(x, y) \in \Omega. \tag{4.1}$$

In this case,

$$u(x, y) = x(x-1)y(y-1), \quad \forall\,(x, y) \in \Omega, \tag{4.2}$$

and its graph is illustrated in the Figure 5.

The numerical results presented below are intended to validate the theoretical concepts. For this, we compare the outputs of the FEM, MsFEM, MHM and MH^{2}M methods with the analytic solution. Then, we present the error convergence graph in the energy norm.

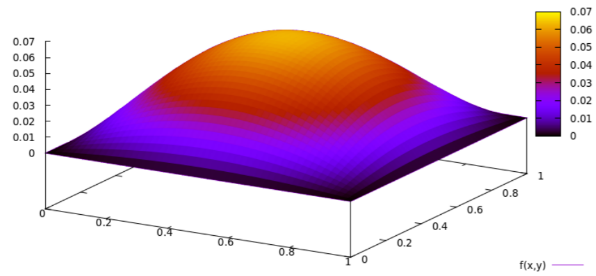

Figure 5 – Analytical solution u of the Poisson problem (2.6).

4.2.1.1 Results from FEM and MsFEM

The FEM was implemented using the following finite element space:

$$V_{\mathcal{H}} := \{v_h \in H^1(\Omega);\ v_h|_K \in \mathbb{P}_1(K),\ \forall\, K \in \mathcal{T}_{\mathcal{H}}\}. \tag{4.3}$$

The solution obtained through this method closely approximate the exact solution if we take $\mathcal{H} = 1/30$, see the cut at $x = 0.5$ illustrated in Figure 6.

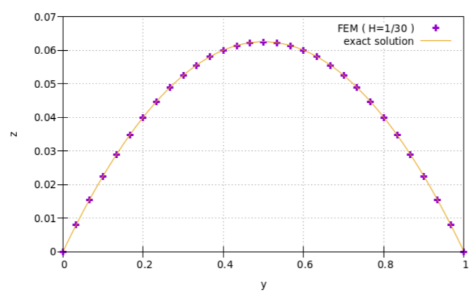

Figure 6 – FEM solution with $\mathcal{H} = 1/30$.

Solving the problem (2.6) with MsFEM, the shape of the mulstiscale base is linear as we can see in Figure 7 on the left. We plot in Figure 7, on the right, the profile of the numerical solution and Figure 8 shows an approximated solution.

Before presenting the numerical results of the MHM and MH^{2}M methods, we need the following definition.

Definition 4.2.1. For each face $F \in \mathcal{E}_{\mathcal{H}}$, we define a regular partition $F^N := \bigcup_{j=1}^{N} f_j^F$, where the integer N indicates the number of face divisions. Let the set $\mathcal{F}^N$ defined as:

$$\mathcal{F}^N := \bigcup_{F \in \mathcal{H}} F^N.$$

Therefore, if $N = 1$ it means that $\mathcal{F}^1 = \mathcal{E}_{\mathcal{H}}$. For $N > 1$ we have $\#(\mathcal{F}^N) = N * \#(\mathcal{E}_{\mathcal{H}})$. —

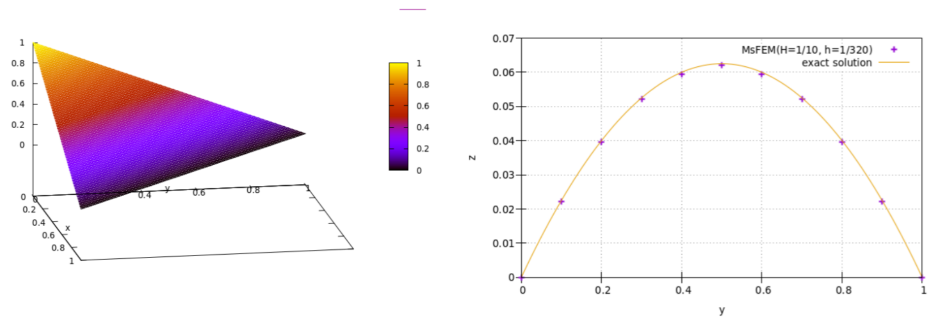

Figure 7 – Representative multiscale base function of the MsFEM with $\mathcal{H} = 1/2$ and $h = \mathcal{H}/128$; and profile of the numerical solution with $\mathcal{H} = 1/10$ and $h = \mathcal{H}/32$.

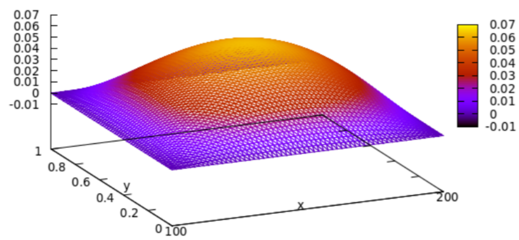

Figure 8 – Numerical solution from MsFEM in which $\mathcal{H} = 1/60$ and $h = \mathcal{H}/4$.

4.2.1.2 Results from the MHM method

Next, we present the numerical results of the MHM method. To this end, we aproximate the Λ space by piecewise constant functions. We denote MHM-N when the space Λ is approximated by the finite element space $\Lambda_{\mathcal{H}_N}$ defined as:

$$\Lambda_{\mathcal{H}_N} := \{\lambda_{\mathcal{H}} \in L^2(\mathcal{F}^N);\ \lambda_{\mathcal{H}}|_{f_j^F} \in \mathbb{P}_0(f_j^F),\ \forall\, j \in \{1, \cdots N\},\ \forall F \in \mathcal{E}_{\mathcal{H}}\}.$$

Therefore, a function $\lambda_{\mathcal{H}} \in \Lambda_{\mathcal{H}_N}$ has N degrees of freedom on each face $F \in \mathcal{E}_{\mathcal{H}}$. To better understand this idea, we ilustrate the representative functions of the spaces $\Lambda_{\mathcal{H}_1}$ and $\Lambda_{\mathcal{H}_2}$ at the face $F \in \mathcal{E}_{\mathcal{H}}$ in Figure 9. Figure 10 shows the representative multiescale base funtions arising from the MHM-1 and MHM-4, respectively.

Increasing the parameter N makes the method more accurate. On the other hand, it demands greater computational power, since the order of the global matrix grows significantly. See in the Figure 11 the profile of the numerical solution from MHM-1 and MHM-2 methods. Then, we plot the numerical solutions for this cases, with the same parameters, in Figure 12.

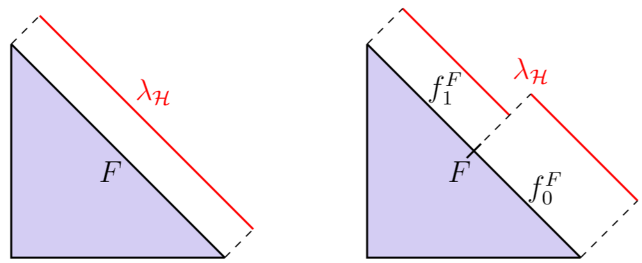

Figure 9 – Representative functions from the spaces $\Lambda_{\mathcal{H}_1}$ and $\Lambda_{\mathcal{H}_2}$ over a face $F \in \mathcal{E}_{\mathcal{H}}$.

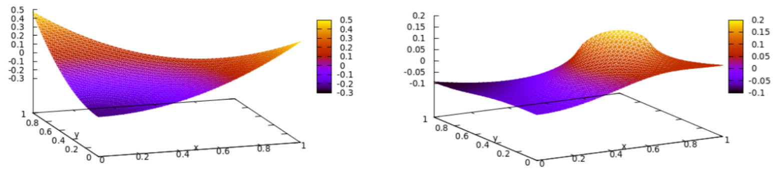

Figure 10 – Representative multiscale base functions from MHM-1 and from MHM-4 with parameters $\mathcal{H} = 1/2,\ h = \mathcal{H}/64$.

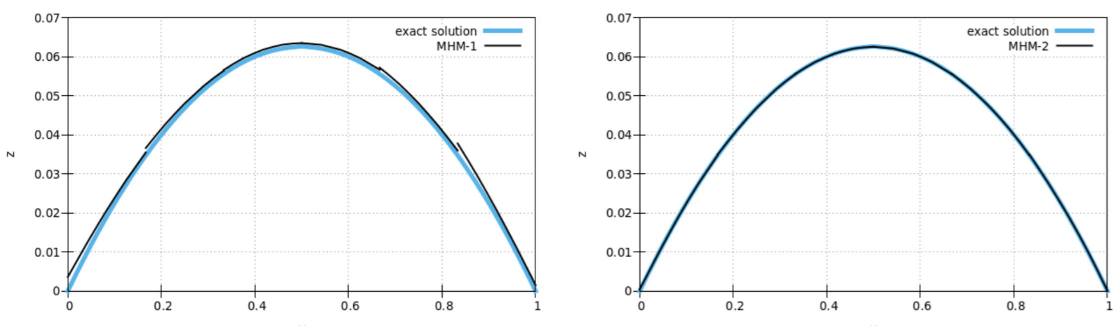

Figure 11 – Prefile of the numerical solution from MHM-1 and MHM-2. The parameters are $\mathcal{H} = 1/6,\ h = \mathcal{H}/8$.

4.2.1.3 Results from the MH^{2}M method

Finally, this method was implemented with the fininte element space $\tilde{\Lambda}_{\mathcal{H}_\Lambda}$ consisting of piecewise constant functions, as introduced in (3.66), but with the partition $\mathcal{E}_{\mathcal{H}}$ replaced by $\mathcal{F}_\Lambda^N$ as presented in the definition (4.2.1). Similarly, we work with the continuous piecewise linear functions for the $\Gamma_{\mathcal{H}_\Gamma}$ space introduced in (3.65), replacing $\mathcal{E}_{\mathcal{H}}$ by $\mathcal{F}_\Gamma^N$. Let $\tilde{\mu} \in \tilde{\Lambda}_{\mathcal{H}_\Lambda}$, $\xi \in \Gamma_{\mathcal{H}_\Gamma}|_F$ and a fixed face $F \in \mathcal{E}_{\mathcal{H}}$.Thus, we use the MH^{2}M-L-R notation when the restrictions $\tilde{\mu}|_F$ and $\xi|_F$ has L and R degrees of freedom, respectively. Figure 13 shows

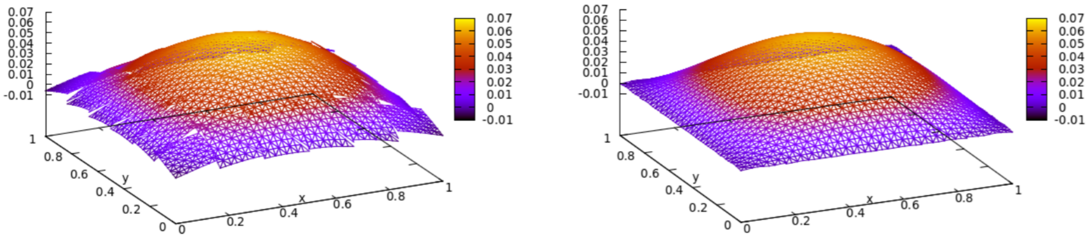

Figure 12 – Numerical solution from MHM-1 and MHM-2, with $\mathcal{H} = 1/6,\ h = \mathcal{H}/8$.

the representative multiscale base functions for different values of R and L.

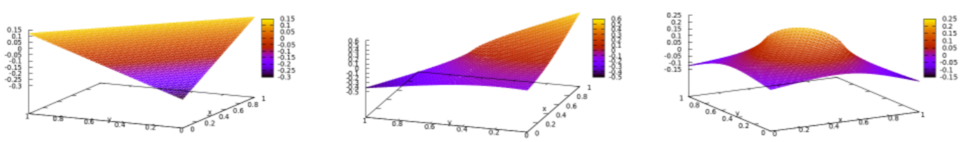

Figure 13 – Representative multiscale base functions from the MH^{2}M-1-2, MH^{2}M-2-2, MH^{2}M-3-2, respectively. The parameters are $\mathcal{H} = 1/2$ and $h = \mathcal{H}/64$.

Note that for MH^{2}M-1-2 the multiscale base function is linear, as in the MsFEM. The Figure 14 shows the accuracy gain as a result of enrichment in $\tilde{\Lambda}_{\mathcal{H}}$ and $\Gamma_{\mathcal{H}}$ spaces and we plot the associated numerical soluton in Figure 15.

We plot in Figure 16 the convergence curves in the energy norm $\|u - u_{\mathcal{H}}\|_{\mathcal{A},\mathcal{T}_{\mathcal{H}}}$. Note that the MH^{2}M-1-2 curve closely approximates the MHM-1 curve. The same holds for MHM-2 and MH^{2}M-2-3. Moreover, the FEM and the MsFEM curves coincide and they have low efficiency compared to MHM and MH^{2}M.

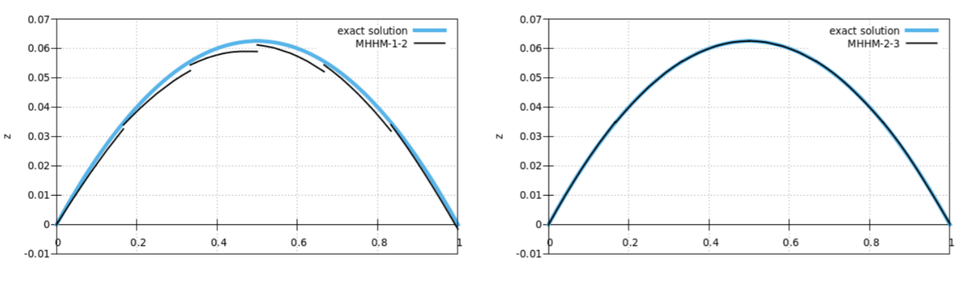

Figure 14 – Profile of a numerical solution from MH^{2}M-1-2 and MH^{2}M-2-3, with $\mathcal{H} = 1/6$ and $h = \mathcal{H}/8$.

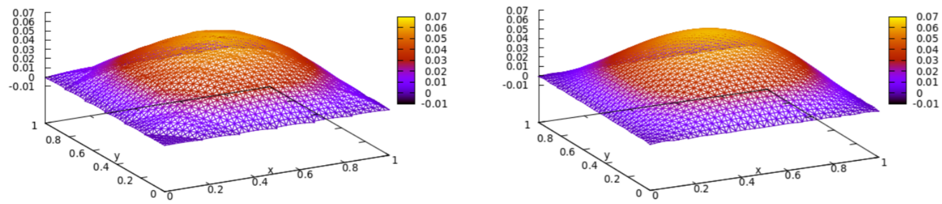

Figure 15 – Numerical solution from the MH^2M-1-2 and MH^2M-2-3, with $\mathcal{H} = 1/6$ and $h = \mathcal{H}/8$.

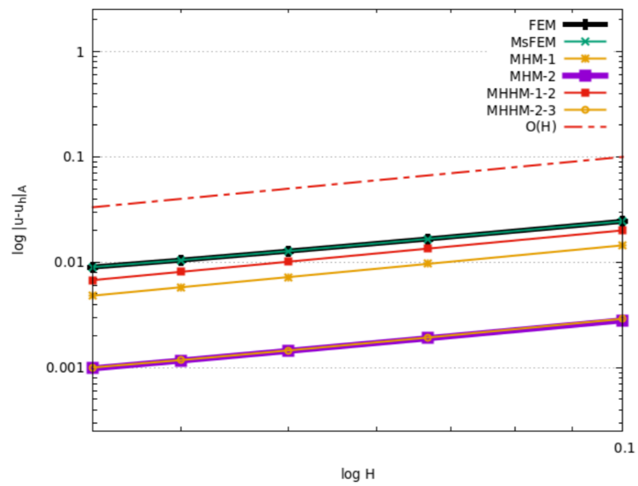

Figure 16 – Mesh based energy error graph with $h = \mathcal{H}/8$.

It is useful to estimate the energy error $\|u - u_{\mathcal{H}}\|_{\mathcal{A},\mathcal{T}_{\mathcal{H}}}$ as a function of the degrees of freedon (DoF). It is a computational cost estimator. See in Figures 17 that the MH^2M-1-2 method offers a result very similar to the MHM-1, at a lower cost, considering the fact that the two methods solve a saddle point problem 2.42 and (1.4) in [2, page 3506]. Surprisingly, we get the same quality with much less degrees of freedom with the elliptical MH^2M-E-1-2 method (2.47). In this case, the FEM and MsFEM methods have a similar computational cost, but a worse approximation compared to the MHM and MH^2M methods.

The good numerical result of the MHM method improves when $\Lambda_{\mathcal{H}_\Lambda}$ is enriched. However, the cost increases in the same proportion. On the other hand, the MH^2M is little affected by this change, as it maintains the same number of degrees of freedom when solves elliptic global problem (3.6), see in Figure 18 the MHM-2 and MH^2M-E-2-2 curves. The MH^2M-2-3 approximates the exact solution as well as the MHM, but with a slightly lower cost.

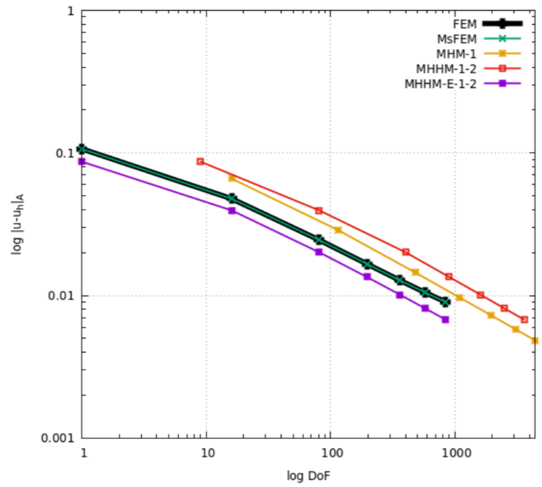

Figure 17 – Mesh based energy error by degrees of freedom graph with $h = \mathcal{H}/8$.

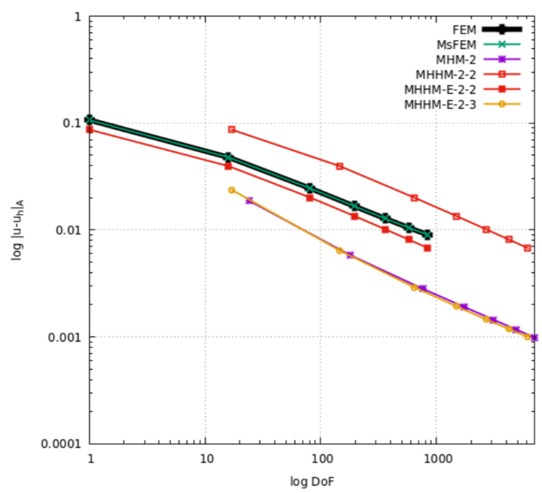

Figure 18 – Mesh based energy error by degrees of freedom graph with $h = \mathcal{H}/8$.

The graph of Figure 19 shows that MH^2M is invariant when increasing the number of degrees of freedom (L) in $\Lambda_{\mathcal{H}_\Lambda}$, while MHM improves. Figure 20 shows the graph of the associated computational cost. Here the number of degrees of freedom (Dof) for the MHM method come from a saddle-point global problem, while the one comes from an elliptical global problem for the MH^2M method.

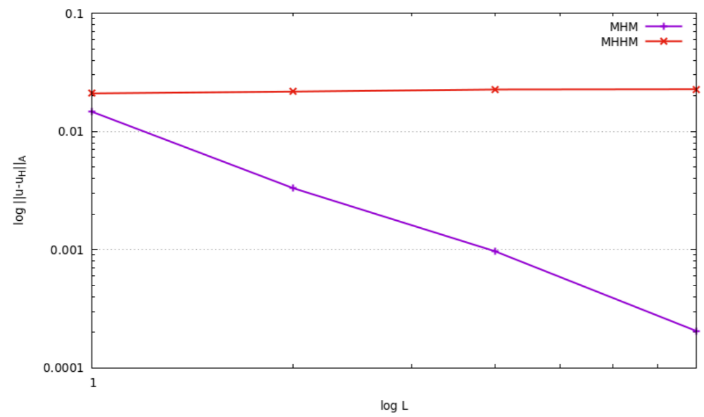

Figure 19 – Energy error based on dimension of $\Lambda_{\mathcal{H}_\Lambda}$ with $\mathcal{H} = 1/10$ and $h = \mathcal{H}/32$.

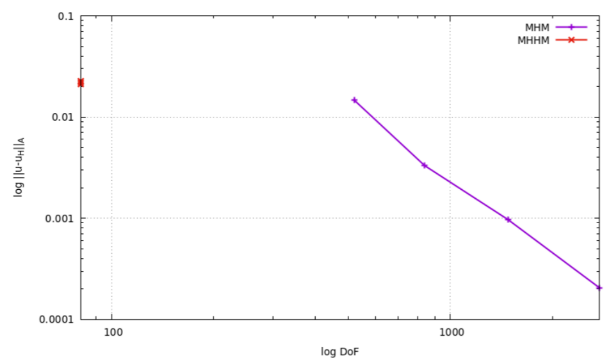

Figure 20 – Error based on dimension of $\Lambda_{\mathcal{H}_\Lambda}$ by degrees of freedom graph, where $\mathcal{H} = 1/10$ and $h = \mathcal{H}/32$.

4.2.2 Two heterogeneous field problems

This section is dedicated to evaluating the robustness of the MH^{2}M. Then, for a given regular function f, we consider again the boundary value problem introduced in (2.6), wich consists of finding u such that:

$$-\nabla \cdot \mathcal{A}\left(\frac{\mathbf{x}}{\varepsilon}\right)\nabla u = f, \quad \text{in } \Omega;$$
$$u = 0 \quad \text{over } \partial\Omega.$$

Here, the diffusion coefficients $\mathcal{A} = \mathcal{A}(\mathbf{x}/\varepsilon)$ models the presence of different types of materials in a structure that we assume to reproduce periodically in the domain Ω, with periodicity $\varepsilon \in]0, 1[$.

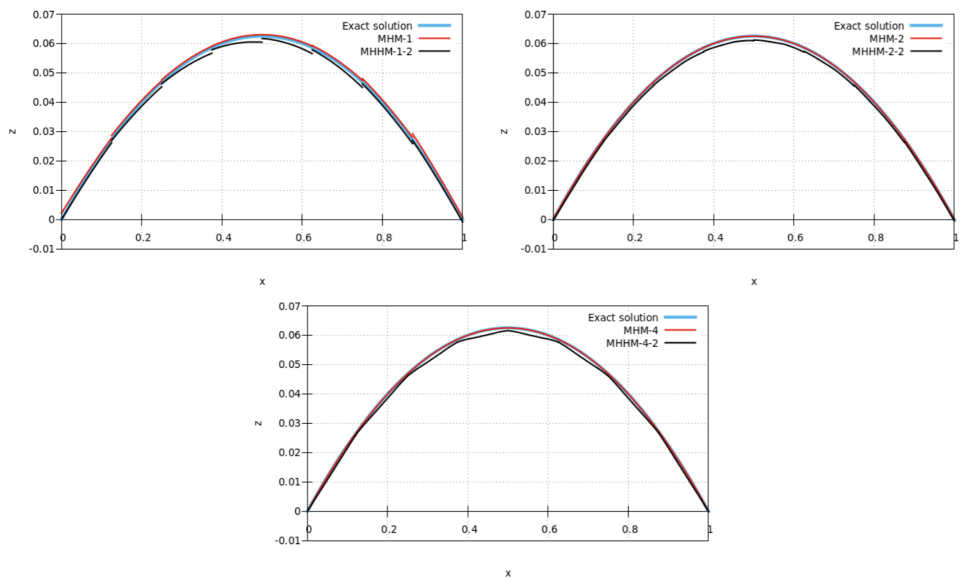

Figure 21 – Profile increasing dimension of $\Lambda_{\mathcal{H}_\Lambda}$, with $\mathcal{H} = 1/8$ and $h = \mathcal{H}/16$.

4.2.2.1 Problem 1

We consider as source term $f \in L^2(\Omega)$ as in (4.1) and tensor $\mathcal{A}(\mathbf{x}/\varepsilon)$ defined as:

$$\mathcal{A}\left(\frac{\mathbf{x}}{\varepsilon}\right) := \frac{2 + \gamma \sin(2\pi x/\varepsilon)}{2 + \gamma \cos(2\pi y/\varepsilon)} + \frac{2 + \sin(2\pi y/\varepsilon)}{2 + \gamma \sin(2\pi x/\varepsilon)}, \quad \forall\, \mathbf{x} := (x, y) \in \Omega, \tag{4.4}$$

where $\gamma = 1.8$. The heterogeneous coefficient (4.4) are illustrated in Figure 22.

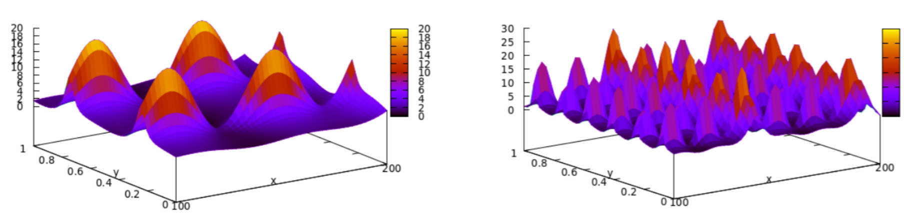

Figure 22 – Plot of $\mathcal{A}(x/\varepsilon)$, for $\varepsilon = 0.5$ and $\varepsilon = 1/17$.

The small scale effects is characterized by the oscillatory behavior of the multiscale base functions at each element. See in Figure 23 the base functions from MsFEM, MHM-4 and MH^{2}M-4-2. Such mulstiscale base functions are approximated at the local level by the standard Galerkin method over the linear continuous polynomial space defined in a triangular sub-mesh.

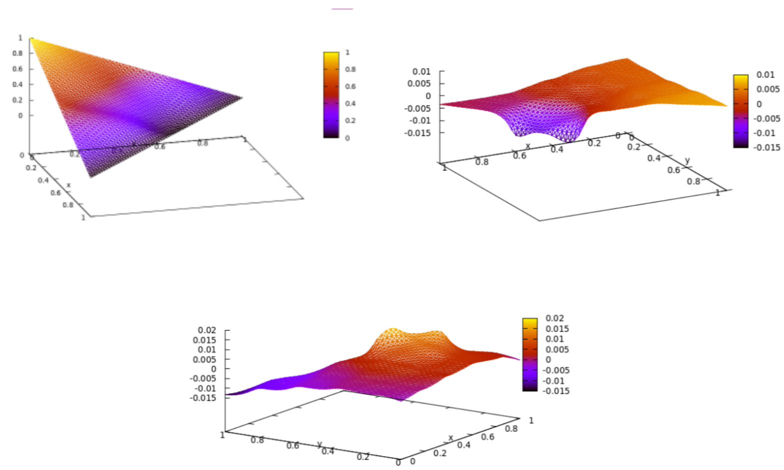

Figure 23 – Representative multiscale base functions from MsFEM, MHM-4 and MH^{2}M-4-2, where $h = 1/64$ and $\varepsilon = 1/17$.

To provide a basis for comparision, we obtain a solution to the problem (2.6) using the Galerkin finite element method with linear elements on a refined mesh of 130 elements in each direction, and call it the "exact solution"; see Figure 24.

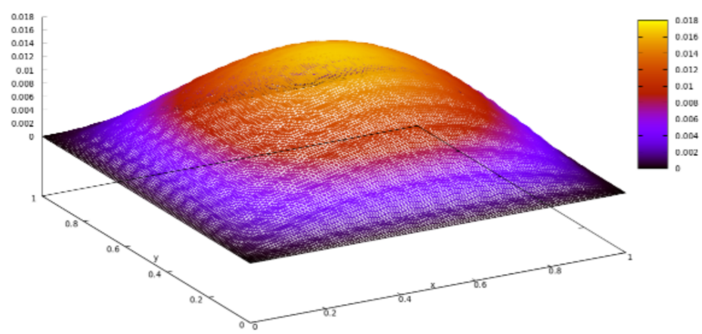

Figure 24 – Plot of the "exact solution" with $\mathcal{H} = 1/130$ and $\varepsilon = 1/17$.

The next two numerical experiments compares the methods in two ways. When $\mathcal{H}$ goes to zero in the first case, and in the second test we fix $\mathcal{H}$ and make $\mathcal{H}_\Lambda$ tend to zero. We compare the results with the ones from [23, page 26] and from the benchmark proposed in [10, page 31]. Figure (25) shows the prefile for the first case. We show in Figure 26 the energy error curves. Note that error increases when $\mathcal{H}$ is of the order of $\varepsilon = 1.0/17.0$ as in [10, 23]. Then, for fixed $\mathcal{H}$, we plot in Figure 27 the profile to $\Lambda_{\mathcal{H}_\Lambda}$ space based convergence. Finally, the numerical result from MH^{2}M-2-2 is ilustrated in Figure 28.

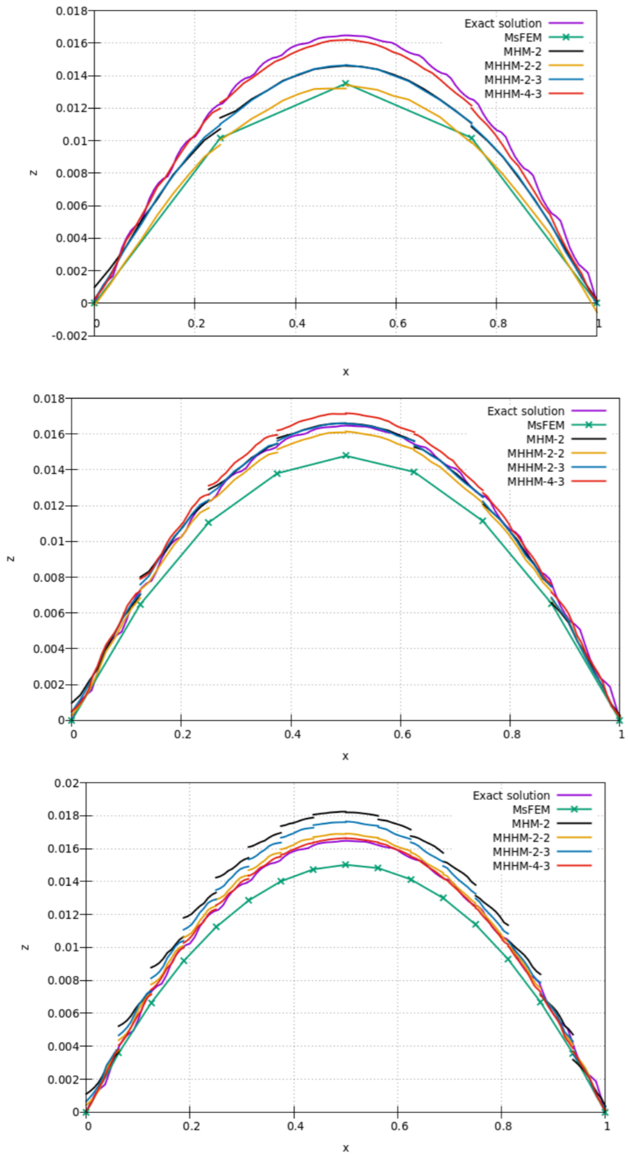

Figure 25 – Profile with parameters $\mathcal{H} = 1/4$, $\mathcal{H} = 1/8$; $\mathcal{H} = 1/16$ and $h = \mathcal{H}/8$, $\varepsilon = 1/17$.

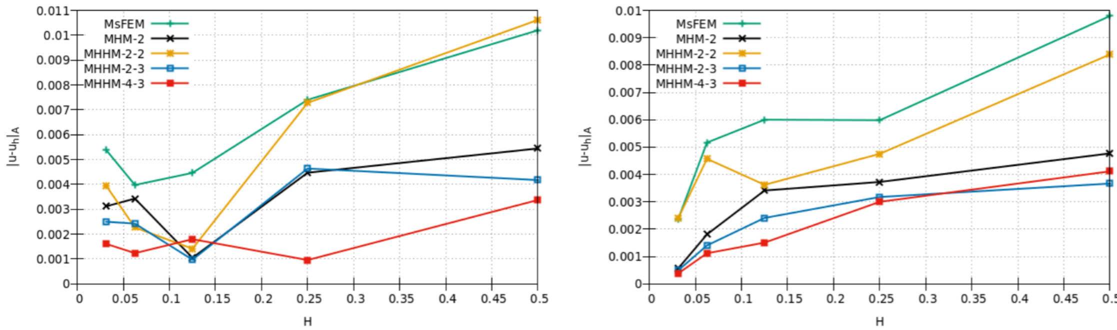

Figure 26 – Energy error graph with $h = \mathcal{H}/8$, $\varepsilon = 1/17$ and $\varepsilon = 1/6$.

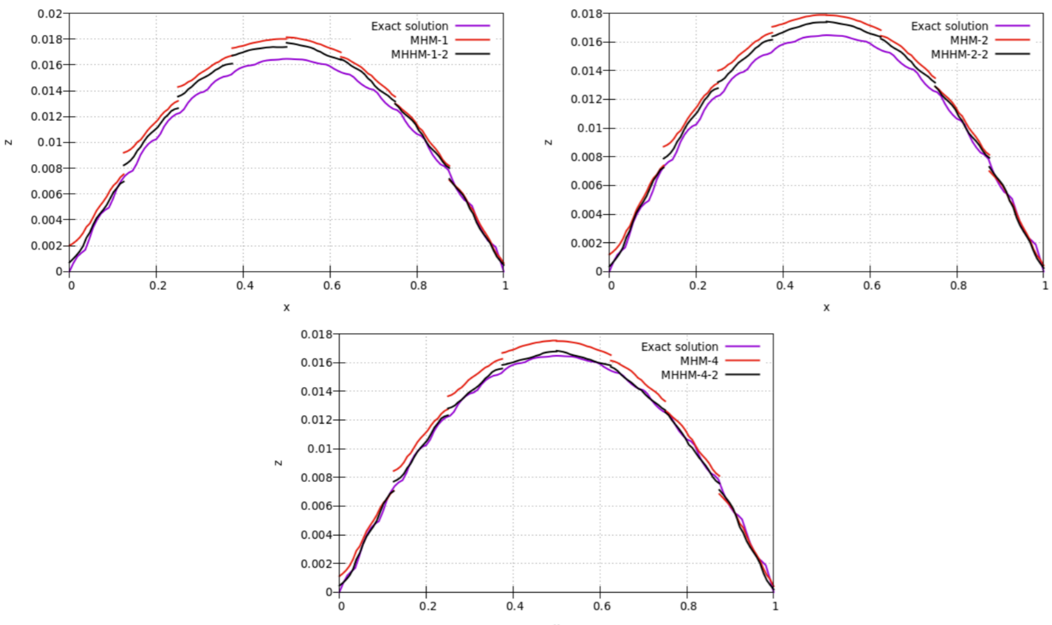

Figure 27 – Profile with parameters $\mathcal{H} = 1/8$, $h = \mathcal{H}/16$, $\varepsilon = 1/17$ and increasing dimension of $\Lambda_{\mathcal{H}_\Lambda}$.

4.2.2.2 Problem 2

To the second test let the model problem (2.6) with the source term $f : \Omega \to \mathbb{R}$ defined as

$$f(x, y) = 2\pi^2 \cos(2\pi x) \cos(2\pi y); \tag{4.5}$$

and the diffusion matrix $\mathcal{A}(\mathbf{x}/\varepsilon)$ is such that

$$\mathcal{A}\left(\frac{\mathbf{x}}{\varepsilon}\right) := \frac{2 + 1.8 \sin\left(\frac{2\pi x}{\varepsilon}\right)}{2 + 1.8 \sin\left(\frac{2\pi y}{\varepsilon}\right)} + \frac{2 + 1.8 \sin\left(\frac{2\pi y}{\varepsilon}\right)}{2 + 1.8 \cos\left(\frac{2\pi x}{\varepsilon}\right)}. \tag{4.6}$$

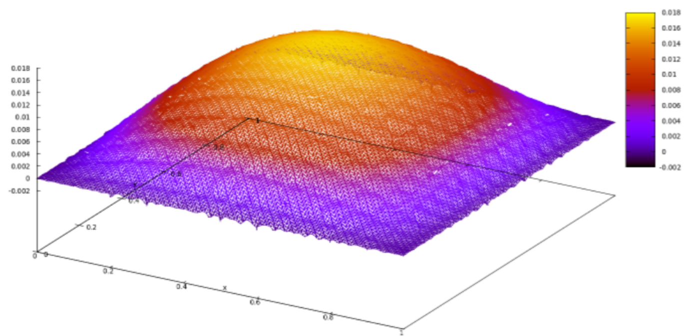

Figure 28 – Numerical solution from MH^{2}M-4-3 method with $\varepsilon = 1/17$, $\mathcal{H} = 1/10$ and $h = \mathcal{H}/16$.

The "exact solution", considered as the solution of the classical Galerkin method with the refinement level $\mathcal{H} = 1/130$ and $\varepsilon = 1/17$, is illustrated in the Figure 29. Its profile cut is shown in the Fiure 30.

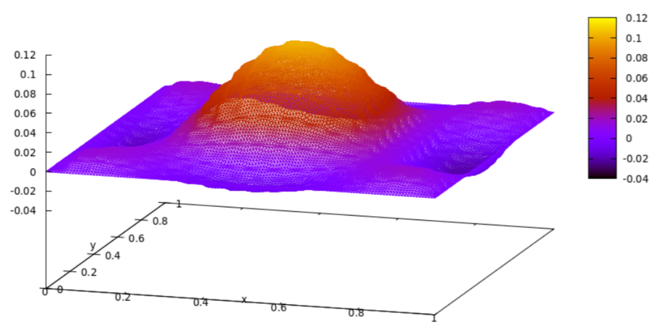

Figure 29 – Plot of the "exact solution" with $\mathcal{H} = 1/130$.

Again, we compare the performances of the methods when the refinement level of the coarse scale $\mathcal{H}$ goes to zero. We plot the profile in Figure 31 and the error curves for $\varepsilon = 1/17$ and $\varepsilon = 1/6$ are shown in Figure 32. Then Figure 33 shows a numerical solution

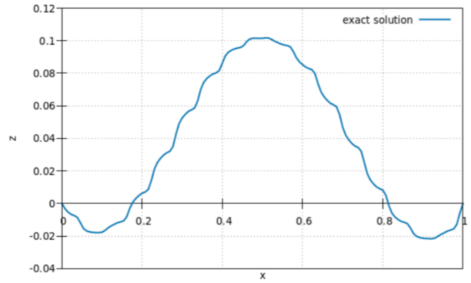

Figure 30 – Profile at $y = 0.5$ of the "exact solution" with $\mathcal{H} = 1/130$.

from the $\mathrm{MH^2M}$ method.

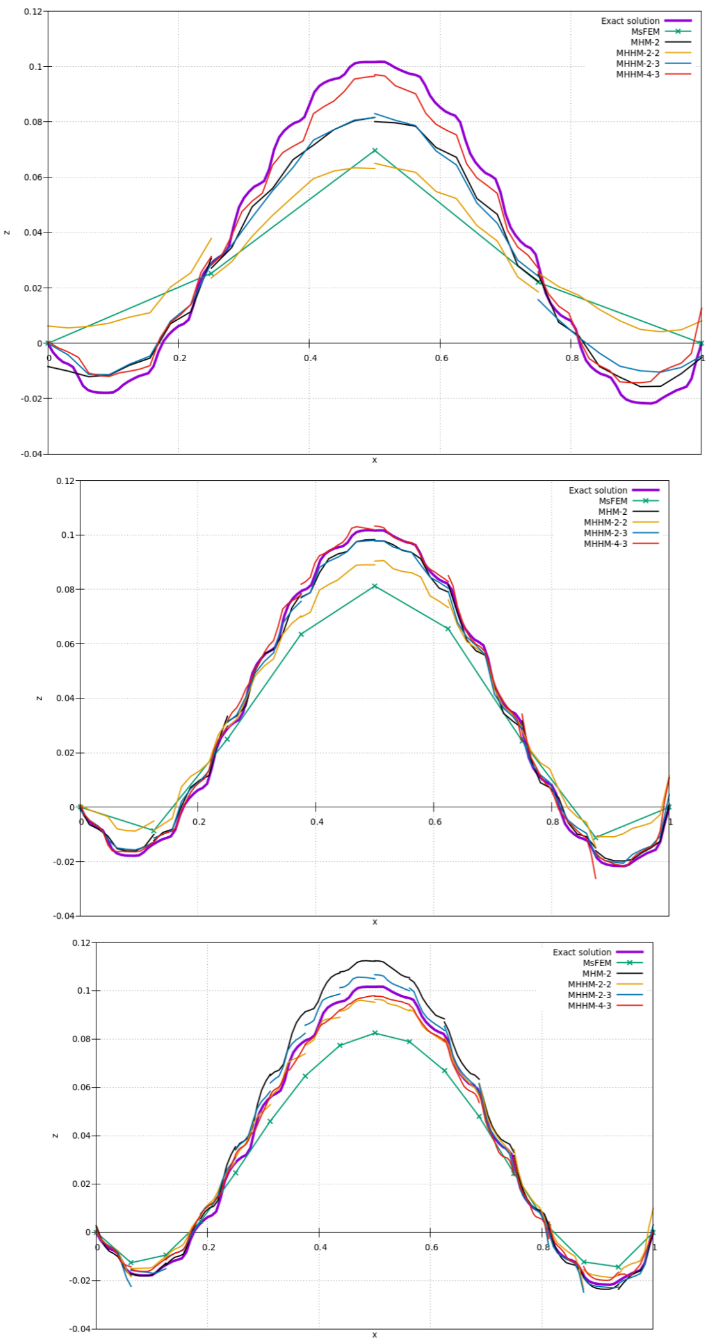

Figure 31 – Profile for $\mathcal{H} = 1/4$, $\mathcal{H} = 1/8$ and $\mathcal{H} = 1/16$; $h = \mathcal{H}/8$ and $\varepsilon = 1/17$.

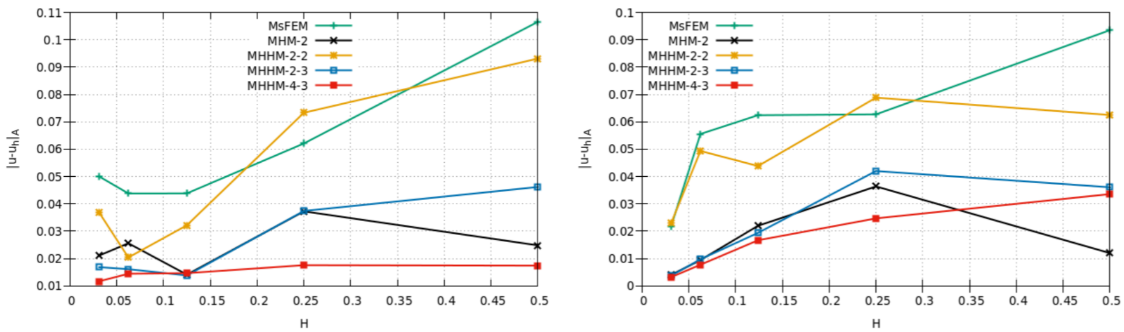

Figure 32 – Mesh based energy error with $h = \mathcal{H}/8$, $\varepsilon = 1/17$ and $\varepsilon = 1/6$.

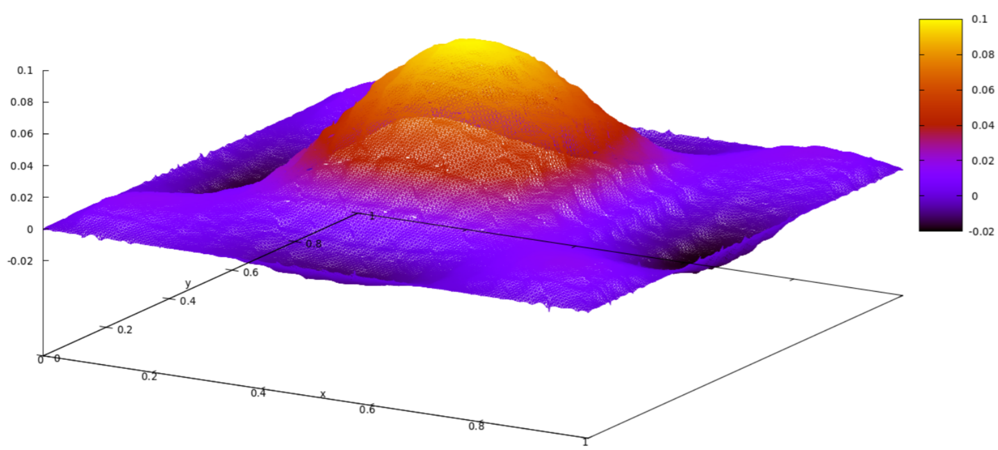

Figure 33 – Numerical solution from MH^{2}M-4-2, with $\mathcal{H} = 1/15$, $h = \mathcal{H}/16$ and $\varepsilon = 1/17$.

5 Conclusion

This work proposes a numerical method, called Multiscale Hybrid-Hybrid-Mixed method (MH^2M), to solve second-order elliptical problem with oscillatory coefficients. This method is derived from the Three-Field Domain Decomposition method, whose continuities of the solution and flux are weakly imposed, after functions and fluxes trial spaces decomposition and two static condensations. Moreover, we relax the flux defining it on each element boundary, so that we can chooce different flux meshes for different elements.

The infinite dimension global problem consists of an simmetric elliptical problem on the traces space, so that its associated matrix is simmetric and positive definite, wich allows resolution with low cost using the Conjugate Gradient method.

Although there is no conformity to the numerical solution, the trace and the flux are conforms, which implies in the mass conservation for the flux of the numerical solution. It is a desired property to flow simulation in porous media.

We show that the continuous and discrete inf-sup conditions are satisfied, as well as the well-posedness of the global problem by the Lax-Milgram lemma.

We deduce the error estimates from the natural norms for specific compatible finite element spaces. Then, numerical experiments confirm the theoretical estimates and show the efficiency and robustness of the MH^2M method with low computational cost, which makes it a competitive multiscale method.

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Appendix

APPENDIX A – Auxiliary results

Corollary A.0.1. *Let Ω be a bounded domain of $\mathbb{R}^n$ with Lipschitz-continuous boundary Γ. Then for each $v \in H^1(\Omega)$ and $u \in H^2(\Omega)$ there holds*

$$\int_\Omega v\Delta u\, dx = -\int_\Omega \nabla u \cdot \nabla v\, dx + \int_\Gamma \frac{\partial u}{\partial n} v\, ds. \tag{A.1}$$

Proof. See Corollary 1.2 [15]. □

Theorem A.0.1. *(Lax-Milgram lemma) Let V be a Hilbert space, $a : V \times V \to \mathbb{R}$ be a bounded V-elliptic bilinear form and $l \in V'$. Then there exists a unique solution to the problem*

$$a(u, v) = l(v), \quad \forall\, v \in V.$$

Proof. See in [24, Theorem 1.5]. □

Lemma A.0.1. *Consider the following spaces*

$$X := \left\{v \in L^2(\Omega);\ v|_K \in H^1(K),\ \forall\, K \in \mathcal{T}_h\right\}, \qquad H_0(div, \Omega) := \{\tau \in H(div, \Omega);\ \tau \cdot \boldsymbol{n} = 0\ on\, \Gamma\}.$$

Then

$$H^1(\Omega) = \left\{v \in X;\ \sum_{K\in\mathcal{T}_h} \langle \tau \cdot \boldsymbol{n}^K, v\rangle_{\partial K} = 0,\ \forall\, \tau \in H_0(div; \Omega)\right\}.$$

Proof. See in [15, Lemma 3.1]. □

Theorem A.0.2. *Let $X := \{v \in L^2(\Omega);\ v|_K \in H^1(K),\ \forall\, K \in \mathcal{T}_{\mathcal{H}}\}$. Then*

$$H^1(\Omega) = \{v \in X;\ v|_{K_i} - v|_{K_j} = 0\ in\ L^2(F),\ \forall\, K_i, K_j \in \mathcal{T}_{\mathcal{H}},\ \{F\} = K_i \cap K_j\}. \tag{A.2}$$

Proof. See in [15, Theorem 3.1]. □

Lemma A.0.2. *Let*

$$Y := \left\{\tau \in \left[L^2(\Omega)\right]^n;\ \tau|_K \in H(div;\ K),\ \forall\, K \in \mathcal{T}_h\right\}.$$

Then

$$H(div;\Omega) = \left\{\tau \in Y;\ \sum_{K\in\mathcal{T}_h} \langle \tau \cdot \boldsymbol{n}^K, v\rangle_{\partial K} = 0,\ \forall\, v \in H_0^1(\Omega)\right\}.$$

Proof. See in [15, Lemma 3.4]. □

Proposition A.0.1. *(Holder's inequality) Let* $x = (x_1, \cdots, x_n)$ *and* $y = (y_1, \cdots, y_n)$ *belongs to* $\mathbb{R}$*. Set:*

$$\|x\|_p := \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}, \quad \|y\|_{p^*} := \left(\sum_{k=1}^n |y_k|^{p^*}\right)^{1/p^*}$$

If p *and* p^* *are conjugates, we then have the Holder inequality:*

$$\max_{x,y\in\mathbb{R}^n} \frac{|\sum_{k=1}^n x_k y_k|}{\|x\|_p \|y\|_{p^*}} = 1.$$

Proof. See in [4, Proposition 1]. □

Definition A.0.1. Suppose Ω has diameter d and is star-shaped with respect to a ball B. Let $\rho_{\max} := \sup\{\rho;\ \Omega$ is star-shaped with respect to a ball of radious $\rho\}$. Then, follows from [8, page 99] that the chunkiness parameter of Ω is defined as:

$$\gamma := \frac{d}{\rho_{\max}}. \tag{A.3}$$

Theorem A.0.3. *(Poincaré inequality) Let Ω be an open set of $\mathbb{R}^n$ which is bounded in at least one space direction. There exists a constant $C > 0$ such that, for every function $v \in H_0^1(\Omega)$,*

$$\int_\Omega |v(x)|^2 \, dx \leq C \int_\Omega |\nabla v(x)| \, dx.$$

—

Proof. See in [1, Proposition 4.3.10]. □

Theorem A.0.4. *(Generalized Poincaré inequality) Let Ω be a bounded domain. Then there exists a positive constant C that depends on the domain Ω and the its boundary Γ such that:*

$$\|v\|_{0,\Omega} \leq C(\Omega, \Gamma) \left(\left| \int_\Gamma v \, ds \right| + |u|_{1,\Omega} \right), \tag{A.4}$$

for all $v \in H^1(\Omega)$. —

Proof. See in [8, page 135]. □

Lemma A.0.3. *(Friedrich's inequality) Suppose Ω is star-shaped with respect to a ball B. Then, for all $u \in W_p^1(\Omega)$,*

$$\|u - \overline{u}\|_{W_p^1(\Omega)} \leq C_{n,\gamma} |u|_{W_p^1(\Omega)},$$

where $\overline{u} = |\Omega|^{-1} \int_\Omega u \, dx$. —

Proof. See in [8, Lemma 4.3.14]. □

Lemma A.0.4. *Let S and $\hat{S}$ be a compact and connected sets of $\mathbb{R}^n$ with Lipschitz-continuous boundaries, and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be the affine mapping given by $F(\hat{x}) = B\hat{x} + v$, $\forall\, \hat{x} \in \mathbb{R}^n$, with $B \in \mathbb{R}^{n \times n}$ invertible and $b \in \mathbb{R}^n$, such that $v \circ F \in H^m(\hat{s})$. Then let m be a nonnegative integer, and let $v \in H^m(S)$. Then $\hat{v} := v \circ F \in H^m(\hat{S})$, and there exists $C := C(m, n) > 0$ such that:*

$$|\hat{v}|_{m,\hat{S}} \leq \hat{C} \|B\|^m |det\ B|^{-1/2} |v|_{m,S}. \tag{A.5}$$

Conversely, if $\hat{v} \in H^m(\hat{S})$ *and we let* $v = \hat{v} \circ F^{-1}$*, then* $v \in H^m(S)$*, and there exists* $\hat{C} := \hat{C}(m,n) > 0$ *such that*

$$|v|_{m,S} \leq \hat{C} \|B^{-1}\|^m |det\ B|^{1/2} |\hat{v}|_{m,\hat{S}}. \tag{A.6}$$

Proof. See in [15, Lemma 3.12]. □

Lemma A.0.5. *Let* S *and* $\hat{S}$ *be compact and connected sets of* $\mathbb{R}^n$ *with Lipschitz-constinuous bundaries, and let* $F : \mathbb{R}^n \to \mathbb{R}^n$ *be the affine mapping given by* $F(\hat{x}) = B\hat{x} + b$*,* $\forall\, \hat{x} \in \mathbb{R}^n$*, with* $B \in \mathbb{R}^{n \times n}$ *invertible and* $b \in \mathbb{R}^n$*, such that* $S = F(\hat{S})$*. next, let*

$$\mathcal{H}_S := \textit{diameter of } S = \max_{x,y \in S} \|x - y \tag{A.7}$$

$$\rho_S := \textit{diameter of largest sphere contained in } S; \tag{A.8}$$

$$\hat{\mathcal{H}} := \textit{diameter of } \hat{S}; \tag{A.9}$$

$$\hat{\rho} := \textit{diameter of largest sphere contained in } \hat{S}. \tag{A.10}$$

Then

$$|det\ B| = \frac{|S|}{|\hat{S}|}, \quad \|B\| \leq \frac{\mathcal{H}_S}{\hat{\rho}} \quad \textit{and} \quad \|B^{-1}\| \leq \frac{\hat{\mathcal{H}}}{\rho_S}. \tag{A.11}$$

Proof. See in [15, Lemma 3.14]. □