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ASYMPTOTICS AND HIERARCHICAL MODELING OF THIN DOMAINS

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Abstract

In this thesis we propose a way to analyze certain classes of dimension reduction models for elliptic problems in thin domains. We consider Poisson equations in thin rectangles and plates, and develop asymptotic expansions for the exact and model solutions, having the thickness as small parameter. The modeling error is then estimated by comparing the respective expansions, and the upper bounds obtained make clear the influence of the order of the model and the thickness on the convergence rates. The techniques developed here allows for estimates in several norms and semi-norms, and also interior estimates (which disregards boundary layers).

Finally, we present several low order dimension reduction models for a clamped linearly elastic plates, the simplest ones being variants of the Reissner–Mindlin models. Unlike many of the previous works on the subject, we impose no restrictive assumptions on the loads and tractions.

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Chapter 1

Introduction

Much investigation has been done in the recent and not so recent past to take advantage of the small thickness to solve or approximate elliptic problems in thin domains. Indeed it is tempting to use *dimension reduction*, i.e., to pose and solve a modified problem in a region with one less dimension and then extend the reduced solution to the more general domain. It is reasonable to expect that the new problem will be simpler than the original one, but it is not easy to predict how far apart are the two solutions. In this dissertation we analyze the approximation properties of some classes of models for elliptic problems in thin domains, not only as the thickness of the domain goes to zero, but also as the "degree" of the models increases, in a sense that we will make clear. To the best of our knowledge, the convergence rates and the techniques employed to obtain them are new.

We assume that the thin domain is of the form $D \times (-\varepsilon, \varepsilon)$, where D is either a one- or two-dimensional smoothly bounded region and $\varepsilon < 1$ is a small positive quantity. Also, for simplicity, we impose Dirichlet conditions on the lateral side $\partial D \times (-\varepsilon, \varepsilon)$, despite the fact that other boundary conditions are also of interest. There is an immense amount of work done for this sort of problem. We present next an overview trying to cover the main techniques and results of which we are aware and which are most closely related to the present thesis. It is clear that references with "mathematical flavor" are prevalent, although many of the main ideas arose in the engineering community. For thorough scholarly reviews we recommend the excellent books of Love [39], and Ciarlet [20].

For bending of linearly elastic plates, the biharmonic (Kirchhoff-Love) model dates back to the 19th century, see [39]. Its derivation was first based on physical arguments and a rigorous validation came only in 1959, when Morgenstern [44] showed that a slightly modified "biharmonic displacement" converges in the relative energy norm to the exact solution. Considering a plate under uniform load, he ingeniously constructed a statically admissible stress field, i.e., one that satisfies the equilibrium equation and the traction condition on the top and bottom of the plate and a displacement field (that includes a boundary corrector) that still satisfies the Dirichlet lateral boundary conditions. The Prager–Synge theorem allowed him then to estimate the difference between the exact and the approximate solutions, *without* knowing the original solution. A convergence rate of $O(\varepsilon^{1/2})$ in the relative energy norm follows from his work. Using basically the same approach, Babuška and Pitkäranta [8] investigated the "plate paradox problem," estimating in the process the errors between the Reissner–Mindlin and biharmonic models and the original solution. Similarly, Rössle et al. [48] used Morgenstern's ideas to show convergence of the $(1, 1, 2)$ model. Also, Chen [19] combined the Prager–Synge theorem with an asymptotic expansion approach to prove new convergence results of the biharmonic model, both in the interior of the domain and globally.

Another popular model for plates is the Reissner–Mindlin model. It is widely used and it is often the choice of the engineering community. One of the reasons for its popularity is that, in finite element implementations, the biharmonic equation requires sophisticated techniques to ensure interelement differentiability. On the other hand, numerical difficulties in the Reissner–Mindlin problem occur when the thickness of the domain goes to zero, and there is the onset of the *locking* phenomenon, see for instance [3], [15]. On the theoretical side, Arnold and Falk [4] gave a complete asymptotic expansion of solutions for the Reissner–Mindlin equations.

Two distinct ways to generate models for elliptic problems in thin domains are by using asymptotic expansions and by using variational techniques. Both put aside physical and other hard to justify considerations, and, for this exact reason, are more suitable to rigorous mathematical justification. In the asymptotic approach, the solution is expressed as a formal sum where the thickness is a parameter and one keeps the first or first few terms of the expansion. For instance, the biharmonic plate model is the "asymptotic limit" of the three-dimensional linearly elastic equations for a plate under bending [21], [20], [28], [27]. One major drawback of this sort of model is that if the thickness is not small enough, one would have to add extra terms in the model to achieve satisfactory results, what would require involved computations of boundary correctors and extra differentiations of terms previously computed. On the other hand, asymptotic expansions give invaluable information about the solutions and we shall use this technique to investigate models that have a nonasymptotic character. For linearly elastic plates, several works [22], [24], [25], [27], [32], [45], [59] developed the first few terms or the complete asymptotic expansion of the displacements and stresses. The book, in two volumes, by Mazja, Nazarow, and Plamenewski [40], [41] discusses several problems related to asymptotic expansions, including systems of elliptic equations in thin domains.

An alternative modeling approach is to project the exact solution into a semi-discretized space (usually a space of functions with polynomial dependence in the transverse direction), resulting in a whole hierarchy of models that approximate the original problem with increasing accuracy as the semi-discrete space gets richer, but maintain the lower dimensional character. For symmetric elliptic problems, one possibility is to use a Ritz projection [56], deriving the *minimum energy models*. See also [53], [6], [7], [38]. Paumier and Raoult [47] analyzed the asymptotic consistency of the minimum energy plate models, specifying the conditions under which they "converge" to the biharmonic one. A great deal of work was done by Schwab and his collaborators on a posteriori error estimation [9], [5], [52], [53] and on various aspects of the boundary layers present in the minimum energy solutions [55], [51], [50], [54].

In a series of three remarkable papers [56], [57], [58], Vogelius and Babuška investigated various aspects of minimum energy methods for scalar elliptic homogeneous problems in a N-dimensional plate, with Neumann boundary condition on the top and bottom of the domain. They started by showing how to optimally choose the semidiscrete subspace that characterizes each model. This space depends only on the coefficients of the differential equation, and a truncated asymptotic expansion of the exact solution belongs to it, if there is no boundary layer present. Then they estimated the rate with respect to the thickness that the solution of the model converges in the energy norm (in the absence of boundary layers), by estimating the difference between the exact solution and the truncated asymptotic expansion of the original solution. As this quantity is certainly bigger than the error of the minimum energy model in the energy norm, they obtained an upper bound for the modeling error. This procedure was extended by Miara [42] to linearly elastic plates under some nontrivial loads and tractions, and in this case the optimal subspace might depend on the data, a clear disadvantage. A recent work by Ovaskainen and Pitkäranta [46] used similar ideas to analyze minimum energy methods for thin linearly elastic strip under traction. One disadvantage of this approach is that it is not clear how treat models that are not energy minimizers. We postpone a more detailed discussion of the relations between the works of Vogelius and Babuška, Schwab and Ovaskainen and Pitkäranta to the final chapter of this thesis.

The above minimum energy models are instances of models derived by variational methods [1], [2], [27]. In linearly elastic plates, if we employ two variants of the Hellinger–

Reissner variational principle, further models come out, including some that are *not* of minimum energy nature, and others that are minimum complementary energy models. In particular, the second Hellinger–Reissner principle gives rise to models that makes the use of the Prager–Synge theorem relatively simple, as a statically admissible stress field results naturally in some cases—there is no need to devise it as in Morgenstern's work [44]. In a joint work with Alessandrini et al. [2] we obtained in this way a $O(\varepsilon^{1/2})$ convergence for one of the models in the relative energy norm under various types of loads and tractions.

In this thesis we propose and apply a method capable of estimating how good the models coming from variational methods are. Our approach is different from the ones employed before: we estimate the modeling error through comparison between the asymptotic expansions of the exact and approximate solutions. We present next, in a simple setting, the principal aspects of this work. Consider the three-dimensional plate $P^\varepsilon = \Omega \times (-\varepsilon, \varepsilon)$, where $\Omega \subset \mathbb{R}^2$ is a smooth, bounded domain. Let $\partial P^\varepsilon_L = \partial\Omega \times (-\varepsilon, \varepsilon)$ be the lateral side of the plate and $\partial P^\varepsilon_\pm = \Omega \times \{-\varepsilon, \varepsilon\}$ its top and bottom. We denote a typical point in P^ε by $\underline{x}^\varepsilon = (\underset{\sim}{x}^\varepsilon, x^\varepsilon_3)$, with $\underset{\sim}{x}^\varepsilon = (x^\varepsilon_1, x^\varepsilon_2) \in \Omega$. We accordingly denote $\underline{\nabla} = (\underset{\sim}{\nabla}, \partial_3) = (\partial_1, \partial_2, \partial_3)$, where the operator ∂_i indicates the partial derivative in the ith direction. Also, $\partial_{ij} = \partial_i\partial_j$ and $\partial^k_j = \partial_j\partial^{k-1}_j$.

Let $u^\varepsilon \in H^1(P^\varepsilon)$ be the weak solution of

$$\begin{aligned} \Delta\, u^\varepsilon &= -f^\varepsilon \qquad \text{in } P^\varepsilon, \\ \frac{\partial u^\varepsilon}{\partial n} &= 0 \qquad \text{on } \partial P^\varepsilon_\pm, \\ u^\varepsilon &= 0 \qquad \text{on } \partial P^\varepsilon_L, \end{aligned} \tag{1.1}$$

where $f^\varepsilon : P^\varepsilon \to \mathbb{R}$, and $\Delta = \partial_{11} + \partial_{22} + \partial_{33}$. In general, the solution of (1.1) will depend on ε in a nontrivial way. In fact the above problem is a singularly perturbed one, and as ε goes to zero it "loses" ellipticity. This causes the onset of boundary layers, as we make clear below.

It is possible to characterize the solution of (1.1) in an alternative way, as the minimizer of the associate energy functional, i.e.,

$$u^\varepsilon = \underset{v\in V(P^\varepsilon)}{\arg\min}\, \mathcal{J}(v), \text{ where } \mathcal{J}(v) = \frac{1}{2}\int_{P^\varepsilon} |\underset{-}{\nabla}\, v|^2\, d\underset{-}{x} - \int_{P^\varepsilon} f^\varepsilon v\, d\underset{-}{x},$$

and $V(P^\varepsilon) = \{v \in H^1(P^\varepsilon) \,:\, v = 0 \text{ on } \partial P_L^\varepsilon\}$.

Aiming to find a "good" approximation for u^ε, we search for

$$\tilde{u}^\varepsilon = \underset{v\in \mathring{H}^1(\Omega;\mathbb{P}_1(-\varepsilon,\varepsilon))}{\arg\min} \mathcal{J}(v), \tag{1.2}$$

where the notation is as follows. For an integer p and a positive real number a, we define $\mathbb{P}_p(-a,a)$ as the space of polynomials of degree p in $(-a,a)$. So $\mathring{H}^1(\Omega;\mathbb{P}_p(-a,a))$ denotes the space of polynomials of degree p with coefficients in $\mathring{H}^1(\Omega)$. The space $\mathring{H}^1(\Omega)$ is the set of functions in the usual Sobolev space $H^1(\Omega)$ with zero trace on $\partial\Omega$. It follows from its definition that $\tilde{u}^\varepsilon$ is the Ritz projection of u^ε into $\mathring{H}^1(\Omega;\mathbb{P}_1(-\varepsilon,\varepsilon))$ and such model is a minimum energy one. Observe that we could have used higher polynomial degrees, yielding higher order models and obtaining a hierarchy of models that furnish increasingly better solutions.

Rewriting (1.2) in variational form, it is not hard to check that if $\tilde{u}^\varepsilon(\underset{-}{x}) = \omega_0(\underset{\sim}{x}) + \omega_1(\underset{\sim}{x})x_3^\varepsilon$, then

$$\begin{gathered}(\partial_{11}+\partial_{22})\omega_0 = -\frac{1}{2}f^0, \quad \frac{2\varepsilon^2}{3}(\partial_{11}+\partial_{22})\omega_1 - 2\omega_1 = -f^1 \qquad \text{in } \Omega,\\ \omega_0 = \omega_1 = 0 \qquad \text{on } \partial\Omega,\end{gathered} \tag{1.3}$$

where

$$f^0(\underset{\sim}{x}^\varepsilon) = \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} f^\varepsilon(\underset{\sim}{x}^\varepsilon, x_3^\varepsilon)\, dx_3^\varepsilon, \quad f^1(\underset{\sim}{x}^\varepsilon) = \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} f^\varepsilon(\underset{\sim}{x}^\varepsilon, x_3^\varepsilon)x_3^\varepsilon\, dx_3^\varepsilon. \tag{1.4}$$

Note that the equations (1.3) are independent of each other. We can express in a unique way any function defined on P^ε as a sum of its even and odd parts with respect to x_3^ε. The even part of f^ε appears only in the equation for ω_0, and the odd part of f^ε shows up in the equation for ω_1. Also, the equation determining ω_1 is singularly perturbed, but this is not the case for the equation determining ω_0. If higher order methods were used, we would have two singularly perturbed independent systems of equations, corresponding to the even and odd parts of $\tilde{u}^\varepsilon$. Similar splitting also occurs in linearized plate models in elasticity, where, for an isotropic plate, the equations decouple into two independent problems corresponding to bending and stretching of a plate.

The natural question of how close $\tilde{u}^\varepsilon$ is to u^ε is not easy to answer due to the complex influence of ε in both the original and model solutions. We resolve this, not by comparing the exact and model solutions directly, but rather by first looking at the difference between the solutions and their truncated asymptotic expansions, and then comparing both asymptotic expansions. This is possible because the same projection used to define each model can be used to find the first terms of the asymptotic expansion of the model. This allows us to compare corresponding terms of the expansions. Schematically, this is how it works:

$$\begin{array}{ccc} u^\varepsilon & \longleftrightarrow & \begin{array}{l}\text{Asymptotic}\\ \text{Expansion of } u^\varepsilon\end{array} \\ \updownarrow & & \updownarrow \\ u^\varepsilon(p) & \longleftrightarrow & \begin{array}{l}\text{Asymptotic}\\ \text{Expansion of } u^\varepsilon(p)\end{array} \end{array}$$

FIGURE 1.1. Scheme of the analysis.

To develop an asymptotic expansion and to be able to conclude estimates that clearly show the influence of ε, it is convenient to define domains and functions that are independent of ε. A classical approach is to define the scaled domain $P = \Omega \times (-1,1)$. A point $\underline{x} = (\underset{\sim}{x}, x_3)$ in P is related to a point $\underline{x}^\varepsilon$ in P^ε by $\underset{\sim}{x} = \underset{\sim}{x}^\varepsilon$, $x_3 = \varepsilon^{-1} x_3^\varepsilon$. We set $f(\underline{x}) = f^\varepsilon(\underline{x}^\varepsilon)$ and assume that f is independent of ε.

Consider then the asymptotic expansion

$$\begin{aligned} u^\varepsilon(\underline{x}^\varepsilon) \sim \zeta^0(\underset{\sim}{x}^\varepsilon) + \varepsilon^2 u^2(\underset{\sim}{x}^\varepsilon, \varepsilon^{-1} x_3^\varepsilon) + \varepsilon^4 u^4(\underset{\sim}{x}^\varepsilon, \varepsilon^{-1} x_3^\varepsilon) + \cdots \\ - \chi(\rho)\big[\varepsilon^2 U^2(\varepsilon^{-1}\rho, \theta, \varepsilon^{-1} x_3^\varepsilon) + \varepsilon^3 U^3(\varepsilon^{-1}\rho, \theta, \varepsilon^{-1} x_3^\varepsilon) + \cdots\big]. \end{aligned}$$

The functions U^2, U^3, etc are boundary correctors, functions that decay exponentially fast away from the lateral boundary, indicating the presence of boundary layers in the original solution. These functions are defined only close to the lateral boundary ∂P_L^ε, and can be expressed in a simpler form if we use a local coordinate system. So, we indicate a point $\underset{\sim}{x}^\varepsilon$ close enough to $\partial\Omega$ by (ρ, θ), where ρ is the distance between $\underset{\sim}{x}^\varepsilon$ and $\partial\Omega$, and θ gives roughly arclength along the boundary, see Chapter 5. Finally, χ is a cutoff function (independent of ε) that equals the unity close to $\partial\Omega$.

All the terms in the asymptotic expansion can be fully characterized (and we do so in Chapter 5), but we describe here the first few ones only. The leading term

$$\zeta^0 = \omega_0,$$

where ω_0 is defined by (1.3). Next, with $\underset{\sim}{x} \in \Omega$ as a parameter, we define u^2 by the following one-dimensional Neumann problem:

$$\begin{aligned} \partial_{33} u^2(\underset{\sim}{x}, x_3) &= -f(\underset{\sim}{x}, x_3) + \frac{1}{2}\int_{-1}^{1} f(\underset{\sim}{x}, x_3)\, dx_3 \qquad \text{in } (-1,1), \\ \frac{\partial u^2}{\partial n}(\underset{\sim}{x}, x_3) &= 0 \qquad \text{on } \{-1,1\}, \\ \int_{-1}^{1} u^2(\underset{\sim}{x}, x_3)\, dx_3 &= 0. \end{aligned} \tag{1.5}$$

Abusing notation, we also allow u^2 to take values in P^ε through the scaling $x_3^\varepsilon = \varepsilon x_3$. Observe that as, by assumption, f is independent of ε, then ζ^0 and u^2 are also independent of ε. Finally, with θ as a parameter, the boundary layer term $U^2(\hat\rho, \theta, x_3)$ solves the Laplacian problem in a semi-infinite strip:

$$\begin{aligned}(\partial_{\hat\rho\hat\rho} + \partial_{33})U^2 = 0 \qquad &\text{in } \mathbb{R}^+ \times (-1,1),\\ \frac{\partial U^2}{\partial n} = 0 \qquad &\text{on } \mathbb{R}^+ \times \{-1,1\},\\ U^2(0,\theta,x_3) = u^2(0,\theta,x_3) \qquad &\text{for } x_3 \in (-1,1).\end{aligned} \tag{1.6}$$

In Chapter 6 we discuss several issues related to problem (1.6) and others of same nature. From the theory developed there, we can conclude that there exists a unique solution in $H^1(\mathbb{R}^+ \times (-1,1))$ for (1.6). Furthermore, this solution decays exponentially to zero as $\hat\rho$ increases.

It is possible to evaluate the difference between a truncated asymptotic expansion with arbitrary number of terms and the exact solution, see Table 5.1. For now, the following estimate suffices:

$$\|u^\varepsilon - \zeta^0 - \varepsilon^2 u^2\|_{H^1(P^\varepsilon)} \le c\varepsilon^2. \tag{1.7}$$

In (1.7), and in the rest of this introduction, we assume that c denotes a positive constant that depends on f and Ω only, and might assume different values in different occurrences. From the triangle inequality,

$$\|u^\varepsilon\|_{H^1(P^\varepsilon)} \ge \|\zeta^0\|_{H^1(P^\varepsilon)} - \varepsilon^2\|u^2\|_{H^1(P^\varepsilon)} - \|u^\varepsilon - \zeta^0 - \varepsilon^2 u^2\|_{H^1(P^\varepsilon)}.$$

Recall that $\zeta^0 = \omega^0$ and from (1.3), (1.4), (1.5), and (1.7) we gather that if $f^0 \ne 0$ then

$$\|u^\varepsilon\|_{H^1(P^\varepsilon)} \ge c_0\varepsilon^{1/2} - c_1\varepsilon^{3/2}.$$

Then

$$\|u^\varepsilon\|_{H^1(P^\varepsilon)} \geq c\varepsilon^{1/2},$$

for ε sufficiently small (and the bound is obvious when ε is not small). Similarly, if $f^0 = 0$, but f itself is a nontrivial function then

$$\|u^\varepsilon\|_{H^1(P^\varepsilon)} \geq c\varepsilon^{3/2}.$$

Under the assumption that f is *not* identically zero, we rewrite the above lower bounds as

$$\|u^\varepsilon\|_{H^1(P^\varepsilon)} \geq c\frac{\varepsilon^{3/2}}{\nu(\varepsilon)}, \quad \text{where } \nu(\varepsilon) = \begin{cases} 1 & \text{if } f^0 = 0, \\ \varepsilon & \text{otherwise.} \end{cases} \tag{1.8}$$

A similar asymptotic expansion holds for the approximation $\tilde{u}^\varepsilon$:

$$\begin{aligned} \tilde{u}^\varepsilon(\underline{x}^\varepsilon) \sim \zeta^0(\underset{\sim}{x}^\varepsilon) + \varepsilon^2\tilde{u}^2(\underset{\sim}{x}^\varepsilon, \varepsilon^{-1}x_3^\varepsilon) + \varepsilon^4\tilde{u}^4(\underset{\sim}{x}^\varepsilon, \varepsilon^{-1}x_3^\varepsilon) + \cdots \\ - \chi(\rho)\big[\varepsilon^2\tilde{U}^2(\varepsilon^{-1}\rho, \theta, \varepsilon^{-1}x_3^\varepsilon) + \varepsilon^3\tilde{U}^3(\varepsilon^{-1}\rho, \theta, \varepsilon^{-1}x_3^\varepsilon) + \cdots\big], \end{aligned}$$

We again describe the first few terms only. Let $\hat{\mathbb{P}}_p(-1,1)$ be the space of polynomials of degree p in $(-1,1)$ with zero average. With $\underset{\sim}{x} \in \Omega$ as a parameter, $\tilde{u}^2(\underset{\sim}{x}, \cdot) \in \hat{\mathbb{P}}_1(-1,1)$ and

$$\int_{-1}^{1} \partial_3\big[u^2(\underset{\sim}{x}, x_3) - \tilde{u}^2(\underset{\sim}{x}, x_3)\big]\partial_3 v(x_3)\, dx_3 = 0 \qquad \text{for all } v \in \hat{\mathbb{P}}_1(-1,1). \tag{1.9}$$

Similarly, for each θ, $\tilde{U}^2$ is the Galerkin projection of U^2 into the space of linear polynomials in x_3 with coefficients in $H^1(\mathbb{R}^+)$.

As for the original solution, it is possible to estimate the difference between $\tilde{u}^\varepsilon$ and a truncated asymptotic expansion with arbitrary number of terms. For instance,

$$\|\tilde{u}^\varepsilon - \zeta^0 - \varepsilon^2\tilde{u}^2\|_{H^1(P^\varepsilon)} \leq c\varepsilon^2. \tag{1.10}$$

We are ready now to bound the modeling error. Using the triangle inequality, (1.7), and (1.10) we have that

$$\begin{aligned}\|u^\varepsilon - \tilde{u}^\varepsilon\|_{H^1(P^\varepsilon)} \leq & \|u^\varepsilon - \zeta^0 - \varepsilon^2 u^2\|_{H^1(P^\varepsilon)} + \varepsilon^2\|u^2 - \tilde{u}^2\|_{H^1(P^\varepsilon)} \\ & + \|\tilde{u}^\varepsilon - \zeta^0 - \varepsilon^2 \tilde{u}^2\|_{H^1(P^\varepsilon)} \leq \varepsilon^2 \|\partial_{x_3^\varepsilon} u^2 - \partial_{x_3^\varepsilon}\tilde{u}^2\|_{L^2(P^\varepsilon)} + c\varepsilon^2.\end{aligned} \tag{1.11}$$

To avoid confusion, we use $\partial_{x_3^\varepsilon}$ to indicate differentiation with respect to x_3^ε. After a change of coordinates, we gather from (1.9) that

$$\|\partial_{x_3^\varepsilon} u^2 - \partial_{x_3^\varepsilon}\tilde{u}^2\|_{L^2(P^\varepsilon)} \leq \varepsilon^{-1/2} \inf_{v \in L^2(\Omega;\hat{\mathbb{P}}_1(-1,1))} \|\partial_3 u^2 - \partial_3 v\|_{L^2(P)}, \tag{1.12}$$

where we define $L^2(\Omega; \hat{\mathbb{P}}_p(-1,1))$ as the space of polynomials in $\hat{\mathbb{P}}_p(-1,1)$ with coefficients in $L^2(\Omega)$. From (1.8), (1.11), and (1.12) we see that the relative error norm

$$\frac{\|u^\varepsilon - \tilde{u}^\varepsilon\|_{H^1(P^\varepsilon)}}{\|u^\varepsilon\|_{H^1(P^\varepsilon)}} \leq c\nu(\varepsilon)\Big(\inf_{v \in L^2(\Omega;\hat{\mathbb{P}}_1(-1,1))} \|\partial_3 u^2 - \partial_3 v\|_{L^2(P)} + \varepsilon^{1/2}\Big), \tag{1.13}$$

In estimate (1.13) we expose how the term with lowest power in ε behaves. Note that the infimum above does not depend on ε. In general, if polynomials of order p are used in (1.2), the final result is

$$\begin{aligned}\frac{\|u^\varepsilon - \tilde{u}^\varepsilon\|_{H^1(P^\varepsilon)}}{\|u^\varepsilon\|_{H^1(P^\varepsilon)}} &\leq c\nu(\varepsilon)\Big(\inf_{v \in L^2(\Omega;\hat{\mathbb{P}}_p(-1,1))} \|\partial_3 u^2 - \partial_3 v\|_{L^2(P)} + \varepsilon^{1/2}\Big) \\ &\leq c\nu(\varepsilon)\big(p^{-1-s}\|f\|_{L^2(\Omega;H^s(-1,1))} + \varepsilon^{1/2}\big),\end{aligned} \tag{1.14}$$

so the model converges as the order of the approximation increases.

Remark 1.1. Note from (1.14) that when $\nu = 1$, or equivalently, when $f^0 = 0$, there is no convergence in ε whatever, only in p. As we show below, this also occurs if we consider the relative L^2 error norm. The lack of convergence is due to the fact that, in general, the asymptotic expansion for the original and model solutions differ already in the first terms, since $\zeta^0 = 0$.

In the above example we used the H^1 norm, but other choices would work as well. For instance, we can consider the L^2 norm in P^ε and then

$$\|u^\varepsilon - \zeta^0 - \varepsilon^2 u^2\|_{L^2(P^\varepsilon)} \leq c\varepsilon^3, \quad \|u^\varepsilon\|_{L^2(P^\varepsilon)} \geq c\frac{\varepsilon^{5/2}}{[\nu(\varepsilon)]^2},$$

where ν is defined in (1.8). Using the same reasoning as in the estimate for the H^1 norm, we can bound the error due to the minimum energy approximation as follows:

$$\frac{\|u^\varepsilon - \tilde{u}^\varepsilon\|_{L^2(P^\varepsilon)}}{\|u^\varepsilon\|_{L^2(P^\varepsilon)}} \leq c[\nu(\varepsilon)]^2\big(\|u^2 - \tilde{u}^2\|_{L^2(P^\varepsilon)} + \varepsilon^{1/2}\big). \tag{1.15}$$

Assuming that we use polynomials of order p in (1.2), we can, from the definition of $\tilde{u}^2$ and a duality argument [14], prove that

$$\|u^2 - \tilde{u}^2\|_{L^2(P)} \leq cp^{-1} \inf_{v\in L^2(\Omega;\hat{\mathbb{P}}_p(-1,1))} \|\partial_3 u^2 - \partial_3 v\|_{L^2(P)} \leq cp^{-2-s}\|f\|_{L^2(\Omega;H^s(-1,1))}.$$

So, again, convergence with respect to p holds in (1.15). Also, if $f^0 = 0$ or equivalently, $\nu = 1$, then there is no convergence in ε. See Remark 1.1.

We now proceed to summarize the contents of each chapter, highlighting the main results. We start by studying the Laplacian problem in a thin, two-dimensional strip. The reason for considering this simpler domain is that, up to technicalities that only make the understanding more arduous, the modus operandi in two or three dimensions is the same. The technicalities involve a flattening of the lateral boundary in the three-dimensional problem, in order to define boundary correctors. It is a cumbersome procedure that we postpone until Chapter 5. So, in Chapter 2 we define a two-dimensional Laplacian problem very similar to (1.1). We carefully describe the asymptotic expansion of the solution, making clear the influence of ε, present several results concerning its terms, and estimate approximation properties of the truncated expansion. Theorem 2.2.2 and Table 2.1 display some convergence estimates.

In Chapter 3, we introduce our first modeling approach, which consists of using a "mixed projection" as we describe next. We first note that the solution of the elliptic problem under consideration, paired up with its derivatives, is the unique critical point, in fact a saddle point, of a certain functional. We seek then approximations for the solution and its derivatives by looking for a critical point of the above mentioned functional, in spaces of functions with polynomial dependence in the transverse direction. This results in equations posed in a lower dimensional, ε-independent domain. It turns out, for the Laplacian problem, that only minimum energy models come up. This does not happen in general. For instance in linearized elasticity, some models generated in essentially the same way will not be of minimum energy type. We proceed next in the chapter as in the example above, first developing an asymptotic expansion and estimating how well it approximates the model solution. We then estimate the modeling errors by using the triangle inequality. The main results are contained in Theorems 3.2.4, 3.3.6, and Table 3.1. We introduce in Chapter 4 a variant modeling technique, where the exact and model solutions are characterized as critical points of a different functional. The models derived will *not* be of minimum energy type, and the error analyses are more involved, but still follow the same basic idea presented in Chapter 3. Theorems 4.2.9 and 4.3.6 contain the convergence results.

Next, in Chapter 5, we extend the results of the previous chapters to the three-dimensional plate P^ε. As we mentioned before, most of the two-dimensional results extend naturally, but some extra difficulties appear. In order of appearance, the main results are Theorem 5.1.2, Table 5.1, Theorems 5.2.1 and 5.2.4, Table 5.2, Theorems 5.3.1, and 5.3.4.

The boundary correctors are naturally defined in a semi-infinite strip, and in Chapter 6 we deal with them, considering not only the boundary correctors for the exact solutions, but also for its approximations. We prove existence, uniqueness, and regularity results, see Theorem 6.1.6. Also, we show that, in general, these functions decay exponentially towards a constant, which we compute explicitly, see Theorems 6.2.5, 6.2.6, 6.2.7, and 6.4.1. To prove such decay, we generalize the work of Horgan and Knowles [33]. We also investigate how close are the boundary layers for the model and exact solutions, see Theorems 6.3.12 and 6.4.8. Most of the technical results needed in this thesis are included in Chapter 6.

Chapter 7 concerns the equations of linear elasticity rather than the Poisson problem considered in the rest of the thesis. We define various models based on the Hellinger–Reissner variational principles for the linearized plate problem under bending and stretching. With Alessandrini et al. [1], [2] we have already introduced these models, but with the exception of two of them, the explicit equations for arbitrary loads were never presented. We do so for the lowest order cases. In Chapter 8, we describe briefly some related works previously done and compare the results therein with the ones obtained in this dissertation. Finally, in appendices A and B we discuss some properties of projection operators and one-dimensional mixed approximations that are needed in several chapters of this thesis.

A main goal of this work is to show convergence of models derived from variational arguments, as exemplified in (1.14) and (1.15). To the best of our knowledge, our convergence results are new.

Although the asymptotic expansions that we develop here are not original, see [41], we try to present them in more detail, and we display error estimates in norms that are not usually considered.

We hope to have contributed to a better understanding of boundary layers through the work developed in Chapter 6. The way we proved decay of solutions is, in our view, simpler than other approaches [35], [23], although is not entirely new [19], [33]. The principal part in this chapter is the investigation of how the boundary layers for the models approximate the boundary layers for the exact solution.

As an application of the variational approach, we present several models for an isotropic elastic plate. These are the lower order cases in the various possible hierarchies of models. For both stretching and bending of plates, we recover classical models (membrane and Reissner–Mindlin)—with added load effects that are not usually considered—and other, more sophisticated models.

We now briefly introduce and explain some basic notation that we use throughout the thesis. For a given open domain D, we denote its outward normal vector by $\underline{n}$ in three dimensions and $\underset{\sim}{n}$ in two dimensions. If s is a real number, then $H^s(D)$ is the Sobolev space of order s, and $\mathring{H}^s(D)$ is the closure in $H^s(D)$ of the set of smooth functions with compact support. For $m \in \mathbb{N}$ and a certain separable Hilbert space E, we denote $H^m(D, E)$ as the space of functions defined on D with values in E such that the E-norm of all partial derivatives of order less or equal to m are in $L^2(D)$. Also, $\hat{L}^2(-a, a)$ is the set of square integrable functions with mean value zero in the domain $(-a, a)$ for a positive number a. And $\mathcal{D}(D)$ denotes the space of $\mathcal{C}^\infty$ functions in D with compact support, while $\mathcal{D}'(D)$ denotes the space of distributions.

As we have already hinted, we use one underbar for 3-vectors and one underbar for 2-vectors. We can then decompose 3-vectors as follows:

$$\underline{u} = \begin{pmatrix} \underset{\sim}{u} \\ u_3 \end{pmatrix}.$$

The indices 2 and 3 denote quantities in the transverse direction in two and three dimensions respectively. Moreover, when necessary, we use ∂_y to denote differentiation with respect to the variable y. We also use the symbol $'$ to denote differentiation of functions of a single variable. Finally, in several upper bounds we will use the constant C, which is always independent of ε but which may assume different values in different locations.

Chapter 2

The Poisson problem in a thin rectangle

In this chapter we study the Poisson problem in a thin rectangle, developing an asymptotic expansion for the solution and presenting rigorous estimates for the difference between the solution itself and its truncated asymptotic expansion.

Section 2.1 – The asymptotic expansion. Consider the rectangle $R^\varepsilon = (-1,1) \times (-\varepsilon, \varepsilon)$ with lateral boundary $\partial R^\varepsilon_L = \{-1,1\} \times (-\varepsilon,\varepsilon)$ and top and bottom boundaries $\partial R^\varepsilon_\pm = (-1,1) \times \{-\varepsilon,\varepsilon\}$. We assume that $u^\varepsilon \in H^1(R^\varepsilon)$ satisfies (in the weak sense)

$$\begin{gathered} \Delta\, u^\varepsilon = -f^\varepsilon \qquad \text{in } R^\varepsilon, \\ \frac{\partial u^\varepsilon}{\partial n} = g^\varepsilon \text{ on } \partial R^\varepsilon_\pm, \quad u^\varepsilon = 0 \text{ on } \partial R^\varepsilon_L, \end{gathered} \tag{2.1.1}$$

where $f^\varepsilon : R^\varepsilon \to \mathbb{R}$ and $g^\varepsilon : \partial R^\varepsilon_\pm \to \mathbb{R}$.

To develop an asymptotic expansion for u^ε (see the introduction) we define the ε-independent domain $R = (-1,1) \times (-1,1)$. A point $\underset{\sim}{x} = (x_1, x_2)$ in R is related to a point $\underset{\sim}{x}^\varepsilon$ in R^ε by $x_1 = x_1^\varepsilon$, $x_2 = \varepsilon^{-1} x_2^\varepsilon$. We accordingly define $\partial R_L = \{-1,1\} \times (-1,1)$, and $\partial R_\pm = (-1,1) \times \{-1,1\}$.

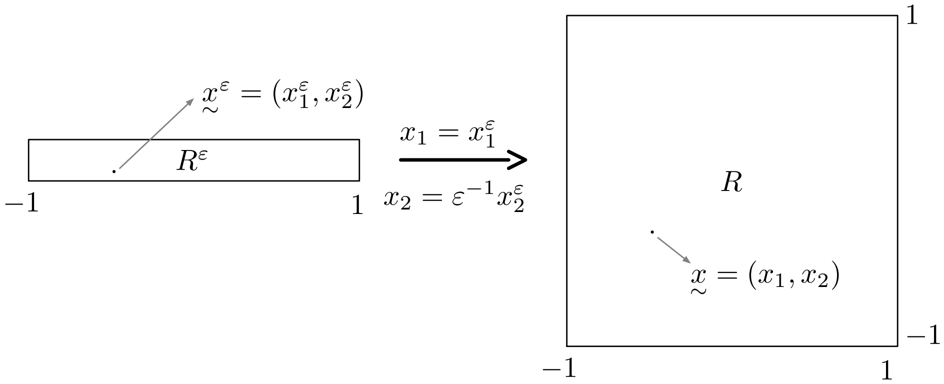

Figure 2.1. Scaling of the rectangle.

In this new domain we define $u(\varepsilon)(\underset{\sim}{x}) = u^\varepsilon(\underset{\sim}{x}^\varepsilon)$, $f(\underset{\sim}{x}) = f^\varepsilon(\underset{\sim}{x}^\varepsilon)$, and $g(\underset{\sim}{x}) = \varepsilon^{-1} g^\varepsilon(\underset{\sim}{x}^\varepsilon)$. We infer from (2.1.1) that

$$\begin{gathered}(\partial_{11} + \varepsilon^{-2}\partial_{22})u(\varepsilon) = -f \qquad \text{in } R,\\ \frac{\partial u(\varepsilon)}{\partial n} = \varepsilon^2 g \qquad \text{on } \partial R_\pm,\\ u(\varepsilon) = 0 \qquad \text{on } \partial R_L.\end{gathered} \tag{2.1.2}$$

We assume that f, g are ε-independent, but this restriction could be relaxed, for instance by assuming that f and g can be represented as a power series in ε, plus a small remainder, see [41].

Consider the asymptotic expansion

$$u(\varepsilon) \sim u^0 + \varepsilon^2 u^2 + \varepsilon^4 u^4 + \cdots, \tag{2.1.3}$$

and formally substitute it in (2.1.2). Grouping together terms with same power in ε we have

$$\begin{gathered}\varepsilon^{-2}\partial_{22}u^0 + \left(\partial_{11}u^0 + \partial_{22}u^2\right) + \varepsilon^2\left(\partial_{11}u^2 + \partial_{22}u^4\right) + \cdots = -f,\\ \frac{\partial u^0}{\partial n} + \varepsilon^2\frac{\partial u^2}{\partial n} + \varepsilon^4\frac{\partial u^4}{\partial n} + \cdots = \varepsilon^2 g \qquad \text{on } \partial R_\pm.\end{gathered}$$

It is then natural to require that

$$\partial_{22}u^0 = 0, \tag{2.1.4}$$

$$\partial_{22}u^2 = -f - \partial_{11}u^0, \tag{2.1.5}$$

$$\partial_{22}u^{2k} = -\partial_{11}u^{2k-2}, \qquad \text{for all } k > 1, \tag{2.1.6}$$

along with the boundary conditions

$$\frac{\partial u^{2k}}{\partial n} = \delta_{k1} g \text{ on } \partial R_\pm, \qquad \text{for all } k \in \mathbb{N}. \tag{2.1.7}$$

Equations (2.1.4)–(2.1.7) define a sequence of Neumann problems on the interval $x_2 \in (-1,1)$ parametrized by $x_1 \in (-1,1)$. If the data for these problems is compatible then the solution can be written as

$$u^{2k}(\underset{\sim}{x}) = \overset{\circ}{u}{}^{2k}(\underset{\sim}{x}) + \zeta^{2k}(x_1), \qquad \text{for all } k \in \mathbb{N}, \tag{2.1.8}$$

where

$$\int_{-1}^{1} \overset{\circ}{u}{}^{2k}(x_1, x_2)\, dx_2 = 0, \tag{2.1.9}$$

with $\overset{\circ}{u}{}^{2k}$ uniquely determined, but ζ^{2k} an arbitrary function of x_1 only. From the Dirichlet boundary condition in (2.1.1), it would be natural to require that $u^{2k} = 0$ on ∂R_L. This is equivalent to imposing

$$\zeta^{2k}(-1) = \zeta^{2k}(1) = 0, \tag{2.1.10}$$

$$\overset{\circ}{u}{}^{2k} = 0 \text{ on } \partial R_L. \tag{2.1.11}$$

However, in general, only (2.1.10) can be imposed and (2.1.11) will not hold. We shall correct this discrepancy latter. Now we show that the functions ζ^{2k}, $\overset{\circ}{u}{}^{2k}$ (and so u^{2k}) are uniquely determined from (2.1.4)–(2.1.10). In fact, (2.1.4) and (2.1.7) yields $\overset{\circ}{u}{}^{0} = 0$. From the compatibility of (2.1.5) and (2.1.7) we see that

$$\partial_{11}\zeta^0(x_1) = -\frac{1}{2}\int_{-1}^{1} f(x_1, x_2)\, dx_2 - \frac{1}{2}[g(x_1, 1) + g(x_1, -1)], \tag{2.1.12}$$

which together with (2.1.10), determines ζ^0 and then, from (2.1.8), u^0. In view of the compatibility condition (2.1.12), $\overset{\circ}{u}{}^{2}$ is fully determined by (2.1.5) and (2.1.7). Next, the Neumann problem (2.1.6), (2.1.7) admits a solution for $k > 1$ if and only if $\partial_{11}\zeta^{2k-2} = 0$. But in view of (2.1.10), this means $\zeta^{2k-2} = 0$, for $k > 1$, and then $\overset{\circ}{u}{}^{2k}$ is uniquely determined from (2.1.6), (2.1.7). Note that $u^0 = \zeta^0$ and $u^{2k} = \overset{\circ}{u}{}^{2k}$ for $k \geq 1$.

Observe that u^0 satisfies all the boundary conditions imposed since $\mathring{u}^0 = 0$ and so (2.1.11) holds for $k = 0$. In general this is not the case for u^2, u^4, etc, as they do not vanish on the lateral side of the domain (although their vertical integrals do). We therefore introduce the boundary corrector U_- and its asymptotic expansion

$$U_- \sim \varepsilon^2 U_-^2 + \varepsilon^4 U_-^4 + \cdots , \tag{2.1.13}$$

where $(\partial_{11} + \varepsilon^{-2}\partial_{22})U_- = 0$ with $\partial U_-/\partial n$ vanishes at $\partial R_\pm$. Note that if we make the change of coordinates $\hat{\rho}_- = \varepsilon^{-1}(1 + x_1)$, we have that $(\partial_{\hat{\rho}_-\hat{\rho}_-} + \partial_{22})U_- = 0$ and the equation no longer depends on ε. This motivates us to pose the boundary corrector problem in the semi-infinite strip $\Sigma = \mathbb{R}^+ \times (-1, 1)$. We impose the vanishing Neumann condition on the union of its top and bottom boundaries $\partial\Sigma_\pm = \mathbb{R}^+ \times \{-1, 1\}$. For positive integers k, define $U_-^{2k}(\hat{\rho}_-, x_2)$ by

$$\begin{aligned} \Delta\, U_-^{2k} &= 0 \qquad \text{in } \Sigma, \\ \frac{\partial U_-^{2k}}{\partial n} &= 0 \qquad \text{on } \partial\Sigma_\pm, \end{aligned} \tag{2.1.14}$$

$$U_-^{2k}(0, x_2) = u^{2k}(-1, x_2) \qquad \text{for } x_2 \in (-1, 1). \tag{2.1.15}$$

Similarly, we set $\hat{\rho}_+ = \varepsilon^{-1}(1 - x_1)$, define U_+^{2k} as the solution of (2.1.14) satisfying the boundary conditions $U_+^{2k}(0, \cdot) = u^{2k}(1, \cdot)$, and define U_+ analogously to (2.1.13). We treat this problem in full detail in Chapter 6.

Combining (2.1.3) and the boundary layer expansions we have that

$$\begin{aligned} u^\varepsilon(\underset{\sim}{x}^\varepsilon) \sim \zeta^0(x_1^\varepsilon) + \sum_{k=1}^{\infty} \varepsilon^{2k} u^{2k}(x_1^\varepsilon, \varepsilon^{-1}x_2^\varepsilon) \\ - \sum_{k=1}^{\infty} \varepsilon^{2k} \big[U_-^{2k}(\hat{\rho}_-, \varepsilon^{-1}x_2^\varepsilon) + U_+^{2k}(\hat{\rho}_+, \varepsilon^{-1}x_2^\varepsilon)\big]. \end{aligned} \tag{2.1.16}$$

We proceed to analyze some properties of the above expansion, introducing first some new notation:

$$
\begin{aligned}
|g|_{C(\partial R_L)} &= |g(-1,-1)| + |g(-1,1)| + |g(1,-1)| + |g(1,1)|, \\
|||(f,g)|||_{N,\partial R_L} &= \sum_{k=0}^{N} \big(\|\partial_1^{2k} f\|_{L^2(\partial R_L)} + |\partial_1^{2k} g|_{C(\partial R_L)} \big), \\
\|v\|_{(m,n,R)} &= \|v\|_{H^m((-1,1);H^n(-1,1))}, \\
|||(f,g)|||_{N,R} &= \|f\|_{(N,0,R)} + \|g\|_{H^N(\partial R_\pm)}.
\end{aligned}
$$

Observe that $|||\cdot|||_{N,\partial R_L}$ is a norm involving derivatives of order of order $2N$.

The following regularity results hold.

Lemma 2.1.1. *Assume that $\tilde{f} \in H^{s-2}(-1,1)$, where $s \geq 2$ is a real number. Assume also that a, b are real numbers such that $b - a = \int_{-1}^{1} \tilde{f}(s)\,ds$. Then there exists a unique $u \in H^1(-1,1) \cap \hat{L}^2(-1,1)$ satisfying*

$$
\begin{aligned}
u'' &= \tilde{f} \qquad \text{in } (-1,1), \\
u'(-1) &= a, \qquad u'(1) = b,
\end{aligned}
$$

weakly. Furthermore $\|u\|_{H^s(-1,1)} \leq \|\tilde{f}\|_{H^{s-2}(-1,1)} + |a| + |b|$.

Lemma 2.1.2. *For any nonnegative integer m, and real number $s \geq 1$, there exists a constant C such that if $\tilde{f} \in H^m((-1,1); H^{s-2}(-1,1))$, $\tilde{g} \in H^m(\partial R_\pm)$, and*

$$
\begin{aligned}
\partial_{22} u(x_1,x_2) &= \tilde{f}(x_1,x_2) \qquad \text{for } x_2 \in (-1,1), \\
\partial_2 u(x_1,x_2) &= \tilde{g}(x_1,x_2) \qquad \text{for } x_2 \in \{-1,1\},
\end{aligned}
\tag{2.1.17}
$$

weakly for almost every $x_1 \in (-1,1)$, then

$$
\begin{aligned}
&\|\partial_1^j u(x_1,\cdot)\|_{H^s(-1,1)} \\
&\qquad \leq C\big(\|\partial_1^j \tilde{f}(x_1,\cdot)\|_{H^{s-2}(-1,1)} + |\partial_1^j \tilde{g}(x_1,-1)| + |\partial_1^j \tilde{g}(x_1,1)|\big), \qquad j = 0,\dots,m, \\
&\|u\|_{(m,s,R)} \leq C\big(\|\tilde{f}\|_{(m,s-2,R)} + \|\tilde{g}\|_{H^m(\partial R_\pm)}\big).
\end{aligned}
$$

Proof. Note that $\partial_1^j u$ solve (2.1.17) weakly for $j = 0, \dots, m$ with $\partial_1^j \tilde{f}$ on the right hand side and $\partial_1^j \tilde{g}$ as boundary condition. Thus, from Lemma 2.1.1, the first inequality follows. Also

$$\begin{aligned}
\|u\|^2_{(m,s,R)} &= \sum_{j=0}^{m} \int_{-1}^{1} \|\partial_1^j u(x_1, \cdot)\|^2_{H^s(-1,1)}\, dx_1 \\
&\leq C \sum_{j=0}^{m} \int_{-1}^{1} \|\partial_1^j \tilde{f}(x_1, \cdot)\|^2_{H^{s-2}(-1,1)} + |\partial_1^j \tilde{g}(x_1, -1)|^2 + |\partial_1^j \tilde{g}(x_1, 1)|^2\, dx_1 \\
&= C\big(\|\tilde{f}\|^2_{(m,s-2,R)} + \|\tilde{g}\|^2_{H^m(\partial R_\pm)}\big).
\end{aligned}$$

□

We can now state regularity results for some of the terms in the asymptotic expansion. For the lemma immediately below, the bound for ζ^0 follows from standard regularity estimates for equation (2.1.12). The bounds for u^2 follow from (2.1.5), (2.1.7) and Lemma 2.1.2.

Lemma 2.1.3. *Let j and m be nonnegative integers and s be a real number such that $s \geq 2$. Then there exists a constant C such that*

$$\|\zeta^0\|_{H^{m+1}(-1,1)} \leq C|||(f,g)|||_{m-1,R},$$

$$\|\partial_1^j u^2(x_1, \cdot)\|_{H^s(-1,1)} \leq C\big(\|\partial_1^j f(x_1, \cdot)\|_{H^{s-2}(-1,1)} + |\partial_1^j g(x_1, -1)| + |\partial_1^j g(x_1, 1)|\big)$$

$$\|u^2\|_{(m,s,R)} \leq C\big(\|f\|_{(m,s-2,R)} + \|g\|_{H^m(\partial R_\pm)}\big).$$

A combination of (2.1.6), (2.1.7), and Lemma 2.1.2 yields the next lemma.

Lemma 2.1.4. *Let j, k, m, be nonnegative integers and s be a real number such that $s \geq 2$, $k \geq 2$. Then there exists a constant C such that*

$$\|\partial_1^j u^{2k}(x_1, \cdot)\|_{H^s(-1,1)} \leq C\|\partial_1^{j+2} u^{2k-2}(x_1, \cdot)\|_{H^{s-2}(-1,1)},$$

$$\|u^{2k}\|_{(m,s,R)} \leq C\|u^{2k-2}\|_{(m+2,s-2,R)}.$$

As a consequence Lemmas 2.1.3 and 2.1.4, the following regularity results hold.

Lemma 2.1.5. *Let j, k, m be nonnegative integers and s be a real number such that $s \geq 2k \geq 2$. Then there exists a constant C such that*

$$\|\partial_1^j u^{2k}(x_1,\cdot)\|_{H^s(-1,1)} \leq C\big(\|\partial_1^{2k-2+j} f(x_1,\cdot)\|_{H^{s-2k}(-1,1)} + |\partial_1^{2k-2+j} g(x_1,-1)| + |\partial_1^{2k-2+j} g(x_1,1)|\big),$$

$$\|u^{2k}\|_{(m,s,R)} \leq C\big(\|f\|_{(m+2k-2,s-2k,R)} + \|g\|_{H^{m+2k-2}(\partial R_\pm)}\big).$$

Proof. We prove this result by induction on k. For $k = 1$, the result follows from Lemma 2.1.3. Assume now that the results holds for $k = \bar{k} \geq 1$. Then, from (2.1.6), (2.1.7) and Lemma 2.1.4, we have that for $k = \bar{k} + 1$,

$$\begin{aligned}\|\partial_1^j u^{2\bar{k}+2}(x_1,\cdot)\|_{H^s(-1,1)} &\leq \|\partial_1^{j+2} u^{2\bar{k}}(x_1,\cdot)\|_{H^{s-2}(-1,1)} \\ &\leq C\big(\|\partial_1^{2\bar{k}+j} f(x_1,\cdot)\|_{H^{s-2\bar{k}-2}(-1,1)} + |\partial_1^{2\bar{k}+j} g(x_1,-1)| + |\partial_1^{2\bar{k}+j} g(x_1,1)|\big),\end{aligned}$$

where we used the inductive hypothesis at the second inequality. Hence the first bound of the lemma holds, and it implies the second one. □

We now describe the properties of the boundary correctors. Existence and uniqueness follow from Theorem 6.1.6 and the exponential decay follows from Lemma 6.2.2 and Theorem 6.2.5, with $M = 0$, and $C_W = 4/\pi^2$, since $u^{2k}(-1,\cdot) \in \hat{L}^2(-1,1)$. The lemma below is a direct application of these results, and similar conclusions hold for U_+^{2k}.

Lemma 2.1.6. *Assume, for a fixed positive integer k, that u^{2k} is defined as above. Then there exists a unique weak solution $U_-^{2k} \in H^1(\Sigma)$ to (2.1.14), (2.1.15). Also, there exists a universal constant C such that*

$$\int_t^\infty \int_{-1}^1 [U_-^{2k}]^2 + |\underset{\sim}{\nabla}\, U_-^{2k}|^2 \, dx_2 \, d\hat{\rho}_- \leq C\|u^{2k}(-1,\cdot)\|^2_{H^{1/2}(-1,1)} \exp(-\pi^2 t/(\pi^2+4)),$$

for every nonnegative real number t.

We now use Lemmas 2.1.6 and 2.1.5 to conclude the result below.

Lemma 2.1.7. *For any positive integer k, there exists a constant C such that*

$$\int_t^\infty \int_{-1}^1 [U_-^{2k}]^2 + |\underset{\sim}{\nabla} U_-^{2k}|^2 \, dx_2 \, d\hat{\rho}_- \\ \leq C\big(\|\partial_1^{2k-2} f\|_{L^2(\partial R_L)} + \|\partial_1^{2k-2} g\|_{C(\partial R_L)}\big) \exp(-\pi^2 t/(\pi^2+4)).$$

for every nonnegative real number t.

We present below upper bounds for some norms of the asymptotic terms in the original (unscaled) domain. To keep the notation simple, we use the same name for functions defined in different domains, if the difference is due to a change of coordinates only. It should always be clear from the context to which domain we are referring. The proofs of the estimates involve a simple exercise of change of coordinates and the use of Lemmas 2.1.3, 2.1.5, and 2.1.7.

Lemma 2.1.8. *For any positive integer k, there exists a constant C such that*

$$\|\zeta^0\|_{H^1(R^\varepsilon)} \leq C\varepsilon^{1/2} |||(f,g)|||_{-1,R},$$

$$\|u^{2k}\|_{L^2(R^\varepsilon)} + \varepsilon \|\partial_{x_2^\varepsilon} u^{2k}\|_{L^2(R^\varepsilon)} \leq C\varepsilon^{1/2} |||(f,g)|||_{2k-2,R},$$

$$\|\partial_1 u^{2k}\|_{L^2(R^\varepsilon)} \leq C\varepsilon^{1/2} |||(f,g)|||_{2k-1,R},$$

$$\begin{aligned} &\|U_-^{2k}\|_{L^2(R^\varepsilon)} + \|U_+^{2k}\|_{L^2(R^\varepsilon)} \\ &\quad + \varepsilon\big(\|\partial_{x_1^\varepsilon} U_-^{2k}\|_{L^2(R^\varepsilon)} + \|\partial_{x_2^\varepsilon} U_-^{2k}\|_{L^2(R^\varepsilon)} + \|\partial_{x_1^\varepsilon} U_+^{2k}\|_{L^2(R^\varepsilon)} + \|\partial_{x_2^\varepsilon} U_+^{2k}\|_{L^2(R^\varepsilon)}\big) \\ &\qquad \leq C\varepsilon\big(\|\partial_1^{2k-2} f\|_{L^2(\partial R_L)} + \|\partial_1^{2k-2} g\|_{C(\partial R_L)}\big). \end{aligned} \tag{2.1.18}$$

Section 2.2 – Error Estimates. We estimate in this section the error between the truncated asymptotic expansion and the real solution:

$$e_N = u^\varepsilon - \zeta^0 - \sum_{k=1}^N \varepsilon^{2k} u^{2k} + \sum_{k=1}^N \varepsilon^{2k} (U_-^{2k} + U_+^{2k}). \tag{2.2.1}$$

In the theorems below, we first bound the error in the H^1 norm of the scaled, ε-independent domain, and then in the original domain.

Theorem 2.2.1. *For any nonnegative integer N, there exists a constant C such that the difference between $u(\varepsilon)$ and its Nth-order asymptotic expansion is bounded as follows*

$$\begin{aligned}\|e_N\|_{H^1(R)} \le C\big[\varepsilon^{2N+3/2}\big(\|\partial_1^{2N} f\|_{L^2(\partial R_L)} + \|\partial_1^{2N} g\|_{C(\partial R_L)}\big)\\ + \varepsilon^{2N+2}|||(f,g)|||_{2N+1,R} + \varepsilon^{3/2}\exp(-\pi^2\varepsilon^{-1}/(2\pi^2+8))|||(f,g)|||_{N,\partial R_L}\big].\end{aligned}$$

Theorem 2.2.2. *For any nonnegative integer N, there exists a constant C such that the difference between the truncated asymptotic expansion and the original solution, measured in the original domain is bounded as follows:*

$$\|e_N\|_{H^1(R^\varepsilon)} \le C\varepsilon^{2N+3/2}\big(|||(f,g)|||_{2N+2,R} + |||(f,g)|||_{N+1,\partial R}\big). \tag{2.2.2}$$

Before we work out the proofs of both theorems, we make some remarks on the results and present some of their consequences.

Remark. The difference between the true solution and the asymptotic series with $2N$ terms is of the same order in the scaled and original domain, but while in the former this is due to the presence of boundary layers, in the latter it is a "global" error. So interior estimates in the scaled domain results in better estimates (a $\varepsilon^{1/2}$ improvement), but in the unscaled one no changes would occur.

Observe that although Theorem 2.2.2 shows that e_N is converging to zero with respect to ε, the rates are deceptive, in the sense that u^ε and the $H^1(R^\varepsilon)$ norm depend on ε as well. A more informative way to measure convergence is through *relative* error norms. Under the assumption that f and g are independent of ε, and that they are not both identically zero, we can conclude from the definitions of ζ^0, u^2 and Theorem 2.2.2 that there exists a constant $C(f,g)$ depending on f and g only such that (cf. (1.7) and (1.8)):

$$\|u^\varepsilon\|_{H^1(R^\varepsilon)} \ge C(f,g)\frac{\varepsilon^{3/2}}{\nu(\varepsilon)}, \quad \text{where } \nu(\varepsilon) = \begin{cases} 1 & \text{if } \zeta^0 = 0, \\ \varepsilon & \text{otherwise.}\end{cases} \tag{2.2.3}$$

We readily conclude from Theorem 2.2.2 that

$$\frac{\|e_N\|_{H^1(R^\varepsilon)}}{\|u^\varepsilon\|_{H^1(R^\varepsilon)}} = \nu(\varepsilon)O(\varepsilon^{2N}).$$

It is easy to estimate the convergence in some other norms as well. For instance, in the $L^2(R^\varepsilon)$ norm, we have from the triangle inequality that

$$\|e_N\|_{L^2(R^\varepsilon)} \le \|e_{N+1}\|_{H^1(R^\varepsilon)} + \varepsilon^{2N+2}\|u^{2N+2}\|_{L^2(R^\varepsilon)} + \varepsilon^{2N+2}\|U_-^{2N+2}\|_{L^2(R^\varepsilon)} + \varepsilon^{2N+2}\|U_+^{2N+2}\|_{L^2(R^\varepsilon)}.$$

Applying previous estimates, we easily conclude that $\|e_N\|_{L^2(R^\varepsilon)} = O(\varepsilon^{2N+5/2})$.

The table below presents various error estimates derived from similar arguments. We assume that f and g are sufficiently smooth functions and we show only the order of the norms with respect ε. "BL" stands for "Boundary Layer" and the "Relative Error" column presents the norm of e_N divided by the norm of u^ε. In parentheses are the *interior estimates* (disregard boundary layers), when these are better than the global ones.

TABLE 2.1. Convergence rates of the truncated asymptotic expansion

	u^ε	BL	e_N	Relative Error
$\|\cdot\|_{L^2(R^\varepsilon)}$	$\nu^{-2}\varepsilon^{5/2}$	ε^3	$\varepsilon^{2N+5/2}$	$\nu^2\varepsilon^{2N}$
$\|\partial_1\cdot\|_{L^2(R^\varepsilon)}$	$\nu^{-3/2}\varepsilon^2(\nu^{-2}\varepsilon^{5/2})$	ε^2	$\varepsilon^{2N+2}(\varepsilon^{2N+5/2})$	$\nu^{3/2}\varepsilon^{2N}(\nu^2\varepsilon^{2N})$
$\|\partial_{x_2^\varepsilon}\cdot\|_{L^2(R^\varepsilon)}$	$\varepsilon^{3/2}$	ε^2	$\varepsilon^{2N+3/2}$	ε^{2N}
$\|\cdot\|_{H^1(R^\varepsilon)}$	$\nu^{-1}\varepsilon^{3/2}$	ε^2	$\varepsilon^{2N+3/2}$	$\nu\varepsilon^{2N}$

We see in the above table that the convergence rates in relative error norms is better when $\nu = \varepsilon$ (i.e., $\zeta^0 \neq 0$) than when $\nu = 1$ ($\zeta^0 = 0$), with one exception. It is also remarkable that the boundary layer "dominates" the relative error only in the L^2 norm of the horizontal derivative. In this case, if $\zeta^0 \neq 0$, the convergence rate is higher in the interior than in the whole domain.

In the rest of this section, we prove Theorems 2.2.1 and 2.2.2.

Definition 2.2.3. Set

$$u_N = \sum_{k=0}^{N} \varepsilon^{2k} u^{2k}, \quad U_N^- = \sum_{k=1}^{N} \varepsilon^{2k} U_-^{2k}, \quad U_N^+ = \sum_{k=1}^{N} \varepsilon^{2k} U_+^{2k},$$
$$u_N^E = u(\varepsilon) - u_N + \chi_- U_N^- + \chi_+ U_N^+,$$

where χ_- is a smooth, ε-independent cut-off function satisfying

$$\chi_-(x_1) = \begin{cases} 1 & \text{if } x_1 < 0, \\ 0 & \text{if } x_1 > \frac{1}{2}, \end{cases}$$

and $\chi_+(x_1) = \chi_-(-x_1)$.

Some results regarding the boundary layer terms are collected below. For simplicity we state and prove results concerning U_N^- only, but these also hold for U_N^+.

Lemma 2.2.4. *For any positive integer N, there exists a constant C such that*

$$\|(1-\chi_-)U_N^-\|_{H^1(R)} + \|\chi_-' U_N^-\|_{L^2(R)} + \varepsilon\|\chi_-' \partial_1 U_N^-\|_{L^2(R)}$$
$$\leq C\varepsilon^{5/2} \exp(-\pi^2\varepsilon^{-1}/(2\pi^2+8))|||(f,g)|||_{N-1,\partial R_L}.$$

Also, for all $v \in H^1(R)$ that vanishes on ∂R_L,

$$\int_R \partial_1 U_N^- \partial_1 v + \varepsilon^{-2}\partial_2 U_N^- \partial_2 v \, d\underset{\sim}{x} = 0. \tag{2.2.4}$$

Proof. Changing coordinates by $\hat{\rho}_- = \varepsilon^{-1}(1+x_1)$ and then using Lemma 2.1.7, we compute for any positive integer $k \leq N$

$$\|\chi_-' U_-^{2k}\|_{L^2(R)}^2 \leq C \int_{-1}^{1}\int_0^{1/2} (U_-^{2k})^2 \, dx_1 \, dx_2 \leq C\varepsilon \int_{-1}^{1}\int_{\varepsilon^{-1}}^{\infty} (U_-^{2k})^2 \, d\hat{\rho}_- \, dx_2$$
$$\leq C\varepsilon\big(\|\partial_1^{2k-2} f\|_{L^2(\partial R_L)} + \|\partial_1^{2k-2} g\|_{C(\partial R_L)}\big)^2 \int_{\varepsilon^{-1}}^{\infty} \exp(-\pi^2\varepsilon^{-1}/(\pi^2+4)) \, d\hat{\rho}_-$$
$$\leq C\varepsilon\big(\|\partial_1^{2k-2} f\|_{L^2(\partial R_L)} + \|\partial_1^{2k-2} g\|_{C(\partial R_L)}\big)^2 \exp(-\pi^2\varepsilon^{-1}/(\pi^2+4)).$$

Using the definition of U_N^- and the triangle inequality,

$$\|\chi'_- U_N^-\|_{L^2(R)} \leq \varepsilon^{5/2} \exp(-\pi^2\varepsilon^{-1}/(2\pi^2+8)) |||(f,g)|||_{N-1,\partial R_L}.$$

The proofs for the other inequalities are analogous. To show that (2.2.4) holds, it is enough to see that

$$\int_R \partial_1 U_-^{2k} \partial_1 v + \varepsilon^{-2} \partial_2 U_-^{2k} \partial_2 v \, d\underset{\sim}{x} = \varepsilon^{-1} \int_{-1}^{1} \int_0^{2\varepsilon^{-1}} \underset{\sim}{\nabla} U_-^{2k} \underset{\sim}{\nabla} \hat{v} \, d\hat{\rho}_- \, dx_2$$
$$= \varepsilon^{-1} \int_{-1}^{1} \int_{\mathbb{R}^+} \underset{\sim}{\nabla} U_-^{2k} \underset{\sim}{\nabla} \hat{v} \, d\hat{\rho}_- \, dx_2 = 0,$$

where we define $\hat{v}(\hat{\rho}_-) = v(x_1)$ if $0 \leq \hat{\rho}_- \leq 2\varepsilon^{-1}$ and $\hat{v} = 0$ otherwise, and then use (2.1.14). $\square$

We obtain now a rough estimate for the asymptotic expansion error. In the proof below, we follow the basic ideas of a similar proof for the elasticity problem [22].

Lemma 2.2.5. *For any positive integer N, there exists a constant C such that*

$$\|u_0^E\|_{H^1(R)} \leq C |||(f,g)|||_{0,R},$$
$$\|u_N^E\|_{H^1(R)} \leq C\big(\varepsilon^{2N} |||(f,g)|||_{2N-1,R}$$
$$+ \varepsilon^{3/2} \exp(-\pi^2\varepsilon^{-1}/(2\pi^2+8)) |||(f,g)|||_{N-1,\partial R_L}\big),$$

where u_N^E is as in Definition 2.2.3.

Proof. We use in this proof that u_N^ε solves the Laplace problem up to arbitrary powers of ε. First note that u_N^E vanishes on ∂R_L. Hence, in view of the Poincaré's inequality,

$$\|u_N^E\|_{H^1(R)}^2 \leq C \int_R (\partial_1 u_N^E)^2 + \varepsilon^{-2} (\partial_2 u_N^E)^2 \, d\underset{\sim}{x}, \tag{2.2.5}$$

and we estimate the right hand side of (2.2.5). We first deal with the simpler case $N = 0$. From the definition of u_0^E,

$$\int_R (\partial_1 u_0^E)^2 + \varepsilon^{-2} (\partial_2 u_0^E)^2 \, d\underset{\sim}{x} = \int_R f u_0^E - \partial_1 u^0 \partial_1 u_0^E \, d\underset{\sim}{x} + \int_{\partial R_\pm} g u_0^E \, dx_1$$
$$\leq C |||(f,g)|||_{0,R} \|u_0^E\|_{H^1(R)},$$

where Lemma 2.1.3 justifies the inequality. Then the case $N = 0$ follows immediately from (2.2.5). We assume now that $N > 0$. Let $v \in H^1(R)$ such that $v = 0$ on ∂R_L. If we define

$$E(N, v) = \int_R \partial_1(u(\varepsilon) - u_N)\partial_1 v + \varepsilon^{-2}\partial_2(u(\varepsilon) - u_N)\partial_2 v \, d\underset{\sim}{x},$$

then, by construction of the asymptotic expansion, we have

$$\begin{aligned} E(N, v) = \int_R f v \, d\underset{\sim}{x} + \int_{\partial R_\pm} g v \, dx_1 - \sum_{k=0}^{N} \varepsilon^{2k} \int_R \big(\partial_1 u^{2k}\partial_1 v + \varepsilon^{-2}\partial_2 u^{2k}\partial_2 v\big) \, d\underset{\sim}{x} \\ = -\varepsilon^{2N} \int_R \partial_1 u^{2N}\partial_1 v \, d\underset{\sim}{x}, \end{aligned}$$

and we conclude from Lemma 2.1.5 that

$$|E(N, v)| \le C\varepsilon^{2N} |||(f, g)|||_{2N-1,R} \|v\|_{H^1(R)}. \tag{2.2.6}$$

We also have

$$\begin{aligned} \left| \int_R \partial_1(\chi_- U_N^-)\partial_1 v - \partial_1 U_N^- \partial_1(\chi_- v) + \varepsilon^{-2}\big[\partial_2(\chi_- U_N^-)\partial_2 v - \partial_2 U_N^- \partial_2(\chi_- v)\big] \, d\underset{\sim}{x} \right| \\ \le \big(\|\chi_-' U_N^-\|_{L^2(R)} + \|\chi_-'\partial_1 U_N^-\|_{L^2(R)}\big)\|v\|_{H^1(R)}. \end{aligned}$$

Hence, by lemma 2.2.4

$$\begin{aligned} \left| \int_R \big[\partial_1(\chi_- U_N^-)\partial_1 v + \varepsilon^{-2}\partial_2(\chi_- U_N^-)\partial_2 v\big] \, d\underset{\sim}{x} \right| \\ \le C\varepsilon^{3/2} \exp(-\pi^2\varepsilon^{-1}/(2\pi^2 + 8)) |||(f, g)|||_{N-1,\partial R_L} \|v\|_{H^1(R)}. \end{aligned} \tag{2.2.7}$$

The upper bound (2.2.7) also applies if we replace $\chi_- U_N^-$ by $\chi_+ U_N^+$. Making $v = u_N^E$ we have

$$\begin{aligned} &\int_R (\partial_1 u_N^E)^2 + (\varepsilon^{-1}\partial_2 u_N^E)^2 \, d\underset{\sim}{x} \\ &\quad = E(N, u_N^E) + \int_R \big[\partial_1\big(\chi_- U_N^- + \chi_+ U_N^+\big)\partial_1 u_N^E + \varepsilon^{-2}\partial_2(\chi_- U_N^- + \chi_+ U_N^+)\partial_2 u_N^E\big] \, d\underset{\sim}{x} \\ &\quad \le C\big(\varepsilon^{2N} |||(f, g)|||_{2N-1,R} + \varepsilon^{3/2} \exp(-\pi^2\varepsilon^{-1}/(2\pi^2 + 8)) |||(f, g)|||_{N-1,\partial R_L}\big) \\ &\qquad \|u_N^E\|_{H^1(R)}. \end{aligned}$$

from (2.2.6) and (2.2.7), and the result follows from (2.2.5). $\square$

The proof of Lemma 2.2.5 does not give sharp estimates in ε, but this is easily fixed, as we show below.

Proof of Theorem 2.2.1. First we improve the result of Lemma 2.2.5 using Lemmas 2.1.5 and 2.1.7

$$\begin{aligned}\|u_N^E\|_{H^1(R)} &\leq \|u_{N+1}^E\|_{H^1(R)} + \varepsilon^{2N+2}\|u^{2N+2} - \chi_- U_-^{2N+2} - \chi_+ U_+^{2N+2}\|_{H^1(R)} \\ &\leq C\big[\varepsilon^{2N+2}|||(f,g)|||_{2N+1,R} + \varepsilon^{3/2}\exp(-\pi^2\varepsilon^{-1}/(2\pi^2+8))|||(f,g)|||_{N,\partial R_L} \\ &\qquad + \varepsilon^{2N+3/2}\big(\|\partial_1^{2N} f\|_{L^2(\partial R_L)} + \|\partial_1^{2N} g\|_{C(\partial R_L)}\big)\big].\end{aligned} \tag{2.2.8}$$

Using the triangle inequality we have that

$$\begin{aligned}\|u(\varepsilon) - u_N + U_N^- + U_N^+\|_{H^1(R)} \\ \leq \|u_N^E\|_{H^1(R)} + \|(1-\chi_-)U_N^-\|_{H^1(R)} + \|(1-\chi_+)U_N^+\|_{H^1(R)}.\end{aligned} \tag{2.2.9}$$

To finish the proof of the theorem, we use (2.2.8) and Lemma 2.2.4 to bound the right hand side of (2.2.9). $\square$

Next we present the convergence in the original domain. The proof is an application of Theorem 2.2.1 and the bounds (2.1.18).

Proof of Theorem 2.2.2. First note that $\|\cdot\|_{H^1(R^\varepsilon)} \leq \varepsilon^{-1/2}\|\cdot\|_{H^1(R)}$. Then, by the triangle inequality, Lemma 2.1.8 and Theorem 2.2.1

$$\begin{aligned}\|e_N\|_{H^1(R^\varepsilon)} &\leq C\big[\|e_{N+1}\|_{H^1(R^\varepsilon)} \\ &\qquad + \varepsilon^{2N+2}\big(\|u^{2N+2}\|_{H^1(R^\varepsilon)} + \|U_-^{2N+2}\|_{H^1(R^\varepsilon)} + \|U_+^{2N+2}\|_{H^1(R^\varepsilon)}\big)\big] \\ &\leq C\big[\varepsilon^{2N+3}\big(\|\partial_1^{2N+2} f\|_{L^2(\partial R_L)} + \|\partial_1^{2N+2} g\|_{C(\partial R_L)}\big) + \varepsilon^{2N+7/2}|||(f,g)|||_{2N+2,R} \\ &\qquad + \varepsilon\exp(-\pi^2\varepsilon^{-1}/(2\pi^2+8))|||(f,g)|||_{N+1,\partial R_L} + \varepsilon^{2N+3/2}|||(f,g)|||_{2N,R} \\ &\qquad + \varepsilon^{2N+2}\big(\|\partial_1^{2N} f\|_{L^2(\partial R_L)} + \|\partial_1^{2N} g\|_{C(\partial R_L)} + |||(f,g)|||_{2N+1,R}\big)\big]\end{aligned}$$

and the result follows. $\square$

Chapter 3

A variational approach for modeling the Poisson problem

We discuss in this chapter a first class of dimensionally reduced models for the two-dimensional Poisson problem (2.1.1). They will be minimum energy models. To derive their convergence estimates, we develop asymptotic expansions of their solutions of the models and then compare to the expansion of the original solution.

Section 3.1 – Derivation of the models. It is possible to characterize the solution u^ε for the Poisson problem (2.1.1) variationally. If we define the spaces $V(R^\varepsilon) = \{v \in H^1(R^\varepsilon) \,:\, v = 0 \text{ on } \partial R^\varepsilon_L\}$ and $\underset{\sim}{S}(R^\varepsilon) = \underset{\sim}{L}^2(R^\varepsilon)$, then, for $\underset{\sim}{\sigma}^\varepsilon = (\sigma_1^\varepsilon, \sigma_2^\varepsilon) = \underset{\sim}{\nabla}\, u^\varepsilon$, the following holds.

SP: $(u^\varepsilon, \underset{\sim}{\sigma}^\varepsilon)$ is the unique critical point of

$$L(v, \underset{\sim}{\tau}) = \frac{1}{2}\int_{R^\varepsilon} |\underset{\sim}{\tau}|^2 \, d\underset{\sim}{x}^\varepsilon + \int_{R^\varepsilon} f^\varepsilon v \, d\underset{\sim}{x}^\varepsilon - \int_{R^\varepsilon} \underset{\sim}{\tau} \cdot \underset{\sim}{\nabla}\, v \, d\underset{\sim}{x}^\varepsilon + \int_{\partial R^\varepsilon_\pm} g^\varepsilon v \, dx_1^\varepsilon \qquad (3.1.1)$$

on $V(R^\varepsilon) \times \underset{\sim}{S}(R^\varepsilon)$.

We call the above variational principle *SP* (standing for "saddle point") as the pair $(u^\varepsilon, \underset{\sim}{\sigma}^\varepsilon)$ will be a saddle point of L. Rewriting SP in a weak form, we have that $u^\varepsilon \in V(R^\varepsilon)$ and $\underset{\sim}{\sigma}^\varepsilon \in \underset{\sim}{S}(R^\varepsilon)$ satisfy

$$\begin{aligned} \int_{R^\varepsilon} \underset{\sim}{\sigma}^\varepsilon \cdot \underset{\sim}{\tau} \, d\underset{\sim}{x}^\varepsilon - \int_{R^\varepsilon} \underset{\sim}{\nabla}\, u^\varepsilon \cdot \underset{\sim}{\tau} \, d\underset{\sim}{x}^\varepsilon = 0 \qquad & \text{for all } \underset{\sim}{\tau} \in \underset{\sim}{S}(R^\varepsilon), \\ \int_{R^\varepsilon} \underset{\sim}{\sigma}^\varepsilon \cdot \underset{\sim}{\nabla}\, v \, d\underset{\sim}{x}^\varepsilon = \int_{R^\varepsilon} f^\varepsilon v \, d\underset{\sim}{x}^\varepsilon + \int_{\partial R^\varepsilon_\pm} g^\varepsilon v \, dx_1^\varepsilon \qquad & \text{for all } v \in V(R^\varepsilon). \end{aligned}$$

We derive our models by considering subspaces of $V(R^\varepsilon)$ and $\underset{\sim}{S}(R^\varepsilon)$ that consist of functions that are polynomials in x_2^ε, and looking for a critical point $(u^\varepsilon(p), \sigma^\varepsilon(p))$

of L within these subspaces. For a function $v \in L^2(R^\varepsilon)$, and an integer p, we write $\deg_2 \le p$ meaning that v is a polynomial of degree at most p in x_2, with coefficients in $L^2(-1,1)$. The interpretation for $p < 0$ is that $v = 0$. For problem (2.1.1), the most reasonable choices of subspaces are $V(R^\varepsilon, p) = \{v \in V(R^\varepsilon) : \deg_2 v \le p\}$ and $\underset{\sim}{S}(R^\varepsilon, p) = \{\underset{\sim}{\tau} \in \underset{\sim}{S}(R^\varepsilon) : \deg_2 \tau_1 \le p, \deg_2 \tau_2 \le p-1\}$, which yield the *SP*($p$) *models*. Since $\underset{\sim}{\nabla} V(R^\varepsilon, p) \subset \underset{\sim}{S}(R^\varepsilon, p)$, then $\underset{\sim}{\sigma}^\varepsilon(p) = \underset{\sim}{\nabla} u^\varepsilon(p)$ is satisfied exactly and we can characterize $u^\varepsilon(p)$ as the minimizer in $V(R^\varepsilon, p)$ of the potential energy

$$\mathcal{J}(v) = \frac{1}{2}\int_{R^\varepsilon} |\underset{\sim}{\nabla} v|^2 \, d\underset{\sim}{x}^\varepsilon - \int_{R^\varepsilon} f^\varepsilon v \, d\underset{\sim}{x}^\varepsilon - \int_{\partial^\varepsilon_\pm} g^\varepsilon v \, dx_1^\varepsilon.$$

We derive now the SP(p) model for arbitrary $p \in \mathbb{N}$. As a basis for $\mathbb{P}_p(-1,1)$ we use the Legendre polynomials L_j, and the first few polynomials are $L_0(z) = 1$, $L_1(z) = z$, $L_2(z) = (3z^2-1)/2$, $L_3(z) = (5z^3 - 3z)/2$. As a basis for $\mathbb{P}_p(-\varepsilon, \varepsilon)$, we use

$$Q_j(z) = \varepsilon^j L_j(\varepsilon^{-1} z). \tag{3.1.2}$$

Define

$$f^k(x_1^\varepsilon) = \varepsilon^{-1} \int_{-\varepsilon}^{\varepsilon} f^\varepsilon(x_1^\varepsilon, x_2^\varepsilon) Q_k(x_2^\varepsilon)\, dx_2^\varepsilon,$$
$$g^0(x_1^\varepsilon) = \frac{1}{2}\big[g^\varepsilon(x_1^\varepsilon, \varepsilon) + g^\varepsilon(x_1^\varepsilon, -\varepsilon)\big], \quad g^1(x_1^\varepsilon) = \frac{1}{2}\big[g^\varepsilon(x_1^\varepsilon, \varepsilon) - g^\varepsilon(x_1^\varepsilon, -\varepsilon)\big],$$

and let $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{O}$ be $(p+1)\times(p+1)$ matrices defined by

$$\begin{aligned} \mathbf{M}_{ij} &= \int_{-\varepsilon}^{\varepsilon} Q_i(x_2^\varepsilon) Q_j(x_2^\varepsilon)\, dx_2^\varepsilon = \frac{2\varepsilon^{2i+1}}{2i+1}\delta_{ij}, \quad \mathbf{O}_{ij} = \int_{-\varepsilon}^{\varepsilon} \partial_2 Q_i(x_2^\varepsilon) \partial_2 Q_j(x_2^\varepsilon)\, dx_2^\varepsilon, \\ \mathbf{N}_{ij} &= \int_{-\varepsilon}^{\varepsilon} \partial_2 Q_i(x_2^\varepsilon) Q_j(x_2^\varepsilon)\, dx_2^\varepsilon = \begin{cases} 0 & \text{if } i \le j \text{ or if } i+j \text{ is even}, \\ 2\varepsilon^{i+j} & \text{otherwise}, \end{cases} \end{aligned} \tag{3.1.3}$$

for $i, j = 0, \dots, p$. Expanding $\partial_2 Q_i$ in terms of the Legendre polynomials, it is a straightforward computation to check that $\mathbf{O} = \mathbf{N}\mathbf{M}^{-1}\mathbf{N}^T$.

Write the SP(p) solution as

$$u^\varepsilon(p)(\underset{\sim}{x}^\varepsilon) = \sum_{j=0}^{p} \omega_j(x_1^\varepsilon) Q_j(x_2^\varepsilon), \quad \underset{\sim}{\sigma}^\varepsilon(p)(\underset{\sim}{x}^\varepsilon) = \begin{pmatrix} \sum_{j=0}^{p} \sigma_1^j(x_1^\varepsilon) Q_j(x_2^\varepsilon) \\ \sum_{j=0}^{p-1} \sigma_2^j(x_1^\varepsilon) Q_j(x_2^\varepsilon) \end{pmatrix}. \tag{3.1.4}$$

Then, since $\underset{\sim}{\sigma}^\varepsilon(p) = \underset{\sim}{\nabla}\, u^\varepsilon(p)$, it follows that $\sigma_1^j = \partial_1 \omega_j$. Also $\sum_{i=0}^{p-1} \sigma_2^i Q_i = \sum_{i=1}^{p} \omega_i Q_i'$, and multiplying both sides by Q_j and integrating from $-\varepsilon$ to ε, we have that $\sigma_2^j = \mathbf{M}_{jj}^{-1} \sum_{i=j+1}^{p} \mathbf{N}_{ij}\omega_i$.

We need now to determine $\omega_0, \ldots, \omega_p$. Define $\mathbf{g} : (-1,1) \to \mathbb{R}^{p+1}$, where $\mathbf{g}_j = g^0$ if j is even and $\mathbf{g}_j = g^1$ if j is odd, and

$$\boldsymbol{\omega}(x_1^\varepsilon) = (\omega_0, \ldots, \omega_p)^T(x_1^\varepsilon), \quad \mathbf{f}(x_1^\varepsilon) = (f^0, \ldots, f^p)^T(x_1^\varepsilon). \tag{3.1.5}$$

From its definition, $u^\varepsilon(p)$ satisfies

$$\int_{R^\varepsilon} \underset{\sim}{\nabla}\, u^\varepsilon(p) \cdot \underset{\sim}{\nabla}(vQ_j)\, d\underset{\sim}{x} = \int_{R^\varepsilon} f^\varepsilon v Q_j + \int_{\partial R^\varepsilon} g^\varepsilon v Q_j \qquad \text{for all } v \in H_0^1(-1,1) \tag{3.1.6}$$

and $j = 0, \ldots, p$. Using (3.1.4) and (3.1.5), we find that the strong equation corresponding to (3.1.6) is

$$\begin{gathered} \mathbf{M}\partial_{11}\boldsymbol{\omega} - \mathbf{O}\boldsymbol{\omega} = -\varepsilon\mathbf{f} - 2\varepsilon^j \mathbf{g}, \\ \boldsymbol{\omega}(-1) = \boldsymbol{\omega}(1) = 0. \end{gathered} \tag{3.1.7}$$

From the way that the matrices $\mathbf{M}$ and $\mathbf{O}$ depend on ε, the influence of ε in (3.1.7) becomes more complex as p increases. Consider the simplest SP(p) model, that is, when $p = 1$. The SP(1) solution is given by

$$u^\varepsilon(\underset{\sim}{x}^\varepsilon) = \omega_0(x_1^\varepsilon) + \omega_1(x_1^\varepsilon) x_2^\varepsilon, \quad \underset{\sim}{\sigma}^\varepsilon(\underset{\sim}{x}^\varepsilon) = \underset{\sim}{\nabla}\, u^\varepsilon(\underset{\sim}{x}^\varepsilon),$$

where

$$\partial_{11}\omega_0 = -\frac{1}{2} f^0 - \varepsilon^{-1} g^0, \quad \frac{2}{3}\varepsilon^2 \partial_{11}\omega_1 - 2\omega_1 = -f^1 - 2g^1, \tag{3.1.8}$$

and $\omega_0 = \omega_1 = 0$ on $\{-1, 1\}$. Consider now $p = 3$. We have that

$$u^\varepsilon(\underset{\sim}{x}^\varepsilon) = \omega_0(x_1^\varepsilon) + \omega_2(x_1^\varepsilon)Q_2(x_2^\varepsilon) + \omega_1(x_1^\varepsilon)x_2^\varepsilon + \omega_3(x_1^\varepsilon)Q_3(x_2^\varepsilon), \quad \underset{\sim}{\sigma}^\varepsilon(\underset{\sim}{x}^\varepsilon) = \underset{\sim}{\nabla}\, u^\varepsilon(\underset{\sim}{x}^\varepsilon).$$

The equations defining ω_0 and ω_2 are

$$\begin{aligned} \partial_{11}\omega_0 &= -\frac{1}{2}f^0 - \varepsilon^{-1}g^0, \\ \frac{2}{5}\varepsilon^2\partial_{11}\omega_2 - 6\omega_2 &= -\varepsilon^{-2}f^2 - 2\varepsilon^{-1}g^0, \end{aligned}$$

where $\omega_0 = \omega_2 = 0$ on $\{-1, 1\}$, and we find ω_1 and ω_3 by solving

$$\begin{aligned} \frac{2}{3}\varepsilon^2\partial_{11}\omega_1 - 2\omega_1 - 2\varepsilon^2\omega_3 &= -f^1 - 2g^1, \\ \frac{2}{7}\varepsilon^4\partial_{11}\omega_3 - 2\omega_1 - 12\varepsilon^2\omega_3 &= -\varepsilon^{-2}f^3 - 2g^1, \end{aligned}$$

where again $\omega_1 = \omega_3 = 0$ on $\{-1, 1\}$.

Section 3.2 – Asymptotic expansion for the solutions of the models. Observe that in SP(p) for $p = 1$, 3, and in fact for arbitrary p, the term ω_0 coincides with ζ^0, cf. (2.1.12). For increasing p, the equations get more sophisticated—(3.1.7) is a singular perturbed system of boundary value ODEs—and $u^\varepsilon(p)$ has a nontrivial behavior, like u^ε, as we show next. We start by developing an asymptotic expansion of $u^\varepsilon(p)$, proceeding as in Chapter 2. We pose the SP(p) problem in the scaled domain R by defining $V(R, p)$ similarly to $V(R^\varepsilon, p)$, and $u(p)(\underset{\sim}{x}) = u^\varepsilon(p)(\underset{\sim}{x}^\varepsilon)$. Hence

$$\int_R \partial_1 u(p)\partial_1 v + \varepsilon^{-2}\partial_2 u(p)\partial_2 v\, d\underset{\sim}{x} = \int_R fv\, d\underset{\sim}{x} + \int_{\partial R_\pm} gv\, dx_1 \qquad \text{for all } v \in V(R, p). \tag{3.2.1}$$

Assuming the asymptotic expansion

$$u(p) \sim u^0(p) + \varepsilon^2 u^2(p) + \varepsilon^4 u^4(p) + \cdots, \tag{3.2.2}$$

and formally substituting it in (3.2.1), we conclude that for all $v \in V(R,p)$,

$$\begin{aligned}\int_R \partial_2 u^0(p)\partial_2 v\, d\underset{\sim}{x} &= 0,\\ \int_R \partial_2 u^2(p)\partial_2 v\, d\underset{\sim}{x} &= \int_R \big(f + \partial_{11}u^0(p)\big) v\, d\underset{\sim}{x} + \int_{\partial R_\pm} gv,\\ \int_R \partial_2 u^{2k}(p)\partial_2 v\, d\underset{\sim}{x} &= \int_R \partial_{11}u^{2k-2}(p) v\, d\underset{\sim}{x},\ \text{for } k>1.\end{aligned} \tag{3.2.3}$$

Repeating the arguments of the Section 2.1, we set $u^0(p) = \zeta^0$ and $u^2(p)(x_1,\cdot)$ as the Galerkin projection of $u^2(x_1,\cdot)$ into $\hat{\mathbb{P}}_p(-1,1)$ for almost every $x_1 \in (-1,1)$, i.e.,

$$\begin{aligned}\int_{-1}^{1} \partial_2 u^2(p)(x_1,x_2)\partial_2 v(x_2)\, dx_2 &= \int_{-1}^{1} [f(x_1,x_2) + \partial_{11}\zeta^0(x_1)] v(x_2)\, dx_2\\ &\quad + g(x_1,-1)v(-1) + g(x_1,1)v(1), \qquad \text{for all } v \in \hat{\mathbb{P}}_p(-1,1).\end{aligned} \tag{3.2.4}$$

For any integer $k \geq 2$, we define $u^{2k}(p) \in \hat{\mathbb{P}}_p(-1,1)$ by

$$\int_{-1}^{1} \partial_2 u^{2k}(p)(x_1,x_2)\partial_2 v(x_2)\, dx_2 = \int_{-1}^{1} \partial_{11}u^{2k-2}(p)(x_1,x_2) v(x_2)\, dx_2, \tag{3.2.5}$$

for all $v \in \hat{\mathbb{P}}_p(-1,1)$, and for almost every $x_1 \in (-1,1)$.

The ansatz (3.2.2) does not satisfy the Dirichlet boundary conditions at ∂R_L and we use then boundary correctors $U_-^{2k}(p)$, $U_+^{2k}(p)$. These functions must be polynomials in x_2 and they are defined in the semi-infinite strip Σ. We need to define the spaces

$$\begin{aligned}V(\Sigma,p) &= \big\{v \in \mathcal{D}'(\Sigma) \,:\, \|\underset{\sim}{\nabla} v\|_{L^2(\Sigma)} + \|v(0,\cdot)\|_{L^2(-1,1)} < \infty,\ \deg_2 v \leq p\big\},\\ V_0(\Sigma,p) &= \big\{v \in V(\Sigma,p) \,:\, v(0,\cdot) = 0\big\},\end{aligned} \tag{3.2.6}$$

where $\mathcal{D}'(\Sigma)$ is the space of distributions on Σ. Then for any positive integer k, define $U_-^{2k}(p) \in V(\Sigma,p)$ as the solutions of

$$\int_\Sigma \underset{\sim}{\nabla} U_-^{2k} \cdot \underset{\sim}{\nabla} v\, d\underset{\sim}{x} = 0 \qquad \text{for all } v \in V_0(\Sigma,p), \tag{3.2.7}$$

$$U_-^{2k}(p)(0,x_2) = u^{2k}(p)(-1,x_2) \qquad \text{for } x_2 \in (-1,1). \tag{3.2.8}$$

We define $U_+^{2k}(p)$ as the solution of (3.2.7) satisfying $U_+^{2k}(p)(0,\cdot) = u^{2k}(p)(1,\cdot)$.

Finally

$$u^\varepsilon(p)(\underset{\sim}{x}^\varepsilon) \sim \zeta^0(x_1^\varepsilon) + \sum_{k\geq 1} \varepsilon^{2k} u^{2k}(p)(x_1^\varepsilon, \varepsilon^{-1}x_2^\varepsilon) - \sum_{k\geq 1} \varepsilon^{2k}[U_-^{2k}(p)(\hat{\rho}_-, \varepsilon^{-1}x_2^\varepsilon) + U_+^{2k}(p)(\hat{\rho}_+, \varepsilon^{-1}x_2^\varepsilon)].$$

We have the following stability result for some terms of the expansion. Its proof is parallel to the one of Lemma 2.1.5 and we do not present it.

Lemma 3.2.1. *Let j, $k \in \mathbb{N}$ be such that $k \geq 1$. Then there exists a constant C such that*

$$\|\partial_1^j u^{2k}(p)(x_1,\cdot)\|_{H^1(-1,1)} \leq C\big(\|\partial_1^{2k-2+j} f(x_1,\cdot)\|_{L^2(-1,1)} + |\partial_1^{2k-2+j} g(x_1,-1)| + |\partial_1^{2k-2+j} g(x_1,1)|\big),$$

for all $p \in \mathbb{N}$.

We present below a result which follows from Theorem 6.2.7 and regards the boundary correctors for the models.

Lemma 3.2.2. *Assume, for a fixed $k \geq 1$, that $u^{2k}(p)$ is defined as above. Then there exists a unique solution $U_-^{2k}(p) \in V(\Sigma, p)$ to (3.2.7), (3.2.8). Also, there exists a universal constant C such that*

$$\int_t^\infty \int_{-1}^1 [U_-^{2k}(p)]^2 + |\underset{\sim}{\nabla}\, U_-^{2k}(p)|^2 \, dx_2 \, d\hat{\rho}_- \leq C\|u^{2k}(p)(-1,\cdot)\|^2_{H^{1/2}(-1,1)} \exp(-\pi^2 t/(\pi^2+4)),$$

for every nonnegative real number t.

We use Lemmas 3.2.2, 3.2.1 and 2.1.5 to conclude the result below. Similar bounds hold for $U_+^{2k}(p)$.

Lemma 3.2.3. *For any positive integer k, there exists a constant C such that*

$$\begin{aligned}\int_t^\infty \int_{-1}^1 \big[U_-^{2k}(p)\big]^2 + |\underset{\sim}{\nabla}\, U_-^{2k}(p)|^2 \, dx_2 \, d\hat{\rho}_- \\ \leq C\big(\|\partial_1^{2k-2} f\|^2_{L^2(\partial R_L)} + \|\partial_1^{2k-2} g\|^2_{C(\partial R_L)}\big) \exp(-\pi^2 t/(\pi^2+4)),\end{aligned} \tag{3.2.9}$$

for every nonnegative real number t.

We present next an estimate, in the $H^1(R^\varepsilon)$ norm, of $u^\varepsilon(p)$ minus its truncated asymptotic expansion. Since the proofs of Chapter 2 work here with minor modifications we refrain from repeating them. We would like to remark that this result gives a bound that is uniform in p, and that the bound is the same (up to a constant) as in Theorem 2.2.2.

Theorem 3.2.4. *For each $N \in \mathbb{N}$, there exists a positive constant C such that*

$$\begin{aligned}\Big\| u^\varepsilon(p) - \zeta^0 - \sum_{k=1}^N \varepsilon^{2k} u^{2k}(p) - \sum_{k=1}^N \varepsilon^{2k}\big[U_-^{2k}(p) + U_+^{2k}(p)\big]\Big\|_{H^1(R^\varepsilon)} \\ \leq C\varepsilon^{2N+3/2}\big(|||(f,g)|||_{2N+2,R} + |||(f,g)|||_{N+1,\partial R_L}\big)\end{aligned}$$

for all $p \in \mathbb{N}$.

Section 3.3 – Estimates for the modeling error. Our next goal is to evaluate how accurate are the models that we derived in Section 3.1. We do that by first comparing some terms of the asymptotic expansion of both u^ε and $u^\varepsilon(p)$. Next, using a triangle inequality and Theorems 2.2.2 and 3.2.4, we find an upper bound for the difference $u^\varepsilon - u^\varepsilon(p)$. Recall that $u^2(p)(x_1,\cdot)$ is simply the Galerkin projection of $u^2(x_1,\cdot)$ into $\hat{\mathbb{P}}_p(-1,1)$ for $x_1 \in (-1,1)$.

Denote by $\hat{\pi}_p^1$ the orthogonal projection operator from $H^1(-1,1) \cap \hat{L}^2(-1,1)$ to $\hat{\mathbb{P}}_p(-1,1)$, with respect to the inner product that induces the norm $|\cdot|_{H^1(-1,1)}$. We use the upper index (x_2) on the projector operators to indicate the action on the variable x_2 only. So, for instance, if $\varphi \in L^2((-1,1); H^1(-1,1))$, then $\hat{\pi}_p^{1(x_2)}\varphi \in L^2((-1,1); \hat{\mathbb{P}}_p(-1,1))$, and

$$\int_R \partial_2(\varphi - \hat{\pi}_p^{1(x_2)}\varphi)\partial_2\psi \, d\underset{\sim}{x} = 0 \qquad \text{for all } \psi \in L^2((-1,1); \hat{\mathbb{P}}_p(-1,1)).$$

We estimate in the following result some approximation properties of the operator $\hat{\pi}_p^{1(x_2)}$. The proof is very similar to the one of Lemma A.4.

Lemma 3.3.1. *For any real number $s \geq 1$, there exists a constant C such that*

$$\|\varphi - \hat{\pi}_p^{1(x_2)}\varphi\|_{L^2(R)} \leq Cp^{-s}\|\varphi\|_{L^2((-1,1);H^s(-1,1))},$$
$$\|\partial_1(\varphi - \hat{\pi}_p^{1(x_2)}\varphi)\|_{L^2(R)} \leq Cp^{-s}\|\varphi\|_{H^1((-1,1);H^s(-1,1))},$$
$$\|\partial_2(\varphi - \hat{\pi}_p^{1(x_2)}\varphi)\|_{L^2(R)} \leq Cp^{1-s}\|\varphi\|_{L^2((-1,1);H^s(-1,1))}.$$

Then, with the notation

$$\begin{aligned} a_s &= \|f\|_{(0,s,R)} + \|g\|_{L^2(\partial R_\pm)}, \quad a_s^b = \|f\|_{H^s(\partial R_L)} + |g|_{C(\partial R_L)}, \\ a_s^1 &= \|f\|_{(1,s,R)} + \|g\|_{H^1(\partial R_\pm)}, \quad a = |||(f,g)|||_{4,R} + |||(f,g)|||_{2,\partial R_L}, \end{aligned} \tag{3.3.1}$$

we have the following estimates.

Lemma 3.3.2. *For any nonnegative real number s, there exists a constant C such that*

$$\|u^2 - u^2(p)\|_{L^2(R^\varepsilon)} \leq C\varepsilon^{1/2}p^{-2-s}a_s,$$
$$\|\partial_1 u^2 - \partial_1 u^2(p)\|_{L^2(R^\varepsilon)} \leq C\varepsilon^{1/2}p^{-2-s}a_s^1,$$
$$\|\partial_{x_2^\varepsilon} u^2 - \partial_{x_2^\varepsilon} u^2(p)\|_{L^2(R^\varepsilon)} \leq C\varepsilon^{-1/2}p^{-1-s}a_s.$$

Proof. As remarked earlier, from equations equations (2.1.5), (2.1.7) and (3.2.4), we have that $u^2(p) = \hat{\pi}_p^{1(x_2)}u^2$. Hence, the estimates of this lemma follow from applications of Lemmas 2.1.5 and 3.3.1, after a change of coordinates. $\square$

We next estimate the convergence for the first boundary correctors. This is not a straightforward issue due to the possible presence of singularities on the exact solutions U_-^2, U_+^2. This singular behavior is, as we show in Section 6.3, direct related to the values of the vertical derivatives of U_-^2, U_+^2 on the corners of the domain Σ. From (2.1.5) and (2.1.7) it is easy to see that

$$|\partial_2 u^2|_{C(\partial R_L)} = |g|_{C(\partial R_L)}, \quad |\partial_2^{2j-1} u^2|_{C(\partial R_L)} = |\partial_2^{2j-3} f|_{C(\partial R_L)}, \tag{3.3.2}$$

for any positive integer $j \geq 2$, if the values above are well defined.

We next specify the convergence rates for the boundary correctors, as in Definition 6.3.10.

Definition 3.3.3. For each nonnegative real number s, we define $J(s)$ by

$$J(s) = \max\{j \in \mathbb{Z} \, : \, 2j < s\}. \tag{3.3.3}$$

If $|g|_{C(\partial R_L)} \neq 0$, set $m = 1$. If $|g|_{C(\partial R_L)} = 0$ and

$$\sum_{j=2}^{J(s+5/2)} |\partial_2^{2j-3} f|_{C(\partial R_L)} \neq 0, \tag{3.3.4}$$

let m be the minimum integer in $\{2, \ldots, J(s+5/2)\}$ such that $|\partial_2^{2m-3} f|_{C(\partial R_L)} \neq 0$. We define in both cases $\mu(s, \delta) = \min\{4m - 2 - \delta, s + 3/2\}$. If $|g|_{C(\partial R_L)} = 0$ and (3.3.4) does not hold, then define $\mu(s, \delta) = s + 3/2$.

The result below is a direct application of Theorem 6.3.12.

Lemma 3.3.4. *For any nonnegative real number s such that $s + 1/2$ is not an even integer, and any arbitrarily small $\delta > 0$, there exists a constant C such that*

$$|U_-^2 - U_-^2(p)|_{H^1(\Sigma)} \leq C p^{-\mu(s,\delta)} \|u^2(-1, \cdot)\|_{H^{s+2}(-1,1)},$$

$$|U_+^2 - U_+^2(p)|_{H^1(\Sigma)} \leq C p^{-\mu(s,\delta)} \|u^2(1, \cdot)\|_{H^{s+2}(-1,1)},$$

where μ is as in Definition 3.3.3.

Hence, the lemma bellow follows from a combination of Lemmas 3.3.4, 2.1.3 and a change of coordinates.

Lemma 3.3.5. *For any nonnegative real number s such that $s+1/2$ is not an even integer, and any arbitrarily small $\delta > 0$, there exists a constant C such that*

$$|U_-^2 - U_-^2(p)|_{H^1(R^\varepsilon)} + |U_+^2 - U_+^2(p)|_{H^1(R^\varepsilon)} \leq Cp^{-\mu(s,\delta)} a_s^b. \tag{3.3.5}$$

where μ is as in Definition 3.3.3.

Remark. Schwab and Wright [SW] study some of the approximation properties of the boundary layer part for minimum energy models. We describe their work briefly here, using $\hat{\rho}$ to represent either $\hat{\rho}_-$ or $\hat{\rho}_+$. They decompose general solutions for (2.1.14) as an infinite sum of functions in the form $U_j(x_2)\exp(\sigma_j\hat{\rho})$, where the pair (U_j, σ_j) is given by either $(\cos(\sigma_j x_2), -j\pi)$ or $(\sin(\sigma_j x_2), -(j+1/2)\pi)$. Similarly, the boundary correctors $U_-^2(p)$, $U_+^2(p)$ are expressed as a finite combination of $U_j(p)(x_2)\exp(\sigma_j(p)\hat{\rho})$, where $U_j(p)$, $\sigma_j(p)$ are the Galerkin projections of U_j, σ_j. They then estimate $|\sigma_j - \sigma_j(p)|$ and $|U_j - U_j(p)|$.

We finally estimate the modeling error in the various components of the H^1 norm. We define $R_0^\varepsilon = I_0 \times (-\varepsilon, \varepsilon)$, where I_0 is an open interval such that $\bar{I}_0 \subset (-1,1)$, and then we can have interior estimates as well.

Theorem 3.3.6. *For any nonnegative real numbers s and s^* such that $s^* + 1/2$ is not an even integer, and any arbitrarily small $\delta > 0$, there exists a constant C such that the* *error between u^ε and the approximation $u^\varepsilon(p)$ given by the SP(p) model is bounded as*

$$\|u^\varepsilon - u^\varepsilon(p)\|_{L^2(R^\varepsilon)} \le C\varepsilon^{5/2}p^{-2-s}a_s + C\varepsilon^3 a,$$

$$\|\partial_1 u^\varepsilon - \partial_1 u^\varepsilon(p)\|_{L^2(R^\varepsilon)} \le C\varepsilon^2 p^{-\mu(s^*,\delta)}a^b_{s^*} + C\varepsilon^{5/2}a,$$

$$\|\partial_1 u^\varepsilon - \partial_1 u^\varepsilon(p)\|_{L^2(R^\varepsilon_0)} \le C\varepsilon^{5/2}p^{-2-s}a^1_s + C\varepsilon^{7/2}a,$$

$$\|\partial_{x_2^\varepsilon} u^\varepsilon - \partial_{x_2^\varepsilon} u^\varepsilon(p)\|_{L^2(R^\varepsilon)} \le C\varepsilon^{3/2}p^{-1-s}a_s + C\varepsilon^2 a,$$

$$\|u^\varepsilon - u^\varepsilon(p)\|_{H^1(R^\varepsilon)} \le C\varepsilon^{3/2}p^{-1-s}a_s + C\varepsilon^2 a,$$

where a_s, a^b_s, a^1_s and a are defined in (3.3.1) *and μ is as in Definition 3.3.3.*

Proof. We prove the fourth estimate. Using the triangle inequality, the following holds:

$$\begin{aligned} \|\partial_{x_2^\varepsilon} u^\varepsilon - \partial_{x_2^\varepsilon} u^\varepsilon(p)\|_{L^2(R^\varepsilon)} &\le \|u^\varepsilon - \zeta^0 - \varepsilon^2 u^2 + \varepsilon^2\big(U^2_- + U^2_+\big)\|_{H^1(R^\varepsilon)} \\ &+ \varepsilon^2\big(\|\partial_{x_2^\varepsilon} u^2 - \partial_{x_2^\varepsilon} u^2(p)\|_{L^2(R^\varepsilon)} + |U^2_- - U^2_-(p)|_{H^1(R^\varepsilon)} + |U^2_+ - U^2_+(p)|_{H^1(R^\varepsilon)}\big) \quad (3.3.6) \\ &+ \|u^\varepsilon(p) - \zeta^0 - \varepsilon^2 u^2(p) + \varepsilon^2[U^2_-(p) + U^2_+(p)]\|_{H^1(R^\varepsilon)}. \end{aligned}$$

From Theorems 2.2.2 and 3.2.4, we have that

$$\begin{aligned} &\|u^\varepsilon - \zeta^0 - \varepsilon^2 u^2 + \varepsilon^2\big(U^2_- + U^2_+\big)\|_{H^1(R^\varepsilon)} \\ &\qquad + \|u^\varepsilon(p) - \zeta^0 - \varepsilon^2 u^2(p) + \varepsilon^2\big[U^2_-(p) + U^2_+(p)\big]\|_{H^1(R^\varepsilon)} \le C\varepsilon^{7/2}a. \end{aligned}$$

The estimate

$$|U^2_- - U^2_-(p)|_{H^1(R^\varepsilon)} + |U^2_+ - U^2_+(p)|_{H^1(R^\varepsilon)} \le Ca$$

come from Lemmas 2.1.8 and 3.2.3. Finally we apply Lemma 3.3.2 to bound $\|\partial_{x_2^\varepsilon} u^2 - \partial_{x_2^\varepsilon} u^2(p)\|_{L^2(R^\varepsilon)}$, and substituting in (3.3.6) we have the result. The other estimates follow from similar arguments. □

Remark 3.3.7. It is possible to show that

$$\|\partial_1 u^\varepsilon - \partial_1 u^\varepsilon(p)\|_{L^2(R^\varepsilon_0)} \le C\varepsilon^{5/2}p^{-2-s}a^1_s + C\varepsilon^{9/2}\big(|||(f,g)|||_{6,R} + |||(f,g)|||_{3,\partial R_L}\big).$$

The proof is still very similar to the one above, the main difference being that an expansion up to terms in ε^4 has to be considered.

We summarize the convergence results in the table below, c.f. Table 2.1. We present only the leading terms of the errors and in parenthesis we show interior estimates if these are better than the global ones. Recall that a_s, a_s^1, and a_s^b are defined in (3.3.1), ν is defined in (2.2.3), and μ is as in Definition 3.3.3.

TABLE 3.1. Convergence estimates for the SP(p) models

	$u^\varepsilon - u^\varepsilon(p)$	Relative Error
$\lVert \cdot \rVert_{L^2(R^\varepsilon)}$	$\varepsilon^{5/2}p^{-2-s}a_s$	$\nu^2 p^{-2-s}a_s$
$\lVert \partial_1 \cdot \rVert_{L^2(R^\varepsilon)}$	$\varepsilon^2 p^{-\mu} a_s^b$ $(\varepsilon^{5/2}p^{-2-s}a_s^1)$	$\nu^{3/2}p^{-\mu}a_s^b$ $(\nu^2 p^{-2-s}a_s^1)$
$\lVert \partial_{x_2^\varepsilon} \cdot \rVert_{L^2(R^\varepsilon)}$	$\varepsilon^{3/2}p^{-1-s}a_s$	$p^{-1-s}a_s$
$\lVert \cdot \rVert_{H^1(R^\varepsilon)}$	$\varepsilon^{3/2}p^{-1-s}a_s$	$\nu p^{-1-s}a_s$

Note that when $\nu = 1$ (i.e., $\zeta^0 = 0$), there exists no convergence in ε in the relative error norm, only in p, see Remark 1.1. Also, the boundary layers slow down the convergence in the L^2 norm of the horizontal derivative, and better rates come up, both in ε (if $\zeta^0 \neq 0$) and p, by considering interior estimates.

A particular case of great importance is when f^ε has a certain polynomial dependence in the vertical direction. In this case, u^{2k} will be a polynomial in x_2^ε for all integers k, see (2.1.4)–(2.1.7) and (2.1.12), and will be equal to $u^{2k}(p)$, if p is big enough. The convergence rates in ε will improve then. For simplicity, we only discuss the case when f^ε is independent of x_2^ε, but it is not hard to generalize the results for an arbitrary polynomial dependence.

Lemma 3.3.8. *If f^ε is a function of $x_1^\varepsilon \in (-1,1)$ only, then, for every nonnegative real number s such that $s + 1/2$ is not an even integer, and every arbitrarily small $\delta > 0$,* *there exists a constant C such that for $p \geq 2$,*

$$\begin{aligned}
\|u - u(p)\|_{H^1(R^\varepsilon)} &\leq C\varepsilon^2 p^{-\mu(s,\delta)} a_s^b + C\varepsilon^3 a, \\
\|u - u(p)\|_{H^1(R_0^\varepsilon)} &\leq C\varepsilon^{2M+3/2} p^{-2M-1-s} \left(\|f\|_{(2M,s,R)} + \|g\|_{H^{2M}(\partial R_\pm)}\right) \\
&\quad + C\varepsilon^{2M+5/2} \left(|||(f,g)|||_{2M+4,R} + |||(f,g)|||_{M+2,\partial R_L}\right),
\end{aligned} \tag{3.3.7}$$

where $M = [p/2]$ and $[x]$ denotes the greatest integer not greater than x.

Proof. From equations (2.1.4)–(2.1.7) we see that $u^{2k}(x_1^\varepsilon, \cdot) \in \mathbb{P}_{2k}(-\varepsilon, \varepsilon)$ for all $k \in \mathbb{N}$. Since $p \geq 2$, then $u^2(p) = u^2$. From (2.1.6), (2.1.7) and (3.2.3), $u^{2k}(p) = u^{2k}$ if $2k \leq p$. Mimicking the proof of Lemma 3.3.6, we conclude the first inequality. To obtain the second inequality, we add and subtract the truncated asymptotic series of both u^ε and $u^\varepsilon(p)$ of order 2M+2 and conveniently use the triangle inequality. As $u^{2k}(p) = u^{2k}$ for $k \leq M$, we are left with

$$\begin{aligned}
\|u - u(p)\|_{H^1(R_\varepsilon^0)} \leq& C\varepsilon^{2M+2} \|\partial_2 u^{2M+2} - \partial_2 u^{2M+2}(p)\|_{L^2(R_\varepsilon^0)} \\
&+ C\varepsilon^{2M+5/2} \left(|||(f,g)|||_{2M+4,R} + |||(f,g)|||_{M+2,\partial R_L}\right),
\end{aligned}$$

and it is enough now to note that $u^{2M+2}(p) = \hat{\pi}_p^1 u^{2M+2}$, see (2.1.6), (2.1.7) and (3.2.3), and then apply Lemma 3.3.1 combined with the regularity results of Lemma 2.1.5. □

Remark. It is possible to improve estimate (3.3.7) even further when f^ε, g^ε are constants. In this case $u^{2k} = 0$ for $k > 1$, and if $p \geq 2$ then $\|u - u(p)\|_{H^1(R_0^\varepsilon)}$ is bounded by a exponentially small quantity.

Chapter 4

An alternative variational approach

We introduce in this chapter an alternative way to derive dimensionally reduced models, originating two new classes of models that are not of minimum energy type. We investigate one of them. The analysis is more involved, but still similar to Chapter 3.

Section 4.1 – Derivation of the models. In Section 3.1 we characterize the solution of the two-dimensional Poisson problem (2.1.1) by what we call SP (saddle point principle). Here we use a variant of SP, which we call SP′. Define the spaces $V'(R^\varepsilon) = L^2(R^\varepsilon)$ and $\underset{\sim}{S}'_{g^\varepsilon}(R^\varepsilon) = \{\underset{\sim}{\sigma} \in \underset{\sim}{H}(\mathrm{div}, R^\varepsilon) : \underset{\sim}{\sigma} \cdot \underset{\sim}{n} = g^\varepsilon \text{ on } \partial R^\varepsilon_\pm\}$. Again, let $\underset{\sim}{\sigma}^\varepsilon = (\sigma_1^\varepsilon, \sigma_2^\varepsilon) = \underset{\sim}{\nabla} u^\varepsilon$, where u^ε is the solution of (2.1.1). Then the following holds.

SP′: $(u^\varepsilon, \underset{\sim}{\sigma}^\varepsilon)$ is the unique critical point of

$$L'(v, \underset{\sim}{\tau}) = \frac{1}{2}\int_{R^\varepsilon} |\underset{\sim}{\tau}|^2 \, d\underset{\sim}{x}^\varepsilon + \int_{R^\varepsilon} f^\varepsilon v \, d\underset{\sim}{x}^\varepsilon + \int_{R^\varepsilon} \mathrm{div}\, \underset{\sim}{\tau} v \, d\underset{\sim}{x}^\varepsilon$$

on $V'(R^\varepsilon) \times \underset{\sim}{S}'_{g^\varepsilon}(R^\varepsilon)$.

Once more $(u^\varepsilon, \underset{\sim}{\sigma}^\varepsilon)$ is a saddle point of L'. We can also write a weak form of SP′ as $(u^\varepsilon, \underset{\sim}{\sigma}^\varepsilon)$ is the unique point in $V'(R^\varepsilon) \times \underset{\sim}{S}'_{g^\varepsilon}(R^\varepsilon)$ such that

$$\int_{R^\varepsilon} \underset{\sim}{\sigma}^\varepsilon \cdot \underset{\sim}{\tau} \, d\underset{\sim}{x}^\varepsilon + \int_{R^\varepsilon} u^\varepsilon \, \mathrm{div}\, \underset{\sim}{\tau} \, d\underset{\sim}{x}^\varepsilon = 0 \qquad \text{for all } \underset{\sim}{\tau} \in \underset{\sim}{S}'_0(R^\varepsilon), \tag{4.1.1}$$

$$\int_{R^\varepsilon} \mathrm{div}\, \underset{\sim}{\sigma}^\varepsilon v \, d\underset{\sim}{x}^\varepsilon = -\int_{R^\varepsilon} f^\varepsilon v \, d\underset{\sim}{x}^\varepsilon \qquad \text{for all } v \in V'(R^\varepsilon), \tag{4.1.2}$$

where $\underset{\sim}{S}'_0(R^\varepsilon)$ is simply $\underset{\sim}{S}'_{g^\varepsilon}(R^\varepsilon)$ with $g^\varepsilon = 0$.

We introduce now two classes of models based on SP′. Choosing subspaces of $V'(R^\varepsilon)$ and $\underset{\sim}{S}'_{g^\varepsilon}(R^\varepsilon)$ composed of functions with polynomial dependence in the transverse direction, and looking for a critical point of L' within these subspaces, we define the SP′

models. For $V'(R^\varepsilon, p) = \{v \in V'(R^\varepsilon) : \deg_2 v \le p\}$ and $\underset{\sim}{S}'_{g^\varepsilon}(R^\varepsilon, p) = \{\underset{\sim}{\tau} \in \underset{\sim}{S}'_{g^\varepsilon}(R^\varepsilon) : \deg_2 \tau_1 \le p, \deg_2 \tau_2 \le p-1\}$ we have the $SP'_1(p)$ *models*. Another option is to choose $\underset{\sim}{S}'_{g^\varepsilon}(R^\varepsilon, p) = \{\underset{\sim}{\tau} \in \underset{\sim}{S}'_{g^\varepsilon}(R^\varepsilon) : \deg_2 \tau_1 \le p, \deg_2 \tau_2 \le p+1\}$, and we define the $SP'_2(p)$ *models*. The solutions of the models, $\underset{\sim}{\sigma}^\varepsilon(p) \in \underset{\sim}{S}'_{g^\varepsilon}(R^\varepsilon, p)$ and $u^\varepsilon(p) \in V'(R^\varepsilon, p)$ satisfy the weak equations

$$\int_{R^\varepsilon} \underset{\sim}{\sigma}^\varepsilon(p) \cdot \underset{\sim}{\tau}\, d\underset{\sim}{x}^\varepsilon + \int_{R^\varepsilon} u^\varepsilon(p) \operatorname{div} \underset{\sim}{\tau}\, d\underset{\sim}{x}^\varepsilon = 0 \qquad \text{for all } \underset{\sim}{\tau} \in \underset{\sim}{S}'_0(R^\varepsilon, p), \tag{4.1.3}$$

$$\int_{R^\varepsilon} \operatorname{div} \underset{\sim}{\sigma}^\varepsilon(p) v\, d\underset{\sim}{x}^\varepsilon = -\int_{R^\varepsilon} f^\varepsilon v\, d\underset{\sim}{x}^\varepsilon \qquad \text{for all } v \in V'(R^\varepsilon, p). \tag{4.1.4}$$

Note that in both $\mathrm{SP}'_1(p)$ and $\mathrm{SP}'_2(p)$ models, $\operatorname{div} \underset{\sim}{S}'_{g^\varepsilon}(R^\varepsilon, p) = V(R^\varepsilon, p)$ and therefore, not only there exists a unique solution for (4.1.3), (4.1.4) (see Lemma 4.2.1), but also $\operatorname{div} \underset{\sim}{\sigma}^\varepsilon(p) = -\pi_{V'} f^\varepsilon$, where $\pi_{V'} f^\varepsilon$ is the orthogonal L^2 projection of f^ε into $V'(R^\varepsilon, p)$. This implies that $\sigma^\varepsilon(p)$ is the minimizer of the complementary energy

$$\mathcal{J}_c(\underset{\sim}{\tau}) = \frac{1}{2} \int_{R^\varepsilon} |\underset{\sim}{\tau}|^2\, d\underset{\sim}{x}$$

over all $\underset{\sim}{\tau} \in \underset{\sim}{S}'_{g^\varepsilon}(R^\varepsilon, p)$ such that $\operatorname{div} \underset{\sim}{\tau} = -\pi_{V'} f^\varepsilon$.

We study here the more sophisticated $\mathrm{SP}'_2(p)$ models, and we describe them for a positive integer p. Recall that Q_k are the Legendre polynomials scaled to $(-\varepsilon, \varepsilon)$. See (3.1.2). We define next polynomials that vanish on $\{-\varepsilon, \varepsilon\}$. Set

$$\tilde{Q}_k(x_2^\varepsilon) = \begin{cases} Q_k(x_2^\varepsilon) - \varepsilon^k & \text{if } k \text{ is even,} \\ Q_k(x_2^\varepsilon) - x_2^\varepsilon \varepsilon^{k-1} & \text{if } k \text{ is odd,} \end{cases} \tag{4.1.5}$$

for $k = 2, \ldots, p+1$. Let $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{N}}$ be $p \times p$ and $p \times (p+1)$ matrices respectively, where

$$\begin{aligned} \tilde{\mathbf{M}}_{ij} &= \int_{-\varepsilon}^{\varepsilon} \tilde{Q}_i(x_2^\varepsilon) \tilde{Q}_j(x_2^\varepsilon)\, dx_2^\varepsilon \qquad \text{for } i, j = 2, \ldots, p+1, \\ \tilde{\mathbf{N}}_{ij} &= \int_{-\varepsilon}^{\varepsilon} \partial_2 \tilde{Q}_i(x_2^\varepsilon) Q_j(x_2^\varepsilon)\, dx_2^\varepsilon \qquad \text{for } i = 2, \ldots, p+1 \text{ and } j = 0, \ldots, p. \end{aligned} \tag{4.1.6}$$

Now write the $\mathrm{SP}_2'(p)$ solution as

$$u^\varepsilon(\underset{\sim}{x}^\varepsilon) = \sum_{j=0}^{p} \omega_j(x_1^\varepsilon) Q_j(x_2^\varepsilon),$$

$$\underset{\sim}{\sigma}^\varepsilon(\underset{\sim}{x}^\varepsilon) = \begin{pmatrix} \sum_{j=0}^{p} \sigma_1^j(x_1^\varepsilon) Q_j(x_2^\varepsilon) \\ \sum_{j=2}^{p+1} \sigma_2^j(x_1^\varepsilon) \tilde{Q}_j(x_2^\varepsilon) \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon^{-1} x_2^\varepsilon g^0 + g^1 \end{pmatrix},$$

and define $\mathbf{q}^e$, $\mathbf{q}^o \in \mathbb{R}^p$ by

$$\mathbf{q}_j^e = \int_\varepsilon^\varepsilon \tilde{Q}_j(x_2^\varepsilon)\, dx_2^\varepsilon, \quad \mathbf{q}_j^o = \varepsilon^{-1} \int_\varepsilon^\varepsilon x_2^\varepsilon \tilde{Q}_j(x_2^\varepsilon)\, dx_2^\varepsilon, \tag{4.1.7}$$

for $j = 2, \ldots, p+1$. Then, using (4.1.3) with $\underset{\sim}{\tau} = (0, \tau_2 \tilde{Q}_i)^T$, where $\tau_2 \in L^2(-1,1)$, we conclude that $\sigma_2^i = -[\tilde{\mathbf{M}}^{-1}(\tilde{\mathbf{N}}\boldsymbol{\omega} + g^1\mathbf{q}^e + g^0\mathbf{q}^o)]_i$ for $i = 2, \cdots, p+1$. Similarly, replacing $\underset{\sim}{\tau} = (\tau_1 Q_i, 0)^T$ in (4.1.3), where $\tau_1 \in H^1(-1,1)$, yields $\sigma_1^i = \partial_1 \omega_i$ for $i = 0, \ldots, p$. Finally, using equation (4.1.4) with vQ_i, $v \in L^2(-1,1)$, as test function, for $i = 0, \ldots, p$, we conclude that

$$\begin{gathered} \mathbf{M}\partial_{11}\boldsymbol{\omega} - \tilde{\mathbf{N}}^T\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{N}}\boldsymbol{\omega} = -\varepsilon\mathbf{f} + \tilde{\mathbf{N}}^T\tilde{\mathbf{M}}^{-1}(g^1\mathbf{q}^e + g^0\mathbf{q}^o) - 2g^0\mathbf{e}_1, \\ \boldsymbol{\omega}(-1) = \boldsymbol{\omega}(1) = 0, \end{gathered} \tag{4.1.8}$$

where $\mathbf{e}_1 = (1, 0, \cdots, 0)$.

Remark. For $p > 1$, the description of the $\mathrm{SP}_1'(p)$ model is very close to the above, the only difference is the dimensions of the matrices involved.

The simplest model in this family is $\mathrm{SP}_2'(1)$, described below:

$$u^\varepsilon(\underset{\sim}{x}^\varepsilon) = \omega_0(x_1^\varepsilon) + \omega_1(x_1^\varepsilon) x_2^\varepsilon,$$

$$\underset{\sim}{\sigma}^\varepsilon(\underset{\sim}{x}^\varepsilon) = \begin{pmatrix} \partial_1\omega_0(x_1^\varepsilon) \\ \varepsilon^{-1} g^0 x_2^\varepsilon \end{pmatrix} + \begin{pmatrix} \partial_1\omega_1(x_1^\varepsilon) x_2^\varepsilon \\ -\frac{5}{4}(\varepsilon^{-2} x_2^{\varepsilon 2} - 1)\omega_1 + \frac{1}{4}(5\varepsilon^{-2} x_2^{\varepsilon 2} - 1) g^1 \end{pmatrix},$$

where

$$\partial_{11}\omega_0 = -\frac{1}{2} f^0 - \varepsilon^{-1} g^0, \quad \frac{2}{3}\varepsilon^2 \partial_{11}\omega_1 - \frac{5}{3}\omega_1 = -f^1 - \frac{5}{3} g^1,$$

and $\omega_0 = \omega_1 = 0$ on $\{-1, 1\}$.

Section 4.2 – Asymptotic expansion for the solutions of the models. For the $\mathrm{SP}_2'(p)$ methods, the relation $\underset{\sim}{\sigma}^\varepsilon = \underset{\sim}{\nabla} u^\varepsilon$ does not hold in general and so we will develop asymptotic expansions for $u^\varepsilon(p)$ and $\underset{\sim}{\sigma}^\varepsilon(p)$ simultaneously. We start by rewriting equations (4.1.3) and (4.1.4) in the scaled domain R. Define

$$u(p)(\underset{\sim}{x}) = u^\varepsilon(p)(\underset{\sim}{x}^\varepsilon), \quad \sigma_1(p)(\underset{\sim}{x}) = \sigma_1^\varepsilon(p)(\underset{\sim}{x}^\varepsilon), \quad \sigma_2(p)(\underset{\sim}{x}) = \varepsilon\sigma_2^\varepsilon(p)(\underset{\sim}{x}^\varepsilon),$$

$$\begin{gathered} V'(R,p) = \left\{v \in L^2(R) \,:\, \deg_2 v \le p\right\}, \\ \underset{\sim}{S}_g'(R,p) = \left\{\underset{\sim}{\tau} \in \underset{\sim}{\mathcal{D}}'(R) \,:\, \underset{\sim}{\tau} \in \underset{\sim}{L}^2(R),\, \partial_1\tau_1 + \varepsilon^{-2}\partial_2\tau_2 \in L^2(R), \right. \\ \left. \underset{\sim}{\tau}\cdot\underset{\sim}{n} = g \text{ on } \partial R_\pm,\, \deg_2 \tau_1 \le p,\, \deg_2\tau_2 \le p+1\right\}. \end{gathered} \tag{4.2.1}$$

Then $u(p) \in V'(R,p)$ and $\underset{\sim}{\sigma}(p) \in \underset{\sim}{S}_{\varepsilon^2 g}'(R,p)$ satisfy

$$\begin{gathered} \int_R \sigma_1(p)\tau_1 + \varepsilon^{-2}\sigma_2(p)\tau_2\, d\underset{\sim}{x} + \int_R u(p)(\partial_1\tau_1 + \varepsilon^{-2}\partial_2\tau_2)\, d\underset{\sim}{x} = 0 \\ \text{for all } \underset{\sim}{\tau} \in \underset{\sim}{S}_0'(R,p), \\ \int_R [\partial_1\sigma_1(p) + \varepsilon^{-2}\partial_2\sigma_2(p)]v\, d\underset{\sim}{x} = -\int_R fv\, d\underset{\sim}{x} \qquad \text{for all } v \in V'(R,p). \end{gathered} \tag{4.2.2}$$

Consider

$$\begin{gathered} u(p) \sim u^0(p) + \varepsilon^2 u^2(p) + \varepsilon^4 u^4(p) + \dots, \\ \underset{\sim}{\sigma}(p) \sim \underset{\sim}{\sigma}^0(p) + \varepsilon^2\underset{\sim}{\sigma}^2(p) + \varepsilon^4\underset{\sim}{\sigma}^4(p) + \dots, \end{gathered} \tag{4.2.3}$$

where $u^{2k}(p) \in L^2(R)$ for any positive integer k. We impose $\underset{\sim}{\sigma}^0(p)$ and $\underset{\sim}{\sigma}^{2k}(p) \in S_0'(R,p)$ for integers $k \ge 2$ and $\underset{\sim}{\sigma}^2(p) \in \underset{\sim}{S}_g'(R,p)$. Then, after the formal process of substituting (4.2.3) into (4.2.2) and grouping together terms with the same powers of ε, we define $u^{-2} = 0$, $\underset{\sim}{\sigma}^{-2} = 0$, and, for $k \in \mathbb{N}$, ask that

$$\begin{gathered} \int_R \sigma_1^{2k-2}(p)\tau_1 + \sigma_2^{2k}(p)\tau_2\, d\underset{\sim}{x} + \int_R [u^{2k-2}(p)\partial_1\tau_1 + u^{2k}(p)\partial_2\tau_2]\, d\underset{\sim}{x} = 0 \\ \text{for all } \underset{\sim}{\tau} \in \underset{\sim}{S}_0'(R,p), \end{gathered} \tag{4.2.4}$$

$$\int_R [\partial_1\sigma_1^{2k-2}(p) + \partial_2\sigma_2^{2k}(p)]v\, d\underset{\sim}{x} = -\delta_{k1}\int_R fv\, d\underset{\sim}{x} \qquad \text{for all } v \in V'(R,p). \tag{4.2.5}$$

To determine $u^{2k}(p)$, $\underset{\sim}{\sigma}^{2k}(p)$ we proceed as in Section 2.1. First, set $k = 0$. From equation (4.2.5) we find that $\partial_2\sigma_2^0(p) = 0$. As $\underset{\sim}{\sigma} \in \underset{\sim}{S}'_0(R,p)$, then $\sigma_2^0(p) = 0$. From (4.2.4), we see that $u^0(p)$ is independent of x_2. Making $k = 1$ and using (4.2.4) with $\tau_2 = 0$, $\sigma_1^0(p) = \partial_1 u^0(p)$. We finally determine $u^0(p)$ completely from the compatibility condition of (4.2.5), and $u^0(p) = \zeta^0$ (see (2.1.10) and (2.1.12)). If we proceed with the same kind of argument, and with $x_1 \in (-1,1)$ as a parameter, $u^2(p)(x_1,\cdot) \in \hat{\mathbb{P}}_p(-1,1)$ and $\sigma_2^2(p)(x_1,\cdot) \in \mathbb{P}_{p+1}(-1,1)$ with $\sigma_2^2(p)(x_1,-1) = -g(x_1,-1)$ and $\sigma_2^2(p)(x_1,1) = g(x_1,1)$ should satisfy

$$\begin{aligned}
&\int_{-1}^{1} \sigma_2^2(p)(x_1,x_2)\tau_2(x_2)\,dx_2 + \int_{-1}^{1} u^2(p)(x_1,x_2)\partial_2\tau_2(x_2)\,dx_2 = 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } \tau_2 \in \mathring{\mathbb{P}}_{p+1}(-1,1), \\
&\int_{-1}^{1} \partial_2\sigma_2^2(p)(x_1,x_2)v(x_2)\,dx_2 = -\int_{-1}^{1} [f(x_1,x_2) + \partial_{11}\zeta^0(x_1)]v(x_2)\,dx_2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } v \in \hat{\mathbb{P}}_p(-1,1), \\
&\sigma_1^2(p) = \partial_1 u^2(p),
\end{aligned} \tag{4.2.6}$$

where $\mathring{\mathbb{P}}_{p+1}(-1,1) = \mathbb{P}_{p+1}(-1,1) \cap \mathring{H}^1(-1,1)$. Also, for any integer $k \geq 2$, we define $\sigma_2^{2k}(p)(x_1,\cdot) \in \mathring{\mathbb{P}}_{p+1}(-1,1)$ and $u^{2k}(p)(x_1,\cdot) \in \hat{\mathbb{P}}_p(-1,1)$ by imposing

$$\begin{aligned}
&\int_{-1}^{1} \sigma_2^{2k}(p)(x_1,x_2)\tau_2(x_2)\,dx_2 + \int_{-1}^{1} u^{2k}(p)(x_1,x_2)\partial_2\tau_2(x_2)\,dx_2 = 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } \tau_2 \in \mathbb{P}_{p+1}^0(-1,1), \\
&\int_{-1}^{1} \partial_2\sigma_2^{2k}(p)(x_1,x_2)v(x_2)\,dx_2 = -\int_{-1}^{1} \partial_{11}u^{2k-2}(p)(x_1,x_2)v(x_2)\,dx_2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } v \in \hat{\mathbb{P}}_{p+1}(-1,1), \\
&\sigma_1^{2k}(p) = \partial_1 u^{2k}(p).
\end{aligned} \tag{4.2.7}$$

Note from (2.1.5), (2.1.7) that $u^2(p)(x_1,\cdot)$ and $\sigma_2^2(p)(x_1,\cdot)$ are mixed approximations for $u^2(x_1,\cdot)$, $\partial_2 u^2(x_1,\cdot)$, with $x_1 \in (-1,1)$ as a parameter.

The condition that $u^\varepsilon(p)$ (or equivalently, $u(p)$) vanishes on ∂R_L^ε (∂R_L) is a natural one in (4.2.4). Nevertheless, this boundary condition is not being imposed at all for the terms of the asymptotic expansion, with the exception of $u^0(p)$, and thus, in general, $u^{2k}(p)$ does not vanish on ∂R_L, for $k \geq 1$. We would like then to have boundary correctors

$$U_-^{2k}(p), \quad \underset{\sim}{\Xi}_-^{2k}(p) = \begin{pmatrix} (\Xi_-^{2k})_1(p) \\ (\Xi_-^{2k})_2(p) \end{pmatrix}$$

such that

$$\begin{aligned} \int_R [(\Xi_-^{2k})_1(p)\tau_1 + \varepsilon^{-2}(\Xi_-^{2k})_2(p)\tau_2]\, d\underset{\sim}{x} + \int_R U_-^{2k}(p)(\partial_1\tau_1 + \varepsilon^{-2}\partial_2\tau_2)\, d\underset{\sim}{x} \\ = -\int_{-1}^{1} u^{2k}(-1, x_2)\tau_1(0, x_2)\, dx_2 \qquad \text{for all } \underset{\sim}{\tau} \in \underset{\sim}{S}_0'(R, p), \\ \int_R [\partial_1(\Xi_-^{2k})_1(p) + \varepsilon^{-2}\partial_2(\Xi_-^{2k})_2(p)]v\, d\underset{\sim}{x} = 0 \qquad \text{for all } v \in V'(R, p). \end{aligned} \tag{4.2.8}$$

Analogously to Section 2.1, we define the boundary corrector problem not by (4.2.8), but by posing an ε-independent problem. Recall that $\hat{\rho}_- = \varepsilon^{-1}(1 + x_1)$, $\Sigma = \mathbb{R}^+ \times (-1, 1)$ and $\partial\Sigma_\pm = \mathbb{R}^+ \times \{-1, 1\}$, and set

$$\begin{aligned} (\hat{\Xi}_-^{2k})_1(p)(\hat{\rho}_-, x_2) = \varepsilon(\Xi_-^{2k})_1(p)(\underset{\sim}{x}), \quad (\hat{\Xi}_-^{2k})_2(p)(\hat{\rho}_-, x_2) = (\Xi_-^{2k})_2(p)(\underset{\sim}{x}), \\ \hat{\underset{\sim}{\Xi}}_-^{2k}(p) = \begin{pmatrix} (\hat{\Xi}_-^{2k})_1(p) \\ (\hat{\Xi}_-^{2k})_2(p) \end{pmatrix}. \end{aligned} \tag{4.2.9}$$

We do not use hat on U_-^{2k}. Define the spaces

$$\begin{aligned} V'(\Sigma, p) = \big\{v \in \mathcal{D}'(\Sigma) \,:\, (1 + \hat{\rho}_-)^{-1}v \in L^2(\Sigma),\, \deg_2 v \leq p\big\}, \\ \underset{\sim}{S}_0'(\Sigma, p) = \big\{\underset{\sim}{\tau} \in \underset{\sim}{\mathcal{D}}'(\Sigma) \,:\, (1 + \hat{\rho}_-)\operatorname{div}\underset{\sim}{\tau} \in L^2(\Sigma), \underset{\sim}{\tau} \in \underset{\sim}{L}^2(\Sigma), \\ \deg_2 \tau_1 \leq p,\, \deg_2 \tau_2 \leq p + 1\big\}. \end{aligned}$$

These are appropriate spaces to pose the boundary corrector problem, as we show in Chapter 6. Posing the problem for the boundary corrector in the semi-infinite strip Σ, we define $U_-^{2k}(p) \in V'(\Sigma,p)$, and $\hat{\underset{\sim}{\Xi}}{}_-^{2k}(p) \in \underset{\sim}{S}_0'(\Sigma,p)$, by requiring that

$$\begin{aligned} \int_\Sigma \hat{\underset{\sim}{\Xi}}{}_-^{2k}(p)\cdot \hat{\underset{\sim}{\tau}}\, d\hat\rho_- dx_2 + \int_\Sigma U_-^{2k}(p)\operatorname{div}\hat{\underset{\sim}{\tau}}\, d\hat\rho_- dx_2 \\ = -\int_{-1}^{1} u^{2k}(p)(-1,x_2)\hat\tau_1(0,x_2)\,dx_2 \qquad \text{for all } \hat{\underset{\sim}{\tau}} \in \underset{\sim}{S}_0'(\Sigma,p), \\ \int_\Sigma \operatorname{div}\hat{\underset{\sim}{\Xi}}{}_-^{2k}(p) v\, d\hat\rho_- dx_2 = 0 \qquad \text{for all } v \in V'(\Sigma,p). \end{aligned} \tag{4.2.10}$$

Looking at (2.1.14), (2.1.15), we note that $U_-^{2k}(p)$, $\hat{\underset{\sim}{\Xi}}{}_-^{2k}(p)$ are mixed approximations for U_-^{2k}, $\underset{\sim}{\nabla} U_-^{2k}$. We define $U_+^{2k}(p)$, $\hat{\underset{\sim}{\Xi}}{}_+^{2k}(p)$ in a similar way. We show latter, see Lemma 4.2.5, that (4.2.10) is well posed.

Finally, the following expansions hold:

$$\begin{aligned} u^\varepsilon(p)(\underset{\sim}{x}^\varepsilon) &\sim \zeta^0(x_1^\varepsilon) + \sum_{k\ge1}\varepsilon^{2k}u^{2k}(p)(x_1^\varepsilon,\varepsilon^{-1}x_2^\varepsilon) \\ &\qquad - \sum_{k\ge1}\varepsilon^{2k}[U_-^{2k}(p)(\hat\rho_-,\varepsilon^{-1}x_2^\varepsilon) + U_+^{2k}(p)(\hat\rho_+,\varepsilon^{-1}x_2^\varepsilon)], \\ \underset{\sim}{\sigma}^\varepsilon(p)(\underset{\sim}{x}^\varepsilon) &\sim \begin{pmatrix} \partial_1\zeta^0(x_1^\varepsilon) \\ 0 \end{pmatrix} + \sum_{k\ge1}\varepsilon^{2k}\begin{pmatrix} \sigma_1^{2k}(p)(x_1^\varepsilon,\varepsilon^{-1}x_2^\varepsilon) \\ \varepsilon^{-1}\sigma_2^{2k}(p)(x_1^\varepsilon,\varepsilon^{-1}x_2^\varepsilon) \end{pmatrix} \\ &\qquad - \sum_{k\ge1}\varepsilon^{2k-1}[\hat{\underset{\sim}{\Xi}}{}_-^{2k}(p)(\hat\rho_-,\varepsilon^{-1}x_2^\varepsilon) + \hat{\underset{\sim}{\Xi}}{}_+^{2k}(p)(\hat\rho_+,\varepsilon^{-1}x_2^\varepsilon)]. \end{aligned} \tag{4.2.11}$$

Now we start to study the above expansions, first presenting results related to the terms in (4.2.11), and ending with estimates of the approximation error of truncated asymptotic expansions.

To be able to discuss problems in mixed form, we present, in an abstract framework, the result that guarantees existence, uniqueness and stability of solutions. Let X be a Hilbert space with associated norm $\|\cdot\|_X$, and let $a(\cdot,\cdot)$ be a symmetric continuous bilinear form on $X\times X$. Let M be another Hilbert space with norm $\|\cdot\|_M$, and $b(\cdot,\cdot)$ a continuous bilinear form on $X\times M$. For $F\in X^*$, the dual of X and $G\in M^*$, the dual of M, we look for $(s,w) \in X \times M$ such that

$$\begin{aligned} a(s,r) + b(r,w) &= F(r) \qquad \text{for all } r \in X, \\ b(s,v) &= G(v) \qquad \text{for all } v \in M. \end{aligned} \tag{4.2.12}$$

The well-posedness of such problem is as follows [16].

Lemma 4.2.1. *Assume that there exists constants α_0, k_0 such that*

$$a(r,r) \geq \alpha_0 \|r\|_X^2 \qquad \textit{for all } r \in \left\{\tilde{r} \in X \,:\, b(\tilde{r},w) = 0, \textit{ for all } w \in M\right\}, \tag{4.2.13}$$

$$\sup_{r \in X} \frac{b(r,v)}{\|r\|_X} \geq k_0 \|v\|_M \qquad \textit{for all } v \in M. \tag{4.2.14}$$

Then there exists a unique $(s,w) \in X \times M$ solving (4.2.12)*. Moreover,*

$$\begin{aligned} \|s\|_X &\leq \frac{1}{k_0}\left(1 + \frac{\|a\|}{\alpha_0}\right)\|G\|_{M^*} + \frac{1}{\alpha_0}\|F\|_{X^*}, \\ \|w\|_M &\leq \frac{\|a\|}{k_0^2}\left(1 + \frac{\|a\|}{\alpha_0}\right)\|G\|_{M^*} + \frac{1}{k_0}\left(1 + \frac{\|a\|}{\alpha_0}\right)\|F\|_{X^*}. \end{aligned}$$

In this and the next section, we need the lemma below regarding one-dimensional mixed problems. This result combines Lemmas B.1 and B.2, which contain detailed proofs.

Lemma 4.2.2. *Given $u \in H^2(-1,1) \cap \hat{L}^2(-1,1)$ and $\sigma = u'$, there exists unique $u(p) \in \hat{\mathbb{P}}_p(-1,1)$ and $\sigma(p) \in \mathbb{P}_{p+1}(-1,1)$ with $\sigma(p)(-1) = \sigma(-1)$, and $\sigma(p)(1) = \sigma(1)$, such that*

$$\begin{aligned} \int_{-1}^{1} [\sigma - \sigma(p)]\tau + [u - u(p)]\tau' \, dx_2 &= 0 \qquad \textit{for all } \tau \in \mathring{\mathbb{P}}_{p+1}(-1,1), \\ \int_{-1}^{1} [\sigma - \sigma(p)]' v \, dx_2 &= 0 \qquad \textit{for all } v \in \hat{\mathbb{P}}_p(-1,1). \end{aligned}$$

Moreover, for any nonnegative real number s, there exists a constant C such that

$$\|\sigma(p)\|_{H^1(-1,1)} \le C\|u\|_{H^2(-1,1)},$$

$$\|u(p)\|_{H^s(-1,1)} \le \begin{cases} C\|u\|_{H^2(-1,1)} & \text{if } 0 \le s \le 5/4, \\ C\|u\|_{H^{3s-7/4}(-1,1)} & \text{if } 5/4 \le s < 7/4, \\ C\|u\|_{H^{4s-7/2}(-1,1)} & \text{if } 7/4 \le s, \end{cases}$$

$$\|u - u(p)\|_{L^2(-1,1)} \le Cp^{-2-s}\|u\|_{H^{s+2}(-1,1)},$$

$$\|u - u(p)\|_{H^{1/2}(-1,1)} \le Cp^{-1-s}\|u\|_{H^{s+2}(-1,1)},$$

$$\|\sigma - \sigma(p)\|_{L^2(-1,1)} \le Cp^{-1-s}\|u\|_{H^{s+2}(-1,1)}, \quad \|\sigma - \sigma(p)\|_{H^1(-1,1)} \le Cp^{-s}\|u\|_{H^{s+2}(-1,1)}.$$

The terms u^{2k} and $\underset{\sim}{\sigma}^{2k}$, for $k \ge 1$, are defined by equations in mixed form, see (4.2.6) and (4.2.7). The lemma below regards one-dimensional problems of such kind, and is an application of Lemma 4.2.2.

Lemma 4.2.3. *For any $\tilde{f} \in L^2(-1,1)$, and a, $b \in \mathbb{R}$, there exists a unique $\tilde{u} \in \hat{\mathbb{P}}_p(-1,1)$ and a unique $\tilde{\sigma} \in \mathbb{P}_{p+1}(-1,1)$ such that $\tilde{\sigma}(-1) = a$, $\tilde{\sigma}(1) = b$ and*

$$\begin{aligned} &\int_{-1}^{1} \tilde{\sigma}\tau + \tilde{u}\tau' \, dx = 0 && \text{for all } \tau \in \mathring{\mathbb{P}}_{p+1}(-1,1), \\ &\int_{-1}^{1} \tilde{\sigma}' v \, dx = \int_{-1}^{1} \tilde{f} v \, dx && \text{for all } v \in \hat{\mathbb{P}}_p(-1,1). \end{aligned} \tag{4.2.15}$$

Moreover, for any nonnegative real number s, there exists a constant C such that

$$\|\tilde{\sigma}\|_{H^1(-1,1)} \le C(\|\tilde{f}\|_{L^2(-1,1)} + |a| + |b|),$$

$$\|\tilde{u}\|_{H^s(-1,1)} \le C(|a| + |b|) + \begin{cases} C\|\tilde{f}\|_{L^2(-1,1)} & \text{if } 0 \le s < 5/4, \\ C\|\tilde{f}\|_{H^{3s-15/4}(-1,1)} & \text{if } 5/4 \le s < 7/4, \\ C\|\tilde{f}\|_{H^{4s-11/2}(-1,1)} & \text{if } 7/4 \le s. \end{cases}$$

Using the above lemma and arguing by induction, as in the proof of Lemma 2.1.5, we have the following result. See also Lemma 3.2.1.

Lemma 4.2.4. *Let j, $k \in \mathbb{N}$ such that $k \geq 1$ and s be a nonnegative real number. Then there exists a positive constant C such that*

$$\|\partial_1^j u^{2k}(p)(x_1,\cdot)\|_{H^1(-1,1)} + \|\partial_1^j \sigma_2^{2k}(p)(x_1,\cdot)\|_{H^1(-1,1)}$$
$$\leq C(\|\partial_1^{2k-2+j} f(x_1,\cdot)\|_{L^2(-1,1)} + |\partial_1^{2k-2+j} g(x_1,-1)| + |\partial_1^{2k-2+j} g(x_1,1)|),$$
$$\|u^2(p)(x_1,\cdot)\|_{H^s(-1,1)}$$
$$\leq C(|g(x_1,-1)| + |g(x_1,1)|) + \begin{cases} C\|f(x_1,\cdot)\|_{L^2(-1,1)} & \text{if } 0 \leq s < 5/4, \\ C\|f(x_1,\cdot)\|_{H^{3s-15/4}(-1,1)} & \text{if } 5/4 \leq s < 7/4, \\ C\|f(x_1,\cdot)\|_{H^{4s-11/2}(-1,1)} & \text{if } 7/4 \leq s, \end{cases}$$

for all $p \in \mathbb{N}$.

We present below the stability result regarding the boundary correctors for the models. The result is a direct consequence of Lemma 6.4.1.

Theorem 4.2.5. *Assume, for any positive integer k, that $u^{2k}(p)$ is defined as above. Then there exist unique $U_-^{2k}(p) \in V'(\Sigma, p)$ and $\hat{\underset{\sim}{\Xi}}{}_-^{2k}(p) \in \underset{\sim}{S}'_0(\Sigma, p)$ satisfying (4.2.10). Furthermore, there exists a universal constant C such that*

$$\|U_-^{2k}(p)\|_{L^2(\Sigma)} + \|(1+\hat{\rho}_-)\operatorname{div}\hat{\underset{\sim}{\Xi}}{}_-^{2k}(p)\|_{L^2(\Sigma)} + \|\hat{\underset{\sim}{\Xi}}{}_-^{2k}(p)\|_{L^2(\Sigma)} \leq C\|u^{2k}(p)\|_{H^{1/2}(\partial R_L)},$$
$$\int_t^\infty \int_{-1}^1 [U_-^{2k}(p)]^2 + |\underset{\sim}{\Xi}{}_-^{2k}(p)|^2 \, dx_2 \, \hat{\rho}_- \leq C\|u^{2k}(p)\|^2_{H^{1/2}(\partial R_L)} \exp(-t/5),$$

for all $p \in \mathbb{N}$ and $t \in \mathbb{R}^+$.

The result below follows from Lemmas 4.2.4 and 4.2.5. Note that the stability constant is independent of p. Similar bounds hold for $U_+^{2k}(p)$ and $\underset{\sim}{\Xi}{}_+^{2k}(p)$.

Lemma 4.2.6. *For any positive integer k, there exists a constant C such that*

$$\begin{aligned} &\|U_-^{2k}(p)\|_{L^2(\Sigma)} + \|(1+\hat{\rho}_-)\operatorname{div}\hat{\underset{\sim}{\Xi}}{}_-^{2k}(p)\|_{L^2(\Sigma)} + \|\hat{\underset{\sim}{\Xi}}{}_-^{2k}(p)\|_{L^2(\Sigma)} \\ &\qquad\qquad \leq C\left(\|\partial_1^{2k-2} f\|_{L^2(\partial R_L)} + |\partial_1^{2k-2} g|_{C(\partial R_L)}\right), \\ &\int_t^\infty \int_{-1}^1 [U_-^{2k}(p)(\underset{\sim}{x})]^2 + |\hat{\underset{\sim}{\Xi}}{}_-^{2k}(p)(\underset{\sim}{x})|^2 \, dx_2 \, d\hat{\rho}_- \\ &\qquad\qquad \leq C(\|\partial_1^{2k-2} f\|^2_{L^2(\partial R_L)} + |\partial_1^{2k-2} g|^2_{C(\partial R_L)}) \exp(-t/5). \end{aligned} \tag{4.2.16}$$

In the remainder of this section, we estimate the convergence rates of the truncated asymptotic expansions. The convergence in ε is the same as in the SP(p) models. The proof is analogous to the one of Theorem 2.2.2, but as it uses the theory of mixed problems, we go through the main steps.

In what follows, we denote by π_p the orthogonal projection from $L^2(-1,1)$ to $\mathbb{P}_p(-1,1)$, and by $\mathring{\pi}^1$ the orthogonal projection operator from $\mathring{H}^1(-1,1)$ to $\mathring{\mathbb{P}}_p(-1,1)$, with respect to the inner product that induces norm $|\cdot|_{H^1(-1,1)}$. We need the following technical result.

Lemma 4.2.7. *If $\tau \in \mathring{H}^1(-1,1)$, then $(\mathring{\pi}^1_{p+1}\tau)' = \pi_p\tau'$ and if $\varphi \in H^1(-1,1)\cap\hat{L}^2(-1,1)$, then $(\hat{\pi}^1_p\varphi)' = \pi_{p-1}\varphi'$.*

Proof. For any $v \in \mathbb{P}_p(-1,1)$, we can write $v = \mathring{v}' + c$, where $\mathring{v}(\hat{\rho}_2) = \int_{-1}^{\hat{\rho}_2} v(s) - c\,ds$, and $c = (1/2)\int_{-1}^{1} v(s)\,ds$. Note that $\mathring{v} \in \mathring{H}^1(-1,1)$ and for $\tau \in \mathring{H}^1(-1,1)$,

$$\int_{-1}^{1} (\mathring{\pi}^1_{p+1}\tau)'v\,d\hat{\rho}_2 = \int_{-1}^{1} (\mathring{\pi}^1_{p+1}\tau)'(\mathring{v}' + c)\,d\hat{\rho}_2 = \int_{-1}^{1} \tau'(\mathring{v}' + c)\,d\hat{\rho}_2 = \int_{-1}^{1} \tau'v\,d\hat{\rho}_2.$$

So $(\mathring{\pi}^1_{p+1}\tau)' = \pi_p\tau'$. The second identity of the lemma follows form similar arguments. $\square$

We denote the orthogonal L^2 projection in the vertical direction by $\pi_p^{(x_2)}$, i.e., if $v \in L^2(R)$, then $\pi_p^{(x_2)}v \in L^2((-1,1);\mathbb{P}_p(-1,1))$ is such that

$$\int_R (\pi_p^{(x_2)}v - v)\psi\,d\underset{\sim}{x} = 0 \qquad \text{for all } \psi \in L^2((-1,1);\mathbb{P}_p(-1,1)).$$

Similar notation holds for $\mathring{\pi}_p^{(x_2)}$.

Before proving the main result, we show how the solutions of a mixed, ε-dependent problem in R behave. Recall that we define $V'(R,p)$ and $\underset{\sim}{S}'_0(R,p)$ in (4.2.1).

Theorem 4.2.8. *Let $\tilde{F} \in (\underset{\sim}{S}'_0(R,p))^*$, the dual space of $\underset{\sim}{S}'_0(R,p)$, and $\tilde{g} \in L^2(R)$. Then there exists unique $u \in V'(R,p)$ and $\underset{\sim}{\sigma} \in \underset{\sim}{S}'_0(R,p)$ such that*

$$\int_R (\sigma_1\tau_1 + \varepsilon^{-2}\sigma_2\tau_2)\,d\underset{\sim}{x} + \int_R u(\partial_1\tau_1 + \varepsilon^{-2}\partial_2\tau_2)\,d\underset{\sim}{x} = \tilde{F}(\underset{\sim}{\tau}) \qquad \text{for all } \underset{\sim}{\tau} \in \underset{\sim}{S}'_0(R,p),$$
$$\int_R (\partial_1\sigma_1 + \varepsilon^{-2}\partial_2\sigma_2)v\,d\underset{\sim}{x} = \int_R \tilde{g}v\,d\underset{\sim}{x} \qquad \text{for all } v \in V'(R,p).$$

Moreover, there is a universal constant C such that

$$\|u\|_{L^2(R)} + \|\sigma_1\|_{L^2(R)} + \varepsilon^{-1}\|\sigma_2\|_{L^2(R)} + \|\partial_1\sigma_1 + \varepsilon^2\partial_2\sigma_2\|_{L^2(R)}$$
$$\leq C\big(\|\tilde{F}\|_{(\underset{\sim}{S}'_0(R,p))^*} + \|\tilde{g}\|_{L^2(R)}\big).$$

Proof. We want to apply Lemma 4.2.1. Let

$$\|v\|_M = \|v\|_{L^2(R)}, \quad M = V'(R,p),$$
$$\|\underset{\sim}{\tau}\|_X^2 = \|\tau_1\|_{L^2(R)}^2 + \varepsilon^{-2}\|\tau_2\|_{L^2(R)}^2 + \|\partial_1\tau_1 + \varepsilon^{-2}\partial_2\tau_2\|_{L^2(R)}^2, \quad X = \underset{\sim}{S}'_0(R,p),$$
$$a(\underset{\sim}{\sigma},\underset{\sim}{\tau}) = \int_R \sigma_1\tau_1 + \varepsilon^{-2}\sigma_2\tau_2\,d\underset{\sim}{x}, \quad b(\underset{\sim}{\tau},v) = \int_R (\partial_1\tau_1 + \varepsilon^{-2}\partial_2\tau_2)v\,d\underset{\sim}{x}.$$

Since $\partial_1\tau_1 + \varepsilon^{-2}\partial_2\tau_2 \in V'(R,p)$ for all $\underset{\sim}{\tau} \in \underset{\sim}{S}'_0(R,p)$, the coercivity hypothesis (4.2.13) holds immediately with $\alpha_0 = 1$. Now we want to show that the inf-sup hypothesis (4.2.14) is also satisfied. Let $v \in V'(R,p)$, and define $V(R) = \{\tilde{v} \in H^1(R) : \tilde{v} = 0 \text{ on } \partial R_L\}$ and $u \in V(R)$ such that

$$\int_R \partial_1 u\partial_1\tilde{v} + \varepsilon^{-2}\partial_2 u\partial_2\tilde{v} = \int_R v\tilde{v}\,d\underset{\sim}{x} \qquad \text{for all } \tilde{v} \in V(R).$$

Then $\|u\|_{H^1(R)} \leq C\|v\|_{L^2(R)}$, where C is a universal constant. Moreover,

$$\int_{-1}^1 \varepsilon^{-2}|u(x_1,\cdot)|_{H^1(-1,1)}^2\,dx_1 = \int_R \varepsilon^{-2}[\partial_2 u(x_1,x_2)]^2\,d\underset{\sim}{x}$$
$$\leq \int_R [\partial_1 u(x_1,x_2)]^2 + \varepsilon^{-2}[\partial_2 u(x_1,x_2)]^2\,d\underset{\sim}{x} = \int_R vu\,d\underset{\sim}{x} \leq \|v\|_{L^2(R)}\|u\|_{L^2(R)}$$
$$\leq C\|v\|_{L^2(R)}^2.$$

Set $\sigma_1 = \partial_1 u$, $\sigma_2 = \partial_2 u$ and $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$. We cannot use $\underset{\sim}{\sigma}$ as our "candidate" for the inf-sup condition, as $\underset{\sim}{\sigma}$ does not belong to $\underset{\sim}{S}'_0(R,p)$ in general. Let $\hat{u}(x_1,x_2) = u(x_1,x_2) - (1/2)\int_{-1}^{1} u(x_1,x_2)\,dx_2$, and, for each $x_1 \in (-1,1)$, define $\sigma_2(p)(x_1,\cdot) \in \mathring{\mathbb{P}}_{p+1}(-1,1)$ and $u(p)(x_1,\cdot) \in \hat{\mathbb{P}}_p(-1,1)$ such that

$$\int_{-1}^{1} [\sigma_2(x_1,x_2) - \sigma_2(p)(x_1,x_2)]\tau(x_2) + [\hat{u}(x_1,x_2) - u(p)(x_1,x_2)]\partial_2\tau(x_2)\,dx_2 = 0$$
$$\text{for all } \tau \in \mathring{\mathbb{P}}_{p+1}(-1,1),$$
$$\int_{-1}^{1} \partial_2[\sigma_2(x_1,x_2) - \sigma_2(p)(x_1,x_2)]\hat{v}(x_2)\,dx_2 = 0 \qquad \text{for all } \hat{v} \in \hat{\mathbb{P}}_p(-1,1).$$

Then $\partial_2\sigma_2(p) = \pi_p^{(x_2)}\partial_2\sigma_2$, and using Lemma 4.2.7 we conclude that $\sigma_2(p) = \mathring{\pi}_{p+1}^{(x_2)}\sigma_2$. It follows that

$$\begin{aligned}\varepsilon^{-2}\|\sigma_2(p)\|^2_{L^2(R)} &= \int_{-1}^{1} \varepsilon^{-2}\|\sigma_2(p)(x_1,\cdot)\|^2_{L^2(-1,1)}\,dx_1\\ \le C\int_{-1}^{1} \varepsilon^{-2}\|\sigma_2(x_1,\cdot)\|^2_{L^2(-1,1)}\,dx_1 &= C\int_{-1}^{1} \varepsilon^{-2}|u(x_1,\cdot)|^2_{H^1(-1,1)}\,dx_1 \le C\|v\|^2_{L^2(R)}.\end{aligned} \tag{4.2.17}$$

Define now $\sigma_1(p) = \pi_p^{(x_2)}\sigma_1$. Then

$$\begin{aligned}\|\sigma_1(p)\|_{L^2(R)} &\le C\|\sigma_1\|_{L^2(R)} \le C\|v\|_{L^2(R)},\\ \|\partial_1\sigma_1(p) + \varepsilon^{-2}\partial_2\sigma_2(p)\|_{L^2(R)} &= \|\pi_p^{(x_2)}\partial_1\sigma_1 + \varepsilon^{-2}\pi_p^{(x_2)}\partial_2\sigma_2\|_{L^2(R)}\\ &\le C\|\partial_1\sigma_1 + \varepsilon^{-2}\partial_2\sigma_2\|_{L^2(R)} = C\|v\|_{L^2(R)}.\end{aligned} \tag{4.2.18}$$

Thus, from (4.2.17) and (4.2.18), $\|\underset{\sim}{\sigma}(p)\|_X \le C\|v\|_{L^2(R)}$. We can finally prove the inf-sup condition, as

$$\begin{aligned}\sup_{\underset{\sim}{\tau}\in \underset{\sim}{S}'_0(R,p)} \int_R v(\partial_1\tau_1 + \varepsilon^{-2}\partial_2\tau_2)\,d\underset{\sim}{x} &\ge \int_R v[\partial_1\sigma_1(p) + \varepsilon^{-2}\partial_2\sigma_2(p)]\,d\underset{\sim}{x}\\ = \int_R v[\pi_p^{(x_2)}\partial_1\sigma_1(p) + \varepsilon^{-2}\pi_p^{(x_2)}\partial_2\sigma_2]\,d\underset{\sim}{x} &= \int_R v(\partial_1\sigma_1 + \varepsilon^{-2}\partial_2\sigma_2)\,d\underset{\sim}{x}\\ &= \|v\|^2_{L^2(R)} \ge C\|v\|_{L^2(R)}\|\underset{\sim}{\sigma}(p)\|_X,\end{aligned}$$

and the condition (4.2.14) follows. Hence the hypotheses of Lemma 4.2.1 are satisfied and we can apply it here to conclude the present result. $\square$

We need the following notation in the proof of the next result. Define

$$u_N(p) = \sum_{k=0}^{N} \varepsilon^{2k} u^{2k}(p), \quad \underset{\sim}{\sigma}_N(p) = \sum_{k=0}^{N} \varepsilon^{2k} \underset{\sim}{\sigma}^{2k}(p),$$
$$U_N(p) = \sum_{k=1}^{N} \varepsilon^{2k} [U_-^{2k}(p) + U_+^{2k}(p)], \quad \hat{\underset{\sim}{\Xi}}_N(p) = \sum_{k=1}^{N} \varepsilon^{2k} [\hat{\underset{\sim}{\Xi}}{}_-^{2k}(p) + \hat{\underset{\sim}{\Xi}}{}_+^{2k}(p)].$$

Theorem 4.2.9. *For each $N \in \mathbb{N}$, there exists a constant C such that*

$$\left\| u^\varepsilon(p) - \zeta^0 - \sum_{k=1}^{N} \varepsilon^{2k} u^{2k}(p) + \sum_{k=1}^{N} \varepsilon^{2k} [U_-^{2k}(p) + U_+^{2k}(p)] \right\|_{L^2(R^\varepsilon)}$$
$$+ \left\| \underset{\sim}{\sigma}^\varepsilon(p) - \begin{pmatrix} \partial_1 \zeta^0 \\ 0 \end{pmatrix} - \sum_{k=1}^{N} \varepsilon^{2k} \begin{pmatrix} \sigma_1^{2k}(p) \\ \varepsilon^{-1} \sigma_2^{2k}(p) \end{pmatrix} + \sum_{k=1}^{N} \varepsilon^{2k-1} [\hat{\underset{\sim}{\Xi}}{}_-^{2k}(p) + \hat{\underset{\sim}{\Xi}}{}_+^{2k}(p)] \right\|_{H(\mathrm{div}, R^\varepsilon)}$$
$$\leq C \varepsilon^{2N+3/2} \left(|||(f,g)|||_{2N+2,R} + |||(f,g)|||_{N+1,\partial R_L} \right).$$

Proof. For $N = 0$, the result is a especial case of Theorem 2.2.2, so we can assume $N \geq 1$. For $\underset{\sim}{\tau} \in \underset{\sim}{S}'_0(R,p)$, let $\hat{\underset{\sim}{\tau}} \in \underset{\sim}{S}'_0(\Sigma,p)$ such that $\|\hat{\underset{\sim}{\tau}}\|_{\underset{\sim}{S}'_0(\Sigma,p)} \leq \|\underset{\sim}{\tau}\|_{H(\mathrm{div},R)}$. Then, from the asymptotic expansion developed above,

$$\int_R [\sigma_1(p) - (\sigma_N)_1(p) + \varepsilon^{-1} (\hat{\Xi}_N)_1(p)] \tau_1 + \varepsilon^{-2} [\sigma_2(p) - (\sigma_N)_2(p) + (\hat{\Xi}_N)_2(p)] \tau_2 \, d\underset{\sim}{x}$$
$$+ \int_R [u(p) - u_N(p) + U_N(p)] (\partial_1 \tau_1 + \varepsilon^{-2} \partial_2 \tau_2) \, d\underset{\sim}{x} = \varepsilon^{-1} \int_{2\varepsilon^{-1}}^{\infty} \int_{-1}^{1} \hat{\underset{\sim}{\Xi}}_N(p) \hat{\underset{\sim}{\tau}} + U_N(p) \, \mathrm{div} \, \hat{\underset{\sim}{\tau}} \, d\hat{\underset{\sim}{\rho}}$$

Also,

$$\int_R \{ \partial_1 [\sigma_1(p) - (\sigma_N)_1(p) + \varepsilon^{-1} (\hat{\Xi}_N)_1(p)] + \varepsilon^{-2} \partial_2 [\sigma_2(p) - (\sigma_N)_2(p) + (\hat{\Xi}_N)_2(p)] \} v \, d\underset{\sim}{x}$$
$$= -\varepsilon^{2N} \int_R \partial_{11} u^{2N}(p) v \, d\underset{\sim}{x} \qquad \text{for all } v \in V'(R,p).$$

Using Lemmas 4.2.4, 4.2.5 and 4.2.8, we have that

$$\begin{aligned}
&\|u(p) - u_N(p) + U_N\|_{L^2(R)} + \|\sigma_1(p) - \sigma_{N1}(p) + \varepsilon^{-1}(\hat{\Xi}_N)_1(p)\|_{L^2(R)} \\
&\qquad + \varepsilon^{-1}\|\sigma_2(p) - (\sigma_N)_2(p) + (\hat{\Xi}_N)_2(p)\|_{L^2(R)} \\
&+ \|\partial_1[\sigma_1(p) - (\sigma_N)_1(p) + \varepsilon^{-1}(\Xi_N)_1(p)] + \varepsilon^{-2}\partial_2[\sigma_2(p) - (\sigma_N)_2(p) + (\Xi_N)_2(p)]\|_{L^2(R)} \\
&\qquad\qquad \le C\varepsilon^{2N}|||(f,g)|||_{2N,R} + C\varepsilon \exp(-\varepsilon^{-1}/5)|||(f,g)|||_{N-1,\partial R_L}.
\end{aligned} \tag{4.2.19}$$

Next, to conclude the final result, we proceed as in Theorems 2.2.1 and 2.2.2, adding and subtracting new terms, using the triangle inequality, estimate (4.2.19), Lemmas 4.2.4 and 4.2.6, and scaling the domain, from R to R^ε. $\square$

Section 4.3 – Estimates for the modeling error. We begin now the last and crucial part of this chapter, the derivation of error estimates for the models SP_2' that we define in Section 4.1. The next lemma results from (4.2.6), (2.1.5), (2.1.7) and Lemma 4.2.2. To simplify the notation, we denote

$$\sigma_2^2(\underset{\sim}{x}) = \partial_{x_2} u^2(\underset{\sim}{x}). \tag{4.3.1}$$

Lemma 4.3.1. *Assume that u^2, σ_2^2, $u^2(p)$, $\sigma_2^2(p)$ are defined as above. Then, with $x_1 \in (-1,1)$ as a parameter,*

$$\int_{-1}^{1} [\sigma_2^2(x_1,x_2) - \sigma_2^2(p)(x_1,x_2)]\tau(x_2) + [u^2(x_1,x_2) - u^2(p)(x_1,x_2)]\tau'(x_2)\,dx_2 = 0$$
$$\text{for all } \tau \in \mathring{\mathbb{P}}_{p+1}(-1,1),$$
$$\int_{-1}^{1} \partial_2[\sigma_2^2(x_1,x_2) - \sigma_2^2(p)(x_1,x_2)]v(x_2)\,dx_2 = 0 \qquad \text{for all } v \in \hat{\mathbb{P}}_p(-1,1).$$

Moreover, for any nonnegative real number s, there exists a constant C such that

$$\|u^2(x_1,\cdot) - u^2(p)(x_1,\cdot)\|_{L^2(-1,1)} \le Cp^{-2-s}\|u^2(x_1,\cdot)\|_{H^{s+2}(-1,1)},$$

$$\|u^2(x_1,\cdot) - u^2(p)(x_1,\cdot)\|_{H^{1/2}(-1,1)} \le Cp^{-1-s}\|u^2(x_1,\cdot)\|_{H^{s+2}(-1,1)},$$

$$\|\sigma_2^2(x_1,\cdot) - \sigma_2^2(p)(x_1,\cdot)\|_{L^2(-1,1)} \le Cp^{-1-s}\|u^2(x_1,\cdot)\|_{H^{s+2}(-1,1)},$$

$$\|\sigma_2^2(x_1,\cdot) - \sigma_2^2(p)(x_1,\cdot)\|_{H^1(-1,1)} \le Cp^{-s}\|u^2(x_1,\cdot)\|_{H^{s+2}(-1,1)}.$$

From Lemmas 4.3.1 and 2.1.3, and integrating in the horizontal direction, we easily conclude the result below. The powers of ε appear from the change of coordinates $\underset{\sim}{x}$ to $\underset{\sim}{x}^\varepsilon$. Recall that a_s, a_s^1 are defined in (3.3.1).

Lemma 4.3.2. *For any nonnegative real number s, there exists a constant C such that*

$$\|u^2 - u^2(p)\|_{L^2(R^\varepsilon)} \le C\varepsilon^{1/2}p^{-2-s}a_s,$$

$$\|\partial_1 u^2 - \partial_1 u^2(p)\|_{L^2(R^\varepsilon)} \le C\varepsilon^{1/2}p^{-2-s}a_s^1,$$

$$\|\sigma_2^2 - \sigma_2^2(p)\|_{L^2(R^\varepsilon)} \le C\varepsilon^{1/2}p^{-1-s}a_s,$$

$$\|\partial_{x_2^\varepsilon}\sigma_2^2 - \partial_{x_2^\varepsilon}\sigma_2^2(p)\|_{L^2(R^\varepsilon)} \le C\varepsilon^{-1/2}p^{-s}a_s,$$

where σ_2^2 is defined in (4.3.1).

We now estimate how well the first term of the boundary layer expansion for $\underset{\sim}{\sigma}^\varepsilon(p)$ approximates the first term of the boundary layer expansion for $\underset{\sim}{\nabla}\, u^\varepsilon$. Similarly to Definition 3.3.3, we specify $\bar{\mu}$ of Definition 6.4.6 in terms of f and g, using (3.3.2).

Definition 4.3.3. Let s be a nonnegative real number. If $|g|_{C(\partial R_L)} \neq 0$, let $\bar{\mu}(s,\delta) = 1-\delta$. If $|g|_{C(\partial R_L)} = 0$, and if there exists an minimum integer $m \in \{2,\dots,J(s+5/2)\}$ such that $|\partial_2^{2m-3} f|_{C(\partial R_L)} \neq 0$, let $\bar{\mu}(s,\delta) = \min\big\{4m-3-\delta, s+3/2\big\}$, otherwise let $\bar{\mu}(s,\delta) = s+3/2$.

The next theorem is a direct application of Lemma 6.4.8. Let

$$\underset{\sim}{\hat{\Xi}}{}^2_-(\hat{\rho}_-, x_2) = \begin{pmatrix} \partial_{\hat{\rho}_-} U^2_-(\hat{\rho}_-, x_2) \\ \partial_{x_2} U^2_-(\hat{\rho}_-, x_2) \end{pmatrix}, \quad \underset{\sim}{\hat{\Xi}}{}^2_+(\hat{\rho}_+, x_2) = \begin{pmatrix} \partial_{\hat{\rho}_+} U^2_+(\hat{\rho}_+, x_2) \\ \partial_{x_2} U^2_+(\hat{\rho}_+, x_2) \end{pmatrix}.$$

Lemma 4.3.4. *For any nonnegative real number s such that $s + 1/2$ is not an even integer, and for any arbitrarily small $\delta > 0$, there exists a constant C such that*

$$\|\underset{\sim}{\hat{\Xi}}{}^2_+ - \underset{\sim}{\hat{\Xi}}{}^2_+(p)\|_{L^2(\Sigma)} + \|\underset{\sim}{\hat{\Xi}}{}^2_- - \underset{\sim}{\hat{\Xi}}{}^2_-(p)\|_{L^2(\Sigma)} \\ \leq C(\|u^2 - u^2(p)\|_{H^{1/2}(\partial R_L)} + p^{-\bar{\mu}(s,\delta)}\|u^2\|_{H^{s+2}(\partial R_L)}).$$

The next result estimate the boundary correctors in R^ε.

Lemma 4.3.5. *For any nonnegative real number s such that $s + 1/2$ is not an even integer, and for any arbitrarily small $\delta > 0$, there exists a constant C such that*

$$\|\underset{\sim}{\hat{\Xi}}{}^2_- - \underset{\sim}{\hat{\Xi}}{}^2_-(p)\|_{L^2(R^\varepsilon)} + \|\underset{\sim}{\hat{\Xi}}{}^2_+ - \underset{\sim}{\hat{\Xi}}{}^2_+(p)\|_{L^2(R^\varepsilon)} \leq C\varepsilon\big(p^{-1-s} + p^{-\bar{\mu}(s,\delta)}\big)a^b_s,$$

where a^b_s is defined in (3.3.1).

Proof. After changing coordinates and using Lemma 4.3.4, we see that

$$\|\underset{\sim}{\hat{\Xi}}{}^2_- - \underset{\sim}{\hat{\Xi}}{}^2_-(p)\|_{L^2(R^\varepsilon)} + \|\underset{\sim}{\hat{\Xi}}{}^2_+ - \underset{\sim}{\hat{\Xi}}{}^2_+(p)\|_{L^2(R^\varepsilon)} \\ \leq C\varepsilon(\|u^2 - u^2(p)\|_{H^{1/2}(\partial R_L)} + p^{-\bar{\mu}(s,\delta)}\|u^2\|_{H^{s+2}(\partial R_L)}).$$

From Lemmas 4.3.1 and 2.1.5,

$$\|u^2 - u^2(p)\|_{H^{1/2}(\partial R_L)} \leq Cp^{-1-s}(\|f\|_{H^s(\partial R_L)} + |g|_{C(\partial R_L)}),$$

and the lemma follows. $\square$

We are ready to compare, in the same way we do in Theorem 3.3.6, the exact and model solutions. Recall that a_s, a^b_s, a^1_s and a are defined in (3.3.1) and $\bar{\mu}$, is as in Definition 4.3.3.

Theorem 4.3.6. *For any nonnegative real numbers s and s^* such that $s^*+1/2$ is not an even integer, and for any arbitrarily small $\delta > 0$, there exists a constant C such that the following bounds hold:*

$$\|u^\varepsilon - u^\varepsilon(p)\|_{L^2(R^\varepsilon)} \leq C\varepsilon^{5/2}p^{-2-s}a_s + C\varepsilon^3 a,$$

$$\|\sigma_1^\varepsilon - \sigma_1^\varepsilon(p)\|_{L^2(R^\varepsilon)} \leq C\varepsilon^2\big(p^{-1-s^*} + p^{-\bar{\mu}(s^*,\delta)}\big)a^b_{s^*} + C\varepsilon^{5/2}a,$$

$$\|\sigma_1^\varepsilon - \sigma_1^\varepsilon(p)\|_{L^2(R_0^\varepsilon)} \leq C\varepsilon^{5/2}p^{-2-s}a^1_s + C\varepsilon^{7/2}a,$$

$$\|\sigma_2^\varepsilon - \sigma_2^\varepsilon(p)\|_{L^2(R^\varepsilon)} \leq C\varepsilon^{3/2}p^{-1-s}a_s + C\varepsilon^2 a,$$

where $\underset{\sim}{\sigma}^\varepsilon = \underset{\sim}{\nabla}\, u^\varepsilon$.

Proof. We prove the fourth estimate only, as the others follow from similar arguments. Using the triangle inequality the following holds:

$$\begin{aligned}\|\sigma_2^\varepsilon - \sigma_2^\varepsilon(p)\|_{L^2(R^\varepsilon)} &\leq \|\sigma_2^\varepsilon - \varepsilon\sigma_2^2 + \varepsilon[(\hat{\Xi}^2_-)_2 + (\hat{\Xi}^2_+)_2]\|_{L^2(R^\varepsilon)}\\ &+ \varepsilon\big(\|\sigma_2^2 - \sigma_2^2(p)\|_{L^2(R^\varepsilon)} + |(\hat{\Xi}^2_-)_2 - (\hat{\Xi}^2_-)_2(p)|_{L^2(R^\varepsilon)} + |(\hat{\Xi}^2_+)_2 - (\hat{\Xi}^2_+)_2(p)|_{L^2(R^\varepsilon)}\big)\\ &+ \|\sigma_2^\varepsilon(p) - \varepsilon\sigma_2^2(p) + \varepsilon[(\hat{\Xi}^2_-)_2(p) + (\hat{\Xi}^2_+)_2(p)]\|_{L^2(R^\varepsilon)}.\end{aligned}$$

From Theorems 2.2.2 and 4.2.9, we have that

$$\begin{aligned}&\|\sigma_2^\varepsilon - \varepsilon\sigma_2^2 + \varepsilon[(\hat{\Xi}^2_-)_2 + (\hat{\Xi}^2_+)_2]\|_{L^2(R^\varepsilon)}\\ &\qquad + \|\sigma_2^\varepsilon(p) - \varepsilon\sigma_2^2(p) + \varepsilon[(\hat{\Xi}^2_-)_2(p) + (\hat{\Xi}^2_+)_2(p)]\|_{L^2(R^\varepsilon)} \leq C\varepsilon^{7/2}a.\end{aligned}$$

The estimate

$$|(\hat{\Xi}^2_-)_2 - (\hat{\Xi}^2_-)_2(p)|_{L^2(R^\varepsilon)} + |(\hat{\Xi}^2_+)_2 - (\hat{\Xi}^2_+)_2(p)|_{L^2(R^\varepsilon)} \leq Ca$$

comes from Lemmas 2.1.8 and 4.2.6. Finally we apply Lemma 4.3.2 to bound $\|\sigma_2^2 - \sigma_2^2(p)\|_{L^2(R^\varepsilon)}$ and the result follows. $\square$

Remark 4.3.7. As in Remark 3.3.7, we have that actually

$$\|\sigma_1^\varepsilon - \sigma_1^\varepsilon(p)\|_{L^2(R_0^\varepsilon)} \leq C\varepsilon^{5/2}p^{-2-s}a_s^1 + C\varepsilon^{9/2}\big(|||(f,g)|||_{6,R} + |||(f,g)|||_{3,\partial R_L}\big).$$

Comparing the results of Theorems 3.3.6 and 4.3.6, we see that the rates of convergence in p (with one exception) and in ε are the same for both SP and SP$'$ models. One should compare the estimates for $\partial_{x_2^\varepsilon}u^\varepsilon(p)$ in the SP models with the ones for $\sigma_2^\varepsilon(p)$ in the SP$'$ models. The results of table 3.1 and the comments afterwards apply here as well, with obvious modifications.

Chapter 5

The Poisson problem in a three-dimensional plate

In this chapter we extend the previous results to a three-dimensional plate. After introducing the Poisson problem, we present the asymptotic expansion for the exact solution. Then we investigate both forms of the variational approach for dimension reduction.

Section 5.1 – The asymptotic expansion for the original solution. Recall that in the introduction we define the three-dimensional plate $P^\varepsilon = \Omega \times (-\varepsilon, \varepsilon)$ and its boundaries $\partial P_L^\varepsilon = \partial\Omega \times (-\varepsilon, \varepsilon)$ and $\partial P_\pm^\varepsilon = \Omega \times \{-\varepsilon, \varepsilon\}$. Assume that $u^\varepsilon \in H^1(P^\varepsilon)$ satisfies (in the weak sense)

$$\begin{aligned} \Delta\, u^\varepsilon &= -f^\varepsilon \qquad \text{in } P^\varepsilon, \\ \frac{\partial u^\varepsilon}{\partial n} &= g^\varepsilon \qquad \text{on } \partial P_\pm^\varepsilon, \\ u^\varepsilon &= 0 \qquad \text{on } \partial P_L^\varepsilon, \end{aligned} \tag{5.1.1}$$

where $f^\varepsilon : P^\varepsilon \to \mathbb{R}$ and $g^\varepsilon : \partial P_\pm^\varepsilon \to \mathbb{R}$.

Our first step to show the influence of ε explicitly is to rewrite (5.1.1) in the scaled domain $P = \Omega \times (-1, 1)$. Let $\partial P_L = \partial\Omega \times (-1, 1)$ and $\partial P_\pm = \Omega \times \{-1, 1\}$. Also, $\underline{x} = (\underset{\sim}{x}, x_3)$ is a typical point of P, with $\underset{\sim}{x} = \underset{\sim}{x}^\varepsilon$ and $x_3 = \varepsilon^{-1} x_3^\varepsilon$. In this new domain we define $u(\varepsilon)(\underline{x}) = u^\varepsilon(\underline{x}^\varepsilon)$, $f(\underline{x}) = f^\varepsilon(\underline{x}^\varepsilon)$, and $g(\underline{x}) = \varepsilon^{-1} g^\varepsilon(\underline{x}^\varepsilon)$. We conclude from (5.1.1) that

$$\begin{aligned} (\partial_{11} + \partial_{22} + \varepsilon^{-2}\partial_{33}) u(\varepsilon) &= -f \qquad \text{in } P, \\ \frac{\partial u(\varepsilon)}{\partial n} &= \varepsilon^2 g \qquad \text{on } \partial P_\pm, \\ u(\varepsilon) &= 0 \qquad \text{on } \partial P_L. \end{aligned} \tag{5.1.2}$$

We again assume that f and g are independent of ε.

Consider the asymptotic expansion

$$u(\varepsilon) \sim u^0 + \varepsilon^2 u^2 + \varepsilon^4 u^4 + \cdots . \tag{5.1.3}$$

Formally substituting (5.1.3) in (5.1.2) and grouping together terms with same power in ε we have

$$\begin{gathered}\varepsilon^{-2}\partial_{33}u^0 + [(\partial_{11}+\partial_{22})u^0 + \partial_{33}u^2] + \varepsilon^2\left[(\partial_{11}+\partial_{22})u^2 + \partial_{33}u^4\right] + \cdots = -f,\\ \frac{\partial u^0}{\partial n} + \varepsilon^2\frac{\partial u^2}{\partial n} + \varepsilon^4\frac{\partial u^4}{\partial n} + \cdots = \varepsilon^2 g \text{ on } \partial P_\pm.\end{gathered}$$

Then, using the same arguments of Section 2.1, we decompose

$$u^{2k}(\underline{x}) = \mathring{u}^{2k}(\underline{x}) + \zeta^{2k}(\underset{\sim}{x}),$$

where $\int_{-1}^{1}\mathring{u}^{2k}(\underline{x})\,dx_3 = 0$, and find that

$$\begin{gathered}(\partial_{11}+\partial_{22})\zeta^0(\underset{\sim}{x}) = -\frac{1}{2}\int_{-1}^{1} f(\underset{\sim}{x},x_3)\,dx_3 - \frac{1}{2}[g(\underset{\sim}{x},1)+g(\underset{\sim}{x},-1)] \qquad \text{in } \Omega,\\ \zeta^0 = 0 \qquad \text{on } \partial\Omega,\end{gathered} \tag{5.1.4}$$

and $\zeta^{2k} = 0$ for $k \geq 1$. Also, for each $\underset{\sim}{x} \in \Omega$,

$$\begin{gathered}\mathring{u}^0 = 0, \quad \partial_{33}\mathring{u}^2 = -f - (\partial_{11}+\partial_{22})\zeta^0,\\ \partial_{33}\mathring{u}^{2k} = -(\partial_{11}+\partial_{22})\mathring{u}^{2k-2} \qquad \text{for all } k \geq 2,\end{gathered}$$

along with the conditions

$$\frac{\partial \mathring{u}^{2k}}{\partial n} = \delta_{k1} g \text{ on } \partial P_\pm, \qquad \text{for all } k \geq 1.$$

Note that $u^0 = \zeta^0$ and u^{2k} for $k \geq 1$ are well determined.

Once again, in general u^{2k} does not vanish on the lateral boundary ∂P_L. We introduce then, formally, the boundary corrector

$$U \sim \varepsilon^2 U^2 + \varepsilon^3 U^3 + \varepsilon^4 U^4 + \cdots , \tag{5.1.5}$$

to correct the values of u^2, u^4, etc. on ∂P_L. We expect also that

$$\begin{aligned} (\partial_{11} + \partial_{22} + \varepsilon^{-2}\partial_{33})U = 0 \qquad &\text{in } P, \\ \frac{\partial U}{\partial n} = 0 \qquad &\text{on } \partial P_{\pm}. \end{aligned} \tag{5.1.6}$$

As in Section 2.1, we hope to pose a boundary corrector problem that is independent of ε. In the two-dimensional domain R, it was enough to define a *stretched* coordinate in the horizontal direction and pose the corrector problem in the semi-infinite strip Σ. We proceed here analogously, using a new system of (boundary-fitted) horizontal coordinates. In this new system, a point close to the boundary $\partial\Omega$ has as coordinates its distance to $\partial\Omega$ and the arclength along the boundary. We are able then to again define a horizontal stretched coordinate, in the normal direction, and pose, after some work, a sequence of problems in, once more, Σ. The geometry of Ω plays an important role, as we see below.

We digress then to introduce the boundary-fitted coordinates, following the notation of Chen [19]. Suppose that $\partial\Omega$ is arclength parametrized by $\underset{\sim}{z}(\theta) = (X(\theta), Y(\theta))$. Let $\underset{\sim}{s} = (X', Y')$, $\underset{\sim}{n} = (Y', -X')$ denote the tangent and the outward normal of $\partial\Omega$, and define

$$\Omega_b = \left\{ \underset{\sim}{z} - \rho \underset{\sim}{n} \,:\, \underset{\sim}{z} \in \partial\Omega,\, 0 < \rho < \rho_0 \right\},$$

where ρ_0 is a positive number smaller than the minimum radius of curvature of $\partial\Omega$. With L denoting the arclength of $\partial\Omega$, then

$$\underset{\sim}{x}(\rho, \theta) = \underset{\sim}{z}(\theta) - \rho \underset{\sim}{n}(\theta).$$

is a diffeomorphism between $(0, \rho_0) \times \mathbb{R}/L$ and Ω_b. Extending $\underset{\sim}{n}$ and $\underset{\sim}{s}$ to Ω_b by

$$\underset{\sim}{n}(\rho, \theta) = \underset{\sim}{n}(\theta), \qquad \underset{\sim}{s}(\rho, \theta) = \underset{\sim}{s}(\theta), \tag{5.1.7}$$

then, for $\alpha = 1$, 2:

$$\partial_\alpha \theta = \frac{s_\alpha}{\hat{J}(\theta)}, \qquad \partial_\alpha \rho = -n_\alpha,$$

where $\hat{J}(\rho, \theta) = 1 - \rho\kappa(\theta)$, and κ is the curvature of $\partial\Omega$. Finally, the change of coordinates yields

$$\partial_\alpha f = \partial_\theta f \partial_\alpha \theta + \partial_\rho f \partial_\alpha \rho, \qquad \text{for } \alpha = 1,\ 2.$$

The expression for the Laplacian in these new coordinates follows:

$$\begin{aligned}(\partial_{11} + \partial_{22})U &= \partial_{\rho\rho} U - \frac{\kappa}{\hat{J}} \partial_\rho U + \frac{1}{\hat{J}^2} \partial_{\theta\theta} U + \frac{\rho\kappa'}{\hat{J}^3} \partial_\theta U \\ &= \partial_{\rho\rho} U + \sum_{j=0}^{\infty} \rho^j \left(a_1^j \partial_\rho U + a_2^j \partial_{\theta\theta} U + a_3^j \partial_\theta U \right),\end{aligned} \tag{5.1.8}$$

where we formally replace each coefficient with its respective Taylor expansion, see [4], and

$$a_1^j = -[\kappa(\theta)]^{j+1}, \quad a_2^j = (j+1)[\kappa(\theta)]^j, \quad a_3^j = \frac{j(j+1)}{2} [\kappa(\theta)]^{j-1} \kappa'(\theta).$$

Defining the new variable $\hat{\rho} = \varepsilon^{-1}\rho$ and using the same name for functions different only up to this change of coordinates, we have from (5.1.8) that

$$(\partial_{11} + \partial_{22})U = \varepsilon^{-2} \partial_{\hat{\rho}\hat{\rho}} U + \sum_{j=0}^{\infty} (\varepsilon\hat{\rho})^j \left(a_1^j \varepsilon^{-1} \partial_{\hat{\rho}} U + a_2^j \partial_{\theta\theta} U + a_3^j \partial_\theta U \right), \tag{5.1.9}$$

Aiming to solve (5.1.6), we formally use (5.1.5) and (5.1.9), collect together terms with same order of ε and for $k \geq 2$, pose the following sequence of problems parametrized by θ:

$$\begin{aligned}(\partial_{\hat{\rho}\hat{\rho}} + \partial_{33}) U^k &= F_k \qquad \text{in } \Sigma, \\ \frac{\partial U^k}{\partial n} &= 0 \qquad \text{on } \partial\Sigma_\pm, \\ U^k(0, \theta, x_3) &= u^k(0, \theta, x_3) \qquad \text{for } x_3 \in (-1, 1),\end{aligned} \tag{5.1.10}$$

where

$$F_k = \sum_{j=0}^{k-2} \hat{\rho}^j \left(a_1^j \partial_{\hat{\rho}} U^{k-j-1} + a_2^j \partial_{\theta\theta} U^{k-j-2} + a_3^j \partial_\theta U^{k-j-2} \right),$$

with the convention that $u^k = 0$ for k odd and $U^0 = U^1 = 0$. From Theorem 6.2.6, we see that U^k decays exponentially towards the constant $c^k(\theta) = \frac{1}{2}\int_\Sigma \hat{\rho} F_k \, d\hat{\rho}\, dx_3$. The next lemma shows that in fact $c_k = 0$ and therefore U^k decays to zero.

Lemma 5.1.1. *Let U^k be defined by* (5.1.10) *for any positive positive k. Then $c^k := \frac{1}{2}\int_\Sigma \hat{\rho} F_k \, d\hat{\rho}\, dx_3 = 0$*

Proof. From the definition of F_k, it is enough to prove that for any positive integer l and for $j = 1, \ldots, k$,

$$\int_\Sigma \hat{\rho}^l \partial_{\hat{\rho}} U^j \, d\hat{\rho}\, dx_3 = \int_\Sigma \hat{\rho}^l U^j \, d\hat{\rho}\, dx_3 = 0. \tag{5.1.11}$$

The proof is by induction in k. For $k = 1$, (5.1.11) obviously holds since $U^1 = 0$. Now assume that (5.1.11) holds for $j = 1, \ldots, k-1$. Then $c_k = 0$ and U^k decays exponentially fast to zero. Using the formula

$$\int_\Sigma u \, \Delta \, v \, d\hat{\rho}\, dx_3 = \int_\Sigma v \, \Delta \, u \, d\hat{\rho}\, dx_3 + \int_{\partial\Sigma} u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \, d\hat{\rho}\, dx_3,$$

with $u = U^k$ and $v = \hat{\rho}^{l+2}/((l+2)(l+1))$, we find that

$$\int_\Sigma \hat{\rho}^l U^k \, d\hat{\rho}\, dx_3 = \int_\Sigma \frac{\hat{\rho}^{l+2}}{(l+2)(l+1)} F_k \, d\hat{\rho}\, dx_3 = 0$$

follows from the definition of F_k and the inductive hypothesis. Similarly, using integration by parts, we have that $\int_\Sigma \hat{\rho}^l \partial_{\hat{\rho}} U^k \, d\hat{\rho}\, dx_3 = 0$. Hence (5.1.11) holds and the lemma follows. $\square$

Combining (5.1.3) and the boundary layer expansion we write

$$u^\varepsilon(\underline{x}^\varepsilon) \sim \zeta^0(\utilde{x}^\varepsilon) + \sum_{k=1}^{\infty} \varepsilon^{2k} u^{2k}(\utilde{x}^\varepsilon, \varepsilon^{-1} x_3^\varepsilon) - \chi(\rho) \sum_{k=2}^{\infty} \varepsilon^k U^k(\varepsilon^{-1}\rho, \theta, \varepsilon^{-1} x_3^\varepsilon), \tag{5.1.12}$$

where $\chi(\rho)$ is a smooth cutoff function identically one if $0 \leq \rho \leq \rho_0/3$ and identically zero if $\rho \geq 2\rho_0/3$.

The results below are similar to the ones stated in Chapter 2. We omit many of the details. Using the convenient hypothesis that f and g are "smooth", we will bound their (arbitrarily high) Sobolev norms by a general constant $C(f,g)$. This allows a simplification of the estimates and no major loss of information occurs. The same ideas of Chapter 2 apply here and the results obtained there can be easily adapted to this three-dimensional problem. We present the convergence estimates of the truncated asymptotic expansion in the $H^1(P^\varepsilon)$ norm without a proof. Let

$$\tilde{e}_N = u^\varepsilon - \zeta^0(\underset{\sim}{x}^\varepsilon) - \sum_{k=1}^{N} \varepsilon^{2k} u^{2k}(\underset{\sim}{x}^\varepsilon, \varepsilon^{-1} x_3^\varepsilon) + \chi(\rho) \sum_{k=2}^{2N} \varepsilon^k U^k(\varepsilon^{-1}\rho, \theta, \varepsilon^{-1} x_3^\varepsilon)$$

Then we have the result below.

Theorem 5.1.2. *For any nonnegative integer N, there exists a constant $C(f,g)$ such that the difference between the truncated asymptotic expansion and the original solution measured in the original domain is bounded as follows:*

$$\|\tilde{e}_0\|_{H^1(P^\varepsilon)} \leq C(f,g)\varepsilon^{3/2}, \quad \|\tilde{e}_N\|_{H^1(P^\varepsilon)} \leq C(f,g)\varepsilon^{2N+1}.$$

As in (2.2.3), if f and g are not both identically zero,

$$\|u^\varepsilon\|_{H^1(P^\varepsilon)} \geq C(f,g)\frac{\varepsilon^{3/2}}{\nu(\varepsilon)}, \quad \text{where } \nu(\varepsilon) = \begin{cases} 1 & \text{if } \zeta^0 = 0, \\ \varepsilon & \text{otherwise,} \end{cases}$$

and then

$$\frac{\|\tilde{e}_0\|_{H^1(P^\varepsilon)}}{\|u^\varepsilon\|_{H^1(P^\varepsilon)}} = O(\nu(\varepsilon)), \quad \frac{\|\tilde{e}_N\|_{H^1(P^\varepsilon)}}{\|u^\varepsilon\|_{H^1(P^\varepsilon)}} = O(\nu(\varepsilon)\varepsilon^{2N-1/2}).$$

We compile in the table below the estimates for the error between the original solution and the truncated asymptotic expansion, for $N \geq 1$. The notation is the same as in Table 2.1.

TABLE 5.1. Convergence rates of the truncated asymptotic expansion

	u^ε	BL	$e_N (N \geq 1)$	Relative Error
$\Vert \cdot \Vert_{L^2(P^\varepsilon)}$	$\nu^{-2}\varepsilon^{5/2}$	ε^3	$\varepsilon^{2N+2}(\varepsilon^{2N+5/2})$	$\nu^2\varepsilon^{2N-1/2}(\nu^2\varepsilon^{2N})$
$\Vert \partial_\rho \cdot \Vert_{L^2(P^\varepsilon)}$	$\nu^{-3/2}\varepsilon^2(\nu^{-2}\varepsilon^{5/2})$	ε^2	$\varepsilon^{2N+1}(\varepsilon^{2N+5/2})$	$\nu^{3/2}\varepsilon^{2N-1}(\nu^2\varepsilon^{2N})$
$\Vert \partial_\theta \cdot \Vert_{L^2(P^\varepsilon)}$	$\nu^{-2}\varepsilon^{5/2}$	ε^3	$\varepsilon^{2N+2}(\varepsilon^{2N+5/2})$	$\nu^2\varepsilon^{2N-1/2}(\nu^2\varepsilon^{2N})$
$\Vert \partial_{x_3^\varepsilon} \cdot \Vert_{L^2(P^\varepsilon)}$	$\varepsilon^{3/2}$	ε^2	$\varepsilon^{2N+1}(\varepsilon^{2N+3/2})$	$\varepsilon^{2N-1/2}(\varepsilon^{2N})$
$\Vert \cdot \Vert_{H^1(P^\varepsilon)}$	$\nu^{-1}\varepsilon^{3/2}$	ε^2	$\varepsilon^{2N+1}(\varepsilon^{2N+3/2})$	$\nu\varepsilon^{2N-1/2}(\nu\varepsilon^{2N})$

Section 5.2 – A variational approach for dimension reduction. We start to consider now some models for problem (5.1.1). Recall that $V(P^\varepsilon) = \{v \in H^1(P^\varepsilon) : v = 0 \text{ on } \partial P_L^\varepsilon\}$ and define $\underline{S}(P^\varepsilon) = \underline{L}^2(P^\varepsilon)$. Similarly to Section 3.1, the pair u^ε and $\underline{\sigma}^\varepsilon = \underline{\nabla}\, u^\varepsilon$ satisfies the following.

SP: $(u^\varepsilon, \underline{\sigma}^\varepsilon)$ is the unique critical point of

$$L(v, \underline{\tau}) = \frac{1}{2}\int_{P^\varepsilon} |\underline{\tau}|^2 \, d\underline{x}^\varepsilon + \int_{P^\varepsilon} f^\varepsilon v \, d\underline{x}^\varepsilon - \int_{P^\varepsilon} \underline{\tau} \cdot \underline{\nabla}\, v \, d\underline{x}^\varepsilon + \int_{\partial P_\pm^\varepsilon} g^\varepsilon v \, d\underset{\sim}{x}^\varepsilon$$

over $V(P^\varepsilon) \times \underline{S}(P^\varepsilon)$.

As before, SP stands for "saddle point" principle. To define the SP(p) models, we seek critical points of $L(\cdot, \cdot)$ in the spaces $V(P^\varepsilon, p) = \{v \in V(P^\varepsilon) : \deg_3 v \leq p\}$ and $\underline{S}(P^\varepsilon, p) = \{\underline{\tau} \in \underline{S}(P^\varepsilon) : \deg_3 \underset{\sim}{\tau} \leq p, \deg_3 \tau_3 \leq p-1\}$, where we define $\deg_3$ analogously to $\deg_2$. These will be minimum energy models, as $\underline{\nabla}\, V(P^\varepsilon, p) \subset \underline{S}(P^\varepsilon, p)$.

We now write explicitly the equations of the SP(p) model. Recall (3.1.2) and (3.1.3) and define

$$\begin{gathered} f^k(\underset{\sim}{x}^\varepsilon) = \varepsilon^{-1} \int_{-\varepsilon}^{\varepsilon} f^\varepsilon(\underset{\sim}{x}^\varepsilon, x_3^\varepsilon) Q_k(x_3^\varepsilon) \, dx_3^\varepsilon, \\ g^0(\underset{\sim}{x}^\varepsilon) = \frac{1}{2}\big[g^\varepsilon(\underset{\sim}{x}^\varepsilon, \varepsilon) + g^\varepsilon(\underset{\sim}{x}^\varepsilon, -\varepsilon)\big], \quad g^1(\underset{\sim}{x}^\varepsilon) = \frac{1}{2}\big[g^\varepsilon(\underset{\sim}{x}^\varepsilon, \varepsilon) - g^\varepsilon(\underset{\sim}{x}^\varepsilon, -\varepsilon)\big]. \end{gathered} \tag{5.2.1}$$

Similarly to Section 3.1, if we write the SP(p) solutions as

$$u^\varepsilon(p)(\underline{x}^\varepsilon) = \sum_{j=0}^{p} \omega_j(\underset{\sim}{x}^\varepsilon) Q_j(x_3^\varepsilon), \quad \underline{\sigma}^\varepsilon(p)(\underline{x}^\varepsilon) = \begin{pmatrix} \sum_{j=0}^{p} \underset{\sim}{\sigma}^j(\underset{\sim}{x}^\varepsilon) Q_j(x_3^\varepsilon) \\ \sum_{j=0}^{p-1} \sigma_3^j(\underset{\sim}{x}^\varepsilon) Q_j(x_3^\varepsilon) \end{pmatrix},$$

then $\underset{\sim}{\sigma}^j = \underset{\sim}{\nabla}\omega_j^\varepsilon$ and $\sigma_3^j = \mathbf{M}_{jj}^{-1}\sum_{i=j+1}^p \mathbf{N}_{ij}\omega_i$. To determine $\omega_0, \ldots, \omega_p$, we define $\mathbf{g} : \Omega \to \mathbb{R}^{p+1}$, where $\mathbf{g}_j = g^0$ if j is even and $\mathbf{g}_j = g^1$ if j is odd, and

$$\boldsymbol{\omega}(\underset{\sim}{x}^\varepsilon) = (\omega_0, \ldots, \omega_p)^T(\underset{\sim}{x}^\varepsilon), \quad \mathbf{f}(\underset{\sim}{x}^\varepsilon) = (f^0, \ldots, f^p)^T(\underset{\sim}{x}^\varepsilon). \tag{5.2.2}$$

Then, from the definition of $u^\varepsilon(p)$,

$$\mathbf{M}(\partial_{11} + \partial_{22})\boldsymbol{\omega} - \mathbf{O}\boldsymbol{\omega} = -\varepsilon\mathbf{f} - 2\varepsilon^j\mathbf{g},$$

$$\boldsymbol{\omega} = 0 \qquad \text{on } \partial\Omega.$$

As before, $u^\varepsilon(p)$ has a nontrivial dependence on ε, which is apparent in its asymptotic expansion, which we develop now. We do not justify the computation of each term in the expansion, verify Section 3.2, we present only the final equations that each term satisfy. In fact, we have that

$$u^\varepsilon(p)(\underline{x}^\varepsilon) \sim \zeta^0(\underset{\sim}{x}^\varepsilon) + \sum_{k=1}^\infty \varepsilon^{2k}u^{2k}(p)(\underset{\sim}{x}^\varepsilon, \varepsilon^{-1}x_3^\varepsilon) - \chi(\rho)\sum_{k=2}^\infty \varepsilon^k U^k(p)(\varepsilon^{-1}\rho, \theta, \varepsilon^{-1}x_3^\varepsilon),$$

where ζ^0 solves (5.1.4) and the other terms are defined as follows. Set $u^2(p)(\underset{\sim}{x}, \cdot) = \hat{\pi}_p^1 \mathring{u}^2(\underset{\sim}{x}, \cdot)$ for almost every $\underset{\sim}{x} \in \Omega$, and hence,

$$\begin{aligned}\int_{-1}^1 \partial_3 u^2(p)(\underset{\sim}{x}, x_3)\partial_3 v(x_3)\,dx_3 &= \int_{-1}^1 [f(\underset{\sim}{x}, x_3) + (\partial_{11} + \partial_{22})\zeta^0(\underset{\sim}{x})]v(x_3)\,dx_3 \\ &\quad + g(\underset{\sim}{x}, -1)v(-1) + g(\underset{\sim}{x}, 1)v(1) \qquad \text{for all } v \in \hat{\mathbb{P}}_p(-1,1).\end{aligned}$$

For any integer $k \geq 2$, we define $u^{2k}(p)(\underset{\sim}{x}, \cdot) \in \hat{\mathbb{P}}_p(-1,1)$ by

$$\int_{-1}^1 \partial_3 u^{2k}(p)(\underset{\sim}{x}, x_3)\partial_3 v(x_3)\,dx_3 = \int_{-1}^1 (\partial_{11} + \partial_{22})u^{2k-2}(p)(\underset{\sim}{x}, x_3)v(x_3)\,dx_3 \qquad \text{for all } v \in \hat{\mathbb{P}}_p(-1,1),$$

and for almost every $\underset{\sim}{x} \in \Omega$.

Recall the definitions of $V(\Sigma,p)$ and $V_0(\Sigma,p)$ in (3.2.6). With $\theta \in \mathbb{R}/L$ as a parameter, the boundary correctors $U^k(p)(\cdot,\theta,\cdot) \in V(\Sigma,p)$ satisfy for $k \geq 2$:

$$\begin{aligned}
\int_\Sigma \underset{\sim}{\nabla}\, U^k(p) \cdot \underset{\sim}{\nabla}\, v\, d\underset{\sim}{x} &= \int_\Sigma F_k(p) v \qquad \text{for all } v \in V_0(\Sigma,p),\\
U^k(p)(0,\theta,x_3) &= u^k(p)(0,\theta,x_3) \qquad \text{for all } x_3 \in (-1,1),\\
F_k(p) &= \sum_{j=0}^{k-1} \hat\rho^j \left(a_1^j \partial_{\hat\rho} U^{k-j-1}(p) + a_2^j \partial_{\theta\theta} U^{k-j-2}(p) + a_3^j \partial_\theta U^{k-j-2}(p) \right),
\end{aligned}$$

where $u^k = 0$ for k odd and $U^0(p) = U^1(p) = 0$. Once again, $U^k(p)$ decays to zero since $\int_\Sigma \hat\rho F_k(p)\, d\hat\rho\, dx_3 = 0$. The proof of this fact is as in Lemma 5.1.1.

We present next estimates of $u^\varepsilon(p)$ minus its truncated asymptotic expansion.

Theorem 5.2.1. *For any positive integer N, there exists a constant $C(f,g)$ such that*

$$\left\| u^\varepsilon(p) - \zeta^0(\underset{\sim}{x}^\varepsilon) - \sum_{k=1}^{N} \varepsilon^{2k} u^{2k}(p) + \chi(\rho) \sum_{k=2}^{2N} \varepsilon^k U^k(p) \right\|_{H^1(P^\varepsilon)} \leq C(f,g)\varepsilon^{2N+1},$$

for all $p \in \mathbb{N}$.

Next, we compare some terms of the asymptotic expansion of both u^ε and $u^\varepsilon(p)$. We need the following notation, which is the three-dimensional equivalent of the notation used in Section 3.3.

Definition 5.2.2. For a nonnegative real number s, let

$$\begin{aligned}
\tilde a_s &= \|f\|_{L^2(\Omega;H^s(-1,1))} + \|g\|_{L^2(\partial P_\pm)}, \quad \tilde a_s^1 = \|f\|_{H^1(\Omega;H^s(-1,1))} + \|g\|_{H^1(\partial P_\pm)},\\
\tilde a_s^b &= \|f(0,\cdot,\cdot)\|_{L^2((0,L);H^s(-1,1))} + \|g(0,\cdot,-1)\|_{L^2(0,L)} + \|g(0,\cdot,1)\|_{L^2(0,L)},
\end{aligned} \tag{5.2.3}$$

where in the definition of $\tilde a_s^b$, we use boundary fitted coordinates. For each fixed $\theta \in [0,L)$, define $\mu(\theta,s,\delta)$ as in Definition 3.3.3, and set

$$\tilde\mu(s,\delta) = \inf_{\theta \in [0,L)} \mu(\theta,s,\delta).$$

The first three estimates in the lemma below hold since $u^2(p)(\underset{\sim}{x},\cdot)$ is the Galerkin projection of $u^2(\underset{\sim}{x},\cdot)$ into $\hat{\mathbb{P}}_p(-1,1)$ for $\underset{\sim}{x} \in \Omega$. The convergence of the boundary correctors is as in Lemma 3.3.5.

Lemma 5.2.3. *For any nonnegative real numbers s and s^* such that $s^*+1/2$ is not an even integer, and for any arbitrarily small $\delta > 0$, there exists a constant C such that*

$$\|u^2 - u^2(p)\|_{L^2(P^\varepsilon)} \le C\varepsilon^{1/2}p^{-2-s}\tilde{a}_s,$$

$$\|\underset{\sim}{\nabla}\, u^2 - \underset{\sim}{\nabla}\, u^2(p)\|_{L^2(P^\varepsilon)} \le C\varepsilon^{1/2}p^{-2-s}\tilde{a}_s^1,$$

$$\|\partial_{x_3^\varepsilon} u^2 - \partial_{x_3^\varepsilon} u^2(p)\|_{L^2(P^\varepsilon)} \le C\varepsilon^{-1/2}p^{-1-s}\tilde{a}_s,$$

$$\|\partial_{x_3^\varepsilon}\{\chi[U^2 - U^2(p)]\}\|_{L^2(P^\varepsilon)} + \|\partial_\rho\{\chi[U^2 - U^2(p)]\}\|_{L^2(P^\varepsilon)} \le Cp^{-\tilde{\mu}(s^*,\delta)}\tilde{a}^b_{s^*}.$$

Finally, we present the convergence results for the SP(p) model. Let $P_0^\varepsilon = \Omega_0 \times (-\varepsilon, \varepsilon)$, where Ω_0 is an open domain such that $\bar{\Omega}_0 \subset \Omega$.

Theorem 5.2.4. *For any nonnegative real numbers s and s^* such that $s^*+1/2$ is not an even integer, and for any arbitrarily small $\delta > 0$, there exist constants C and $C(f,g)$ such that the error between u^ε and the approximation $u^\varepsilon(p)$ given by the SP(p) model is bounded as*

$$\|u^\varepsilon - u^\varepsilon(p)\|_{L^2(P^\varepsilon)} \le C\varepsilon^{5/2}p^{-2-s}\tilde{a}_s + C(f,g)\varepsilon^3,$$

$$\|\partial_\rho[u^\varepsilon - u^\varepsilon(p)]\|_{L^2(P^\varepsilon)} \le C\varepsilon^2 p^{-\tilde{\mu}(s^*,\delta)}\tilde{a}^b_{s^*} + C(f,g)\varepsilon^{5/2},$$

$$\|\partial_\theta[u^\varepsilon - u^\varepsilon(p)]\|_{L^2(P^\varepsilon)} \le C\varepsilon^{5/2}p^{-2-s}\tilde{a}^1_s + C(f,g)\varepsilon^3,$$

$$\|\underset{\sim}{\nabla}\, u^\varepsilon - \underset{\sim}{\nabla}\, u^\varepsilon(p)\|_{L^2(P_0^\varepsilon)} \le C\varepsilon^{5/2}p^{-2-s}\tilde{a}^1_s + C(f,g)\varepsilon^{9/2},$$

$$\|\partial_{x_3^\varepsilon} u^\varepsilon - \partial_{x_3^\varepsilon} u^\varepsilon(p)\|_{L^2(P^\varepsilon)} \le C\varepsilon^{3/2}p^{-1-s}\tilde{a}_s + C(f,g)\varepsilon^2,$$

$$\|u^\varepsilon - u^\varepsilon(p)\|_{H^1(P^\varepsilon)} \le C\varepsilon^{3/2}p^{-1-s}\tilde{a}_s + C(f,g)\varepsilon^2.$$

We summarize the convergence results in the table below, c.f. table 2.1. We present only the leading terms of the errors and in parenthesis we show interior estimates if those are better than the global ones.

TABLE 5.2. Convergence estimates for the SP(p) models

	$u^\varepsilon - u^\varepsilon(p)$	Relative Error
$\\|\cdot\\|_{L^2(P^\varepsilon)}$	$\varepsilon^{5/2}p^{-2-s}\tilde{a}_s$	$\nu^2 p^{-2-s}\tilde{a}_s$
$\\|\partial_\rho\cdot\\|_{L^2(P^\varepsilon)}$	$\varepsilon^2 p^{-\tilde{\mu}}\tilde{a}_s^b\ (\varepsilon^{5/2}p^{-2-s}\tilde{a}_s^1)$	$\nu^{3/2}p^{-\tilde{\mu}}\tilde{a}_s^b\ (\nu^2 p^{-2-s}\tilde{a}_s^1)$
$\\|\partial_\theta\cdot\\|_{L^2(P^\varepsilon)}$	$\varepsilon^{5/2}p^{-2-s}\tilde{a}_s^1$	$\nu^2 p^{-2-s}\tilde{a}_s^1$
$\\|\partial_{x_3^\varepsilon}\cdot\\|_{L^2(P^\varepsilon)}$	$\varepsilon^{3/2}p^{-1-s}\tilde{a}_s$	$p^{-1-s}\tilde{a}_s$
$\\|\cdot\\|_{H^1(P^\varepsilon)}$	$\varepsilon^{3/2}p^{-1-s}\tilde{a}_s$	$\nu p^{-1-s}\tilde{a}_s$

Section 5.3 – An alternative variational approach. We now present the SP$'(p)$ models for (5.1.1) and the results related to it. Let $V'(P^\varepsilon) = L^2(P^\varepsilon)$ and $\underline{S}'_g(P^\varepsilon) = \{\underline{\sigma} \in \underline{H}(\text{div}, P^\varepsilon) : \underline{\sigma}\cdot\underline{n} = g^\varepsilon \text{ on } \partial P^\varepsilon_\pm\}$. Then we have the following principle.

SP$'$: $(u^\varepsilon, \underline{\sigma}^\varepsilon)$ is the unique critical point of

$$L'(v,\underline{\tau}) = \frac{1}{2}\int_{P^\varepsilon} |\underline{\tau}|^2\, d\underline{x}^\varepsilon + \int_{P^\varepsilon} f^\varepsilon v\, d\underline{x}^\varepsilon + \int_{P^\varepsilon} \text{div}\,\underline{\tau} v\, d\underline{x}^\varepsilon$$

in $V'(P^\varepsilon)\times \underline{S}'_g(P^\varepsilon)$. Looking for critical points in the spaces $V'(P^\varepsilon,p) = \{v \in V'(P^\varepsilon) : \deg_3 v \le p\}$ and $\underline{S}'_g(P^\varepsilon,p) = \{\underline{\tau} \in \underline{S}'_g(P^\varepsilon) : \deg_3 \underset{\sim}{\tau} \le p,\ \deg_3 \tau_3 \le p-1\}$ we derive the SP$'_1(p)$ models. Another option is to choose $\underline{S}'_g(P^\varepsilon,p) = \{\underline{\tau} \in \underline{S}'_g(P^\varepsilon) : \deg_3 \underset{\sim}{\tau} \le p,\ \deg_3 \tau_3 \le p+1\}$ instead, yielding the SP$'_2(p)$ models. As in Chapter 4, for both SP$'_1(p)$ and SP$'_2(p)$ models, $\text{div}\,\underline{S}'_g(P^\varepsilon,p) = V(P^\varepsilon,p)$ and $\sigma^\varepsilon(p)$ is the minimizer of the complementary energy

$$J_c(\underline{\tau}) = \frac{1}{2}\int_{P^\varepsilon} |\underline{\tau}|^2\, d\underset{\sim}{x}$$

over all $\underline{\tau} \in \underline{S}'_g(P^\varepsilon,p)$ such that $\text{div}\,\underline{\tau} = -\pi_{V'}f^\varepsilon$, where here, $\pi_{V'}f^\varepsilon$ is the orthogonal L^2 projection on f^ε into $V'(P^\varepsilon,p)$.

We present next some results regarding the SP$'_2(p)$ models, omiting the motivations and the proofs. These are parallel to the arguments of Chapter 4.

Recalling (3.1.2), (3.1.3), (4.1.5), (4.1.6), (4.1.7), and if we write the $\mathrm{SP}_2'(p)$ solution for a positive integer p as

$$u^\varepsilon(\underline{x}^\varepsilon) = \sum_{j=0}^{p} \omega_j(\underset{\sim}{x}^\varepsilon) Q_j(x_3^\varepsilon),$$

$$\underline{\sigma}^\varepsilon(\underline{x}^\varepsilon) = \begin{pmatrix} \sum_{j=0}^{p} \underset{\sim}{\sigma}^j(\underset{\sim}{x}^\varepsilon) Q_j(x_3^\varepsilon) \\ \sum_{j=2}^{p+1} \sigma_3^j(\underset{\sim}{x}^\varepsilon) \tilde{Q}_j(x_3^\varepsilon) \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon^{-1} x_3^\varepsilon g^0 + g^1 \end{pmatrix},$$

then $\sigma_3^i = -[\tilde{\mathbf{M}}^{-1}(\tilde{\mathbf{N}}\boldsymbol{\omega} + g^1\mathbf{q}^e + g^0\mathbf{q}^o)]_i$ for $i = 2, \cdots, p+1$ and $\underset{\sim}{\sigma}^i = \underset{\sim}{\nabla}\, \omega_i$ for $i = 0, \ldots, p$. Finally,

$$\mathbf{M}(\partial_{11} + \partial_{22})\boldsymbol{\omega} - \tilde{\mathbf{N}}^T \tilde{\mathbf{M}}^{-1} \tilde{\mathbf{N}} \boldsymbol{\omega} = -\varepsilon \mathbf{f} + \tilde{\mathbf{N}}^T \tilde{\mathbf{M}}^{-1}(g^1 \mathbf{q}^e + g^0 \mathbf{q}^o) - 2g^0 \mathbf{e}_1,$$

$$\boldsymbol{\omega} = 0 \qquad \text{on } \partial\Omega.$$

For the $\mathrm{SP}_2'(p)$ methods, the asymptotic expansions for $u^\varepsilon(p)$ and $\underset{\sim}{\sigma}^\varepsilon(p)$ are

$$u^\varepsilon(p)(\underline{x}^\varepsilon) \sim \zeta^0(\underset{\sim}{x}^\varepsilon) + \sum_{k=1}^{\infty} \varepsilon^{2k} u^{2k}(p)(\underset{\sim}{x}^\varepsilon, \varepsilon^{-1} x_3^\varepsilon) + \text{boundary correctors},$$

$$\underline{\sigma}(p)(\underline{x}^\varepsilon) \sim \begin{pmatrix} \underset{\sim}{\nabla}\, \zeta^0 \\ 0 \end{pmatrix} (\underset{\sim}{x}^\varepsilon) + \sum_{k=1}^{\infty} \varepsilon^{2k} \begin{pmatrix} \underset{\sim}{\sigma}^{2k}(p) \\ \varepsilon^{-1} \sigma_3^{2k}(p) \end{pmatrix} (\underset{\sim}{x}^\varepsilon, \varepsilon^{-1} x_3^\varepsilon)$$
$$+ \text{ boundary correctors.}$$

Equations (5.1.4) define ζ^0. The other terms are determined as below. With $\underset{\sim}{x} \in \Omega$ as a parameter, $u^2(p)(\underline{x}) \in \hat{\mathbb{P}}_p(-1,1)$ and $\sigma_3^2(p)(\underset{\sim}{x}, \cdot) \in \mathbb{P}_{p+1}(-1,1)$ with $\sigma_3^2(p)(\underset{\sim}{x}, -1) = -g(\underset{\sim}{x}, -1)$ and $\sigma_3^2(p)(\underset{\sim}{x}, 1) = g(\underset{\sim}{x}, 1)$ satisfy

$$\int_{-1}^{1} \sigma_3^2(p)(\underset{\sim}{x}, x_3)\tau_3(x_3)\, dx_3 + \int_{-1}^{1} u^2(p)(\underset{\sim}{x}, x_3)\partial_3 \tau_3(x_3)\, dx_3 = 0$$
$$\text{for all } \tau_3 \in \mathring{\mathbb{P}}_{p+1}(-1,1),$$

$$\int_{-1}^{1} \partial_3 \sigma_3^2(p)(\underset{\sim}{x}, x_3) v(x_3)\, dx_3 = -\int_{-1}^{1} [f(\underset{\sim}{x}, x_3) + (\partial_{11} + \partial_{22})\zeta^0(\underset{\sim}{x})] v(x_3)\, dx_3$$
$$\text{for all } v \in \hat{\mathbb{P}}_p(-1,1).$$

Note that $u^2(p)$, $\sigma_3^2(p)$ are mixed method approximations of u^2, $\partial_3 u^2$ (with $\underset{\sim}{x} \in \Omega$ as a parameter).

For all integers $k \geq 2$, let $\sigma_3^{2k}(p)(\underset{\sim}{x}, \cdot) \in \mathring{\mathbb{P}}_{p+1}(-1,1)$ and $u^{2k}(p)(\underset{\sim}{x}, \cdot) \in \hat{\mathbb{P}}_p(-1,1)$ be such that

$$\int_{-1}^{1} \sigma_3^{2k}(p)(\underset{\sim}{x}, x_3)\tau_3(x_3)\, dx_3 + \int_{-1}^{1} u^{2k}(p)(\underset{\sim}{x}, x_3)\partial_3\tau_3(x_3)\, dx_3 = 0$$

$$\text{for all } \tau_3 \in \mathring{\mathbb{P}}_{p+1}(-1,1),$$

$$\int_{-1}^{1} \partial_3\sigma_3^{2k}(p)(\underset{\sim}{x}, x_3)v(x_3)\, dx_3 = -\int_{-1}^{1} (\partial_{11} + \partial_{22})u^{2k-2}(p)(\underset{\sim}{x}, x_3)v(x_3)\, dx_3$$

$$\text{for all } v \in \hat{\mathbb{P}}_{p+1}(-1,1).$$

Also, $\underset{\sim}{\sigma}^{2k} = \underset{\sim}{\nabla}\, u^{2k}(p)$. We present some details regarding the boundary corrector problem. We expect a pair of correctors $U(p)$, $\underline{\Xi}(p)$ with trace $U_0(p)$ on ∂P_L^ε to satisfy

$$\begin{aligned} &\int_{P^\varepsilon} \underline{\Xi}(p) \cdot \underline{\tau} + U(p)\,\mathrm{div}\,\underline{\tau}\, d\underline{x} = \int_{\partial P_L^\varepsilon} U_0(p)\underline{\tau} \cdot \underline{n}\, d\underset{\sim}{x}\, dx_3 \qquad \text{for all } \underline{\tau} \in \underline{S}_0'(P^\varepsilon, p), \\ &\int_{P^\varepsilon} \mathrm{div}\,\underline{\Xi}(p) v\, d\underline{x} = 0 \qquad \text{for all } v \in V'(P^\varepsilon, p). \end{aligned} \tag{5.3.1}$$

We use (5.1.7) to define

$$\Xi_n(p)(\underline{x}^\varepsilon) = \underset{\sim}{\Xi}(p)(\underline{x}^\varepsilon) \cdot \underset{\sim}{n}(\underset{\sim}{x}^\varepsilon), \qquad \Xi_s(p)(\underline{x}^\varepsilon) = \underset{\sim}{\Xi}(p)(\underline{x}^\varepsilon) \cdot \underset{\sim}{s}(\underset{\sim}{x}^\varepsilon),$$

$$\tau_n(p)(\underline{x}^\varepsilon) = \underset{\sim}{\tau}(p)(\underline{x}^\varepsilon) \cdot \underset{\sim}{n}(\underset{\sim}{x}^\varepsilon), \qquad \tau_s(p)(\underline{x}^\varepsilon) = \underset{\sim}{\tau}(p)(\underline{x}^\varepsilon) \cdot \underset{\sim}{s}(\underset{\sim}{x}^\varepsilon),$$

in Ω_b. Then, a long but straightforward computation shows that

$$\mathrm{div}\,\underset{\sim}{\Xi}(p) = \partial_\rho \Xi_n(p) + \frac{1}{\hat{J}}\partial_\theta \Xi_s(p) - \frac{\kappa}{\hat{J}}\Xi_n(p).$$

Hoping that the correctors will decay very quickly, we, in a first step, pose (5.3.1) in Ω_b using the boundary fitted coordinates $(\rho, \theta, x_3^\varepsilon)$. Next, we use the "stretched" (in the normal and vertical directions) variables $(\hat{\rho}, \theta, x_3)$ in order to pose a ε-independent sequence of corrector problems, and define

$$\hat{\Xi}_n(p)(\hat{\rho}, \theta, x_3) = \varepsilon\Xi_n(p)(\rho, \theta, x_3^\varepsilon), \qquad \hat{\Xi}_s(p)(\hat{\rho}, \theta, x_3) = \Xi_s(p)(\rho, \theta, x_3^\varepsilon),$$

$$\hat{\Xi}_3(p)(\hat{\rho}, \theta, x_3) = \varepsilon\Xi_3(p)(\rho, \theta, x_3^\varepsilon).$$

Similar definitions hold for $\hat{\tau}_n(p)$, $\hat{\tau}_s(p)$ and $\hat{\tau}_3(p)$. The motivation for multiplying $\Xi_n(p)$ and $\Xi_3(p)$ by ε is that we expect them to "behave" as ε^{-1}, after all they approximate $\partial_\rho U$ and $\partial_3 U$ in P^ε. All the above described transformations lead to

$$\int_{\hat{Q}} \big[\varepsilon^{-2}\hat{\Xi}_n(p)\hat{\tau}_n + \hat{\Xi}_s(p)\hat{\tau}_s + \varepsilon^{-2}\hat{\Xi}_3(p)\hat{\tau}_3 + U(p)\big(\varepsilon^{-2}\partial_{\hat{\rho}}\hat{\tau}_n + \frac{1}{\hat{J}}\partial_\theta\hat{\tau}_s + \varepsilon^{-2}\partial_3\hat{\tau}_3\big)\big]\hat{J}$$

$$- \varepsilon^{-1}\kappa U(p)\hat{\tau}_n \, d\hat{Q} = \int_0^{2\pi}\int_{-1}^{1} U_0(p)(0,\theta,x_3)\hat{\tau}_n(0,\theta,x_3)\, dx_3\, d\theta,$$

$$\int_{\hat{Q}} \big[\varepsilon^{-2}\partial_{\hat{\rho}}\hat{\Xi}_n(p) + \frac{1}{\hat{J}}\partial_\theta\hat{\Xi}_s + \varepsilon^{-2}\partial_3\hat{\Xi}_3 - \varepsilon^{-1}\frac{\kappa}{\hat{J}}\hat{\Xi}_n(p)\big] v\hat{J}\, d\hat{Q} = 0,$$

where $\hat{Q} = \mathbb{R}^+ \times (0,2\pi) \times (-1,1)$ is a semi-infinite quadrilateral domain with the union of its top and bottom boundaries given by $\partial\hat{Q}_\pm = \mathbb{R}^+ \times (0,2\pi) \times \{-1,1\}$, and

$$\underline{\tau} \in \big\{\underline{\tau} \in \underline{H}(\mathrm{div},\hat{Q}) \,:\, \tau_3 = 0 \text{ on } \partial\hat{Q}_\pm,\ \deg_3 \underset{\sim}{\tau} \le p,\ \deg_3 \tau_3 \le p+1\big\},$$

$$v \in \big\{L^2(\hat{Q}) \,:\, \deg_3 v \le p\big\}.$$

Replacing $\hat{\tau}_n$ by $\hat{\tau}_n/\hat{J}$, $\hat{\tau}_3$ by $\hat{\tau}_3/\hat{J}$, and v by $v/\hat{J}$, formally substituting the Taylor series of the coefficients and

$$U(p)(\underline{x}) \sim \varepsilon^2 U^2(p)(\underline{x}) + \varepsilon^3 U^3(p)(\underline{x}) + \varepsilon^4 U^4(p)(\underline{x}) + \dots,$$

$$\hat{\underline{\Xi}}(p)(\underline{x}) \sim \varepsilon^2 \hat{\underline{\Xi}}^2(p)(\underline{x}) + \varepsilon^3 \hat{\underline{\Xi}}^3(p)(\underline{x}) + \varepsilon^4 \hat{\underline{\Xi}}^4(p)(\underline{x}) + \dots$$

$$U_0(p) \sim \varepsilon^2 u^2(p) + \varepsilon^3 u^3(p) + \varepsilon^4 u^4(p) + \dots,$$

where $u^k(p) = 0$ for k odd, we arrive at the following sequence of problems, parametrized by $\theta \in \mathbb{R}/L$ and defined in the semi-infinite strip Σ:

$$\int_\Sigma \hat{\Xi}_n^k(p)\hat{\tau}_n + \hat{\Xi}_3^k(p)\hat{\tau}_3 + U^k(p)\left(\partial_{\hat{\rho}}\hat{\tau}_n + \partial_3\hat{\tau}_3\right)\, d\hat{\rho}\, dx_3$$

$$= -\int_{\gamma_0} u^k(p)(0,\theta,x_3)\hat{\tau}_n(0,x_3)\, dx_3 \qquad \text{for all } \underset{\sim}{\tau} \in \underset{\sim}{S}'_0(\Sigma,p),$$

$$\int_\Sigma \left[\partial_{\hat{\rho}}\hat{\Xi}_n^k(p) + \partial_3\hat{\Xi}_3^k(p)\right] v\, d\hat{\rho}\, dx_3 = \int_\Sigma G_k(p) v\, d\hat{\rho}\, dx_3 \qquad \text{for all } v \in V'(\Sigma,p),$$

$$\hat{\Xi}_s^k(p) = \hat{\rho}\kappa(\theta)\hat{\Xi}_s^{k-1}(p) + \partial_\theta U^k(p),$$

$$G_k(p) = \sum_{j=0}^{k-2} \hat{\rho}^j \left(a_1^j \hat{\Xi}_n^{k-j-1}(p) + a_2^j \partial_{\theta\theta} U^{k-j-2}(p) + a_3^j \partial_\theta U^{k-j-2}(p)\right).$$

Finally,

$$u^\varepsilon(p)(\underline{x}^\varepsilon) \sim \zeta^0(\underset{\sim}{x}^\varepsilon) + \sum_{k\geq 1} \varepsilon^{2k} u^{2k}(p)(\underset{\sim}{x}^\varepsilon, \varepsilon^{-1}x_3^\varepsilon) - \chi(\rho)\sum_{k\geq 2}\varepsilon^k U^k(p)(\varepsilon^{-1}\rho, \theta, \varepsilon^{-1}x_3^\varepsilon),$$

$$\underline{\sigma}^\varepsilon(p)(\underline{x}^\varepsilon) \sim \begin{pmatrix} \underset{\sim}{\nabla}\zeta^0 \\ 0 \end{pmatrix}(\underset{\sim}{x}^\varepsilon) + \sum_{k\geq 1}\varepsilon^{2k}\begin{pmatrix} \underset{\sim}{\sigma}^{2k}(p) \\ \varepsilon^{-1}\sigma_3^{2k}(p) \end{pmatrix}(\underset{\sim}{x}^\varepsilon, \varepsilon^{-1}x_3^\varepsilon)$$
$$- \chi(\rho)\sum_{k\geq 2}\varepsilon^k \begin{pmatrix} \varepsilon^{-1}\hat{\Xi}_n^k(p)\underset{\sim}{n} + \hat{\Xi}_s^k(p)\underset{\sim}{s} \\ \varepsilon^{-1}\hat{\Xi}_3^k(p) \end{pmatrix}(\varepsilon^{-1}\rho, \theta, \varepsilon^{-1}x_3^\varepsilon).$$

We present next the various error estimates, vide Chapter 4.

Theorem 5.3.1. *For any nonnegative integer N, there exists a constant $C(f,g)$ such that*

$$\left\| u^\varepsilon(p) - \zeta^0 - \sum_{k=1}^{N}\varepsilon^{2k}u^{2k}(p) + \chi(\rho)\sum_{k=2}^{2N}\varepsilon^k U^k(p) \right\|_{L^2(P^\varepsilon)}$$
$$+ \left\| \underline{\sigma}^\varepsilon(p) - \begin{pmatrix} \underset{\sim}{\nabla}\zeta^0 \\ 0 \end{pmatrix} - \sum_{k=1}^{N}\varepsilon^{2k}\begin{pmatrix} \underset{\sim}{\sigma}^{2k}(p) \\ \varepsilon^{-1}\sigma_3^{2k}(p) \end{pmatrix} \right.$$
$$\left. + \chi(\rho)\sum_{k=2}^{2N}\varepsilon^k \begin{pmatrix} \varepsilon^{-1}\hat{\Xi}_n^k(p)\underset{\sim}{n} + \hat{\Xi}_s^k(p)\underset{\sim}{s} \\ \varepsilon^{-1}\hat{\Xi}_3^k(p) \end{pmatrix} \right\|_{L^2(P^\varepsilon)} \leq C(f,g)\varepsilon^{2N+1}$$

The next two lemmas estimate the difference between the first terms of the asymptotic expansion of u^ε and $u^\varepsilon(p)$. As in Section 5.2, the following definition is necessary.

Definition 5.3.2. For each fixed nonnegative real number s and $\theta \in [0, l)$, define $\bar{\mu}(\theta, s, \delta)$ as in Definition 4.3.3. Then set

$$\tilde{\bar{\mu}}(s, \delta) = \inf_{\theta \in [0,L)} \bar{\mu}(\theta, s, \delta).$$

Lemma 5.3.3. *For any nonnegative real numbers s and s^* such that $s^* + 1/2$ is not an even integer, and for any arbitrarily small $\delta > 0$, there exists a constant C such that*

$$\|u^2 - u^2(p)\|_{L^2(P^\varepsilon)} \leq C\varepsilon^{1/2} p^{-2-s} \tilde{a}_s,$$

$$\| \underset{\sim}{\nabla} u^2 - \underset{\sim}{\nabla} u^2(p)\|_{L^2(P^\varepsilon)} \leq C\varepsilon^{1/2} p^{-2-s} \tilde{a}_s^1,$$

$$\|\sigma_3^2 - \sigma_3^2(p)\|_{L^2(P^\varepsilon)} \leq C\varepsilon^{1/2} p^{-1-s} \tilde{a}_s,$$

$$\|\partial_{x_3^\varepsilon} \sigma_3^2 - \partial_{x_3^\varepsilon} \sigma_3^2(p)\|_{L^2(P^\varepsilon)} \leq C\varepsilon^{-1/2} p^{-s} \tilde{a}_s,$$

$$|\chi[\hat{\Xi}_n^2 - \hat{\Xi}_n^2(p)]|_{L^2(P^\varepsilon)} + |\chi[\hat{\Xi}_3^2 - \hat{\Xi}_3^2(p)]|_{L^2(P^\varepsilon)} \leq C\varepsilon\big(p^{-1-s^*} + p^{-\tilde{\bar{\mu}}(s^*,\delta)}\big)\tilde{a}_{s^*}^b,$$

where $\sigma_3^2(\underline{x}) = \partial_{x_3} u^2(\underline{x})$, $\hat{\Xi}_3^2(\underline{x}) = \partial_{x_3} U^2(\underline{x})$.

We end this chapter by presenting the convergence results for the SP$'(p)$ model.

Theorem 5.3.4. *For any nonnegative real numbers s and s^* such that $s^* + 1/2$ is not an even integer, and for any arbitrarily small $\delta > 0$, there exist constants C and $C(f, g)$ such that the following bounds hold:*

$$\|u^\varepsilon - u^\varepsilon(p)\|_{L^2(P^\varepsilon)} \leq C\varepsilon^{5/2} p^{-2-s} \tilde{a}_s + C(f, g)\varepsilon^3,$$

$$\|\underset{\sim}{\sigma}^\varepsilon \cdot \underset{\sim}{n} - \underset{\sim}{\sigma}^\varepsilon(p) \cdot \underset{\sim}{n}\|_{L^2(P^\varepsilon)} \leq C\varepsilon^2\big(p^{-1-s^*} + p^{-\tilde{\bar{\mu}}(s^*,\delta)}\big)\tilde{a}_{s^*}^b + C(f, g)\varepsilon^{5/2},$$

$$\|\underset{\sim}{\sigma}^\varepsilon \cdot \underset{\sim}{s} - \underset{\sim}{\sigma}^\varepsilon(p) \cdot \underset{\sim}{s}\|_{L^2(P^\varepsilon)} \leq C\varepsilon^{5/2} p^{-2-s} \tilde{a}_s^1 + C(f, g)\varepsilon^3,$$

$$\|\underset{\sim}{\sigma}^\varepsilon - \underset{\sim}{\sigma}^\varepsilon(p)\|_{L^2(P_0^\varepsilon)} \leq C\varepsilon^{5/2} p^{-2-s} \tilde{a}_s^1 + C(f, g)\varepsilon^{9/2},$$

$$\|\sigma_3^\varepsilon - \sigma_3^\varepsilon(p)\|_{L^2(P^\varepsilon)} \leq C\varepsilon^{3/2} p^{-1-s} \tilde{a}_s + C(f, g)\varepsilon^2,$$

where $\underline{\sigma}^\varepsilon(\underline{x}^\varepsilon) = \underline{\nabla}\, u^\varepsilon(\underline{x}^\varepsilon)$.

Chapter 6

The Poisson problem in a semi-infinite strip

We discuss in this chapter several issues related to the Poisson problem

$$\begin{aligned} \Delta U &= -f \qquad \text{in } \Sigma, \\ \frac{\partial U}{\partial n} &= 0 \qquad \text{on } \partial\Sigma_\pm, \\ U(0,\cdot) &= U_0(\cdot), \end{aligned} \tag{6.1}$$

where $\Sigma = \mathbb{R}^+ \times (-1,1)$ is a planar semi-infinite strip, with top and bottom $\partial\Sigma_\pm = \mathbb{R}^+ \times \{-1,1\}$. We prove that the problem given by (6.1), as well as a Galerkin, and a mixed approximation, are well posed, and that the corresponding solutions decay exponentially towards a constant. We also bound the difference between the exact and approximate solutions.

Section 6.1 – Well-posedness. We prove in this section that there exists a unique solution for (6.1) in a weighted Sobolev space that contains functions with a certain algebraic growth. Stability also holds, as we show below. We indicate an arbitrary point in Σ by $\underset{\sim}{\hat{\rho}} = (\hat{\rho}_1, \hat{\rho}_2)$. We assume that f belongs to an appropriate weighted Sobolev space, to be defined, which guarantees that f decays "rapidly enough" along Σ. It will be useful to consider the sets

$$\Sigma(t,s) = \big\{ \underset{\sim}{\hat{\rho}} \in \Sigma \,:\, t < \hat{\rho}_1 < s \big\}, \quad \text{and} \quad \gamma_t = \big\{ \underset{\sim}{\hat{\rho}} \in \Sigma \,:\, \hat{\rho}_1 = t \big\},$$

for $0 \le t \le s \le \infty$.

Our first goal in this section is to show that if f decays exponentially to zero as $\hat{\rho}_1 \to \infty$, then $\underset{\sim}{\nabla} U$ also decays exponentially to zero and U tends exponentially fast to the constant $c_\infty(U)/2$, which is easy to calculate formally:

$$c_\infty(U) = \int_\Sigma \hat{\rho}_1 f(\underset{\sim}{\hat{\rho}})\, d\underset{\sim}{\hat{\rho}} + \int_{\gamma_0} U \, d\hat{\rho}_2. \tag{6.1.1}$$

Next, for $f = 0$, and a Dirichlet condition at γ_0, we will study the properties of two distinct approximations for U, in spaces with polynomial dependence in the vertical direction. The first approximation is the standard Galerkin projection, and the second one is given by a mixed model, and in both cases we show stability and convergence results. The results presented here are used throughout this thesis to analyze the boundary layers which appear in solutions of problems in thin domains.

With $\mathcal{D}'(\Sigma)$ as the space of distributions on Σ, and $w(\underset{\sim}{\hat{\rho}}) = (1+\hat{\rho}_1)^{-1}$, we endow the weighted Sobolev spaces

$$L^2_w(\Sigma) = \left\{v \in \mathcal{D}'(\Sigma) \,:\, wv \in L^2(\Sigma)\right\} \qquad \text{and}$$

$$V(\Sigma) = \left\{v \in \mathcal{D}'(\Sigma) \,:\, v \in L^2_w(\Sigma),\, \underset{\sim}{\nabla}\, v \in \underset{\sim}{L}^2(\Sigma)\right\}$$

with the norms $\|v\|_{L^2_w(\Sigma)} = \|wv\|_{L^2(\Sigma)}$ and $\|v\|_{V_w(\Sigma)} = (\|v\|^2_{L^2_w(\Sigma)} + \|\underset{\sim}{\nabla}\, v\|^2_{L^2(\Sigma)})^{1/2}$. Denote by $\mathcal{D}(\overline{\Sigma})$ the space of restrictions to Σ of the functions of $\mathcal{D}(\mathbb{R}^2)$. It follows from standard arguments [26] that $\mathcal{D}(\overline{\Sigma})$ is dense in $V(\Sigma)$. Also, it is possible to define a bounded trace operator from $V(\Sigma)$ onto $H^{1/2}(\gamma_0)$. Indeed, for $U \in V(\Sigma)$, let the restriction $r(U) = U|_{\Sigma(0,1)} \in H^1(\Sigma(0,1))$. Now, the trace operator $\tilde{\nu}\colon H^1(\Sigma(0,1)) \to H^{1/2}(\gamma_0)$ is bounded and surjective. We have finally $\tilde{\nu} \circ r\colon V(\Sigma) \to H^{1/2}(\gamma_0)$.

For future reference, we present some estimates that follow from Hardy's inequality, see [36] for a proof.

Lemma 6.1.1. *Let $d < 1$, and $U \in H^1_0(\mathbb{R}^+)$ be such that*

$$\int_{\mathbb{R}^+} (c+\hat{\rho}_1)^d [U'(\hat{\rho}_1)]^2 \, d\hat{\rho}_1 < \infty,$$

where c is a nonnegative constant. Then the following inequality holds:

$$\int_{\mathbb{R}^+} (c+\hat{\rho}_1)^{d-2} U^2(\hat{\rho}_1) \, d\hat{\rho}_1 \le \left(\frac{2}{d-1}\right)^2 \int_{\mathbb{R}^+} (c+\hat{\rho}_1)^d [U'(\hat{\rho}_1)]^2 \, d\hat{\rho}_1.$$

Applying the previous result with $d = 2(1-\alpha)$ and $c = 1$ the following holds.

Corollary 6.1.2. *Let $U \in H_0^1(\mathbb{R}^+)$ and $\alpha \geq 1$. Then the following inequality holds:*

$$\int_{\mathbb{R}^+} (1+\hat{\rho}_1)^{-2\alpha} U^2(\hat{\rho}_1)\, d\hat{\rho}_1 \leq \left(\frac{2}{2\alpha - 1}\right)^2 \int_{\mathbb{R}^+} [U'(\hat{\rho}_1)]^2\, d\hat{\rho}_1.$$

We proceed now to establish an equivalent norm on $V(\Sigma)$.

Lemma 6.1.3. *Assume that $v \in V(\Sigma)$. Then*

$$\frac{1}{3}\|v\|_{V_w(\Sigma)} \leq \left(\| \underset{\sim}{\nabla} v\|^2_{L^2(\Sigma)} + \|v\|^2_{L^2(\gamma_0)}\right)^{1/2} \leq C\|v\|_{V_w(\Sigma)}.$$

Proof. Since the trace operator is bounded, the second inequality follows easily from the definition of $\|v\|_{V_w(\Sigma)}$. We advance now to prove the first inequality, when $v \in \mathcal{D}(\overline{\Sigma})$. The general case follows from the density of $\mathcal{D}(\overline{\Sigma})$ in $V(\Sigma)$. Applying Corollary 6.1.2 with $\alpha = 1$, we have that

$$\int_{\Sigma} (1+\hat{\rho}_1)^{-2}[v(\hat{\underset{\sim}{\rho}}) - v(0,\hat{\rho}_2)]^2\, d\hat{\underset{\sim}{\rho}} \leq 4\int_{-1}^{1}\int_{\mathbb{R}^+} (\partial_1 v)^2\, d\hat{\rho}_1\, d\hat{\rho}_2. \tag{6.1.2}$$

On the other hand,

$$\int_{\Sigma} (1+\hat{\rho}_1)^{-2} v^2(0,\hat{\rho}_2)\, d\hat{\underset{\sim}{\rho}} = \int_{-1}^{1} v^2(0,\hat{\rho}_2) \int_{\mathbb{R}^+} (1+\hat{\rho}_1)^{-2}\, d\hat{\rho}_1\, d\hat{\rho}_2 = \|v\|^2_{L^2(\gamma_0)}, \tag{6.1.3}$$

and then using (6.1.2) and (6.1.3) we conclude that

$$\begin{aligned}\|v\|^2_{V_w(\Sigma)} &= \|(1+\hat{\rho}_1)^{-1}v\|^2_{L^2(\Sigma)} + \| \underset{\sim}{\nabla} v\|^2_{L^2(\Sigma)} \\ &\leq 2\|(1+\hat{\rho}_1)^{-1}[v - v(0,\cdot)]\|^2_{L^2(\Sigma)} + 2\|(1+\hat{\rho}_1)^{-1}v(0,\cdot)\|^2_{L^2(\Sigma)} + \| \underset{\sim}{\nabla} v\|^2_{L^2(\Sigma)} \\ &\leq 9\| \underset{\sim}{\nabla} v\|^2_{L^2(\Sigma)} + 2\|v\|^2_{L^2(\gamma_0)}\end{aligned}$$

and the result holds. $\square$

It follows from the above lemma that $V(\Sigma)$ is a Hilbert space equipped with the inner product

$$a(u,v) = \int_{\Sigma} \underset{\sim}{\nabla} u \cdot \underset{\sim}{\nabla} v\, d\hat{\underset{\sim}{\rho}} + \int_{\gamma_0} uv\, d\hat{\rho}_2,$$

and we denote the induced norm by $\|\cdot\|_{V(\Sigma)}$.

Remark. The function $(1+\hat{\rho}_1)^\alpha \in V(\Sigma)$, for $\alpha < 1/2$, but functions that are linear in $\hat{\rho}_1$ do not belong to $V(\Sigma)$.

We define $V_0(\Sigma)$ as the subspace of $V(\Sigma)$ of functions with vanishing traces on γ_0, i.e., $V_0(\Sigma) = \{v \in V(\Sigma) \,:\, v = 0 \text{ on } \gamma_0\}$. Let $V^*(\Sigma)$ denote the dual space of $V_0(\Sigma)$, and $\|\cdot\|_{V^*(\Sigma)}$ denote the dual norm. The following existence and uniqueness result holds.

Theorem 6.1.4. *Suppose that $F \in V^*(\Sigma)$, and that $U_0 \in H^{1/2}(\gamma_0)$. Then there exists unique $U \in V(\Sigma)$ satisfying*

$$\begin{aligned} U &= U_0 \qquad \text{on } \gamma_0, \\ a(U,v) &= F(v) \qquad \text{for all } v \in V_0(\Sigma). \end{aligned} \tag{6.1.4}$$

Moreover,

$$\|U\|_{V(\Sigma)} \le \|F\|_{V^*(\Sigma)} + C\|U_0\|_{H^{1/2}(\gamma_0)}.$$

Proof. In the case $U_0 = 0$, the result follows from the Riesz representation theorem. For non-homogeneous boundary conditions, define $\tilde{U} \in H^1(\Sigma)$ such that $\tilde{U} = U_0$ on γ_0, $\tilde{U}(\underset{\sim}{\hat{\rho}}) = 0$ for $\hat{\rho}_1 > 1$ and $\|\underset{\sim}{\nabla}\tilde{U}\|_{L^2(\Sigma)} \le C\|U_0\|_{H^{1/2}(\gamma_0)}$. Let $w \in V_0(\Sigma)$ be the solution of

$$a(w,v) = F(v) - a(\tilde{U}, v) \qquad \text{for all } v \in V_0(\Sigma).$$

So $\|w\|_{V(\Sigma)} \le \|F\|_{V^*(\Sigma)} + \|\underset{\sim}{\nabla}\tilde{U}\|_{L^2(\Sigma)}$, and $U = \tilde{U} + w$ satisfies (6.1.4). Using the triangle inequality we have that

$$\|U\|_{V(\Sigma)} \le \|\tilde{U}\|_{V(\Sigma)} + \|w\|_{V(\Sigma)} \le \|F\|_{V^*(\Sigma)} + C\|U_0\|_{H^{1/2}(\gamma_0)},$$

and the proof is complete. $\square$

We show next that functions with a certain algebraic decay belong to the dual space of $V_0(\Sigma)$. In fact, we have the following result.

Lemma 6.1.5. *Assume that $\alpha \geq 1$. Then for all $f \in L^2_{w^{-\alpha}}(\Sigma)$ and for all $v \in V_0(\Sigma)$ we have that*

$$\int_\Sigma f\, v\, d\hat{\underset{\sim}{\rho}} \leq \frac{2}{2\alpha - 1} \|f\|_{L^2_{w^{-\alpha}}(\Sigma)} \|v\|_{V(\Sigma)}.$$

Proof. Using the Cauchy–Schwartz inequality to estimate

$$\int_\Sigma f\, v\, d\hat{\underset{\sim}{\rho}} \leq \|(1+\hat{\rho}_1)^\alpha f\|_{L^2(\Sigma)} \|(1+\hat{\rho}_1)^{-\alpha} v\|_{L^2(\Sigma)},$$

and then applying Corollary 6.1.2 we conclude the proof. $\square$

If we define $F(v) = \int_\Sigma f\, v\, d\hat{\underset{\sim}{\rho}}$ where $f \in L^2_{w^{-\alpha}}(\Sigma)$, then

$$\|F\|_{V^*(\Sigma)} \leq \frac{2}{2\alpha - 1} \|f\|_{L^2_{w^{-\alpha}}(\Sigma)},$$

and we can apply Theorem 6.1.4 to conclude the following result.

Theorem 6.1.6. *Assume that $f \in L^2_{w^{-\alpha}}(\Sigma)$, where $\alpha \geq 1$, and let $U_0 \in H^{1/2}(\gamma_0)$. Then there exists unique $U \in V(\Sigma)$ such that*

$$\begin{aligned} U &= U_0 && \text{on } \gamma_0, \\ a(U, v) &= \int_\Sigma f\, v\, d\hat{\underset{\sim}{\rho}} && \text{for all } v \in V_0(\Sigma). \end{aligned} \tag{6.1.5}$$

Moreover,

$$\|U\|_{V(\Sigma)} \leq \frac{2}{2\alpha - 1} \|f\|_{L^2_{w^{-\alpha}}(\Sigma)} + C\|U_0\|_{H^{1/2}(\gamma_0)}.$$

Note that (6.1.5) is a weak formulation of (6.1). We are interested in the case when f decays exponentially, and thus we shall assume that f is a measurable function such that

$$|f(\hat{\underset{\sim}{\rho}})| \leq M \exp(-c_0 \hat{\rho}_1) \qquad \text{for almost every } \hat{\underset{\sim}{\rho}} \in \Sigma, \tag{6.1.6}$$

where M is a nonnegative and c_0 is a positive real number. Certainly, functions with property (6.1.6) belong to $L^2_{w^{-\alpha}}(\Sigma)$, for arbitrary α, and then Theorem 6.1.6 applies.

Section 6.2 – Exponential decay of solutions. Now that the question of existence and uniqueness of solution for (6.1) is answered, we proceed to analyze the behavior of the solution and its approximations. We show next that they converge to a constant function exponentially quickly, in a sense that will be made clear. In this section, U is not necessarily a solution of (6.1), but it might be the projection of the solution into some particular space. Similarly, $\underset{\sim}{\sigma}$ might be either the gradient of the solution or its approximation. As we see below, sufficient conditions for such exponential decay are that $U \in L^2_w(\Sigma)$, $\underset{\sim}{\sigma} \in \underset{\sim}{L}^2(\Sigma)$ and that U, $\underset{\sim}{\sigma}$ satisfy for $0 \le t \le s < \infty$:

$$\int_{\Sigma(t,s)} |\underset{\sim}{\sigma}|^2 \, d\underset{\sim}{\hat{\rho}} = \int_{\Sigma(t,s)} fU \, d\underset{\sim}{\hat{\rho}} - \int_{\gamma_t} \sigma_1 U \, d\hat{\rho}_2 + \int_{\gamma_s} \sigma_1 U \, d\hat{\rho}_2, \tag{C1}$$

$$\int_{\Sigma(t,s)} f \, d\underset{\sim}{\hat{\rho}} = \int_{\gamma_t} \sigma_1 \, d\hat{\rho}_2 - \int_{\gamma_s} \sigma_1 \, d\hat{\rho}_2, \tag{C2}$$

$$-\int_{\Sigma(0,t)} \hat{\rho}_1 f \, d\underset{\sim}{\hat{\rho}} = \int_{\gamma_0} U \, d\hat{\rho}_2 + \int_{\gamma_t} (t\sigma_1 - U) \, d\hat{\rho}_2, \tag{C3}$$

$$\int_{\gamma_t} U^2 \, d\hat{\rho}_2 \le C_W \int_{\gamma_t} \sigma_2^2 \, d\hat{\rho}_2 + \frac{1}{2} \left(\int_{\gamma_t} U \, d\hat{\rho}_2 \right)^2 \qquad \text{for some } C_W \ge 0. \tag{C4}$$

The constant C_W in the condition (C4) mimics the Wirtinger inequality (the one-dimensional version of the Poincaré's inequality, see [43]).

Lemma 6.2.1 (Wirtinger inequality). *If $u \in H^1(a,b) \cap \hat{L}^2(a,b)$, then*

$$\int_a^b u(x)^2 \, dx \le \left(\frac{b-a}{\pi} \right)^2 \int_a^b u'(x)^2 \, dx.$$

Lemma 6.2.2. *If (6.1.6) is valid, $U \in V(\Sigma)$ satisfies (6.1), and $\underset{\sim}{\sigma} = \underset{\sim}{\nabla} U$, then identities (C1)–(C4) hold with $C_W = 4/\pi^2$.*

Proof. Conditions (C1)–(C3) follow from Green's identity. Next, if we define

$$I(t) = \int_{\gamma_t} U \, d\hat{\rho}_2, \tag{6.2.1}$$

then the Wirtinger's inequality yields condition (C4) as

$$\int_{\gamma_t} U^2\, d\hat{\rho}_2 = \int_{\gamma_t} \left(U(t,\hat{\rho}_2) - \frac{I(t)}{2} \right)^2 d\hat{\rho}_2 + \frac{I(t)^2}{2} \leq C_W \int_{\gamma_t} |\partial_2 U|^2\, d\hat{\rho}_2 + \frac{I(t)^2}{2}.$$

□

In the following two lemmas we show that results similar to (C1)–(C3) are valid in unbounded sections of Σ as well.

Lemma 6.2.3. *Assume that* (6.1.6) *holds,* $U \in L^2_w(\Sigma)$*,* $\underset{\sim}{\sigma} \in \underset{\sim}{L}^2(\Sigma)$ *and that conditions (C2), (C3) are satisfied. Then for* $t \geq 0$

$$\int_{\Sigma(t,\infty)} f\, d\underset{\sim}{\hat{\rho}} = \int_{\gamma_t} \sigma_1\, d\hat{\rho}_2, \tag{6.2.2}$$

$$\int_{\gamma_t} U\, d\hat{\rho}_2 = c_\infty(U) + \int_{\Sigma(t,\infty)} (t - \hat{\rho}_1) f(\underset{\sim}{\hat{\rho}})\, d\underset{\sim}{\hat{\rho}}. \tag{6.2.3}$$

Proof. If we define $P(s) = \int_{\gamma_s} \sigma_1\, d\hat{\rho}_2$, then in view of (C2) we have that

$$P(s) = \int_{\gamma_t} \sigma_1\, d\hat{\rho}_2 - \int_{\Sigma(t,s)} f\, d\underset{\sim}{\hat{\rho}}.$$

Thus P is a continuous function and $\lim_{s\to\infty} P(s) = d$, where d is the constant

$$d = \int_{\gamma_t} \sigma_1\, d\hat{\rho}_2 - \int_{\Sigma(t,\infty)} f\, d\underset{\sim}{\hat{\rho}}.$$

As $|\underset{\sim}{\sigma}| \in L^2(\Sigma)$ and (6.1.6) holds, then $P(s) \in L^2(\mathbb{R}^+)$. Hence $d = 0$ and identity (6.2.2) follows. Now, to conclude (6.2.3), we use (C3) and then equations (6.1.1), (6.2.2). □

Lemma 6.2.4. *Assume that* (6.1.6) *holds,* U*,* $|\underset{\sim}{\sigma}| \in L^2(\Sigma)$ *and that condition (C1) is satisfied. Then for* $t \geq 0$

$$\int_{\Sigma(t,\infty)} |\underset{\sim}{\sigma}|^2\, d\underset{\sim}{\hat{\rho}} = \int_{\Sigma(t,\infty)} f\, U\, d\underset{\sim}{\hat{\rho}} - \int_{\gamma_t} \sigma_1 U\, d\hat{\rho}_2. \tag{6.2.4}$$

Proof. Define the function $H(s) = \int_{\gamma_s} \sigma_1 U \, d\hat{\rho}_2$. Then, by (C1):

$$H(s) = \int_{\Sigma(t,s)} |\underset{\sim}{\sigma}|^2 \, d\underset{\sim}{\hat{\rho}} - \int_{\Sigma(t,s)} f \, U \, d\underset{\sim}{\hat{\rho}} + \int_{\gamma_t} \sigma_1 U \, d\hat{\rho}_2.$$

Hence H is continuous and

$$\lim_{s\to\infty} H(s) = \int_{\Sigma(t,\infty)} |\underset{\sim}{\sigma}|^2 \, d\underset{\sim}{\hat{\rho}} - \int_{\Sigma(t,\infty)} f \, U \, d\underset{\sim}{\hat{\rho}} + \int_{\gamma_t} \sigma_1 U \, d\hat{\rho}_2.$$

It follows from its definition that $2|H(s)| \leq \|\sigma_1\|^2_{L^2(\gamma_s)} + \|U\|^2_{L^2(\gamma_s)}$, and then $H \in L^1(\mathbb{R}^+)$. Therefore $\lim_{s\to\infty} H(s) = 0$ and (6.2.4) holds. $\square$

The estimate

$$\left| \int_{\Sigma(t,\infty)} (t - \hat{\rho}_1) f(\underset{\sim}{\hat{\rho}}) \, d\underset{\sim}{\hat{\rho}} \right| \leq \frac{2M}{c_0^2} \exp(-c_0 t) \tag{6.2.5}$$

follows solely from (6.1.6) and will be useful further ahead.

Theorem 6.2.5. *Assume that* (6.1.6) *holds, that* $U \in L^2_w(\Sigma)$, $\underset{\sim}{\sigma} \in \underset{\sim}{L}^2(\Sigma)$ *satisfy (C1)–(C4), and also that* $c_\infty(U) = 0$. *Then there exists a constant* C *depending only on* c_0 *and* C_W *such that*

$$\begin{aligned} \int_{\Sigma(t,\infty)} U^2 \, d\underset{\sim}{\hat{\rho}} &\leq \left(CM^2 + C_W \int_\Sigma |\underset{\sim}{\sigma}|^2 \, d\underset{\sim}{\hat{\rho}} \right) \exp(-t/c_1), \\ \int_{\Sigma(t,\infty)} |\underset{\sim}{\sigma}|^2 \, d\underset{\sim}{\hat{\rho}} &\leq \left(CM^2 + \int_\Sigma |\underset{\sim}{\sigma}|^2 \, d\underset{\sim}{\hat{\rho}} \right) \exp(-t/c_1), \end{aligned} \tag{6.2.6}$$

where $c_1 = \max\{1 + C_W, 1/c_0\}$.

Proof. First of all, recall the definition (6.2.1), and then by (6.2.3), (6.2.5):

$$|I(t)| \leq \frac{2M}{c_0^2} \exp(-c_0 t). \tag{6.2.7}$$

If we define the function $E(t) = \int_{\Sigma(t,\infty)} |\underset{\sim}{\sigma}|^2\, d\underset{\sim}{\hat\rho}$, then $E'(t) = -\int_{\gamma_t} |\underset{\sim}{\sigma}|^2\, d\hat\rho_2$ and (C4) yields

$$\int_{\gamma_t} U^2\, d\hat\rho_2 \leq -C_W E'(t) + \frac{I(t)^2}{2}, \tag{6.2.8}$$

$$\begin{aligned}\int_{\Sigma(t,\infty)} U^2\, d\underset{\sim}{\hat\rho} &\leq \int_t^\infty \left(C_W \int_{\gamma_{\hat\rho_1}} |\underset{\sim}{\sigma}|^2\, d\hat\rho_2 + \frac{1}{2} I(\hat\rho_1)^2 \right) d\hat\rho_1 \\ &= C_W E(t) + \frac{1}{2}\int_t^\infty I(\hat\rho_1)^2\, d\hat\rho_1.\end{aligned} \tag{6.2.9}$$

We can now bound the growth of the energy. From (6.2.7) and (6.2.9), we conclude that $U \in L^2(\Sigma)$. Using Lemma 6.2.4, we gather that:

$$\begin{aligned} E(t) &= -\int_{\gamma_t} \sigma_1 U\, d\hat\rho_2 + \int_{\Sigma(t,\infty)} f\, U\, d\underset{\sim}{\hat\rho} \\ &\leq \frac{1}{2}\int_{\gamma_t} \sigma_1^2\, d\hat\rho_2 + \frac{1}{2}\int_{\gamma_t} U^2\, d\hat\rho_2 + \frac{\alpha}{2}\int_{\Sigma(t,\infty)} U^2\, d\underset{\sim}{\hat\rho} + \frac{1}{2\alpha}\int_{\Sigma(t,\infty)} f^2\, d\underset{\sim}{\hat\rho} \\ &\leq -\frac{(1+C_W)}{2} E'(t) + \frac{I(t)^2}{4} + \frac{\alpha C_W}{2} E(t) + \frac{\alpha}{4}\int_t^\infty I(\hat\rho_1)^2\, d\hat\rho_1 + \frac{1}{2\alpha}\int_{\Sigma(t,\infty)} f^2\, d\underset{\sim}{\hat\rho}, \end{aligned} \tag{6.2.10}$$

where (6.2.8) and (6.2.9) were used in the last inequality. Choose $\alpha = (C_W)^{-1}$ in (6.2.10) to conclude that (recall that $E'(t)$ is nonpositive):

$$c_1 E'(t) \leq (1 + C_W) E'(t) \leq -E(t) + G(t), \tag{6.2.11}$$

where

$$c_1 = \max\{1 + C_W, \frac{1}{c_0}\}, \quad G(t) = \frac{I(t)^2}{2} + \frac{1}{2C_W}\int_t^\infty I(\hat\rho_1)^2\, d\hat\rho_1 + C_W \int_{\Sigma(t,\infty)} f^2\, d\underset{\sim}{\hat\rho}. \tag{6.2.12}$$

We estimate now the energy norm. Define $W(t)$ such that

$$\begin{aligned} W'(t) &= -\frac{W(t)}{c_1} + \frac{G(t)}{c_1}, \\ W(0) &= E(0). \end{aligned}$$

Then

$$E(t) \leq W(t) = \frac{1}{c_1}\exp(-t/c_1)\int_0^t \exp(\hat{\rho}_1/c_1)G(\hat{\rho}_1)\,d\hat{\rho}_1 + E(0)\exp(-t/c_1). \qquad (6.2.13)$$

In this next step we prove that in fact the integral in (6.2.13) is bounded and then we infer the exponential decay of E. Indeed, using (6.2.12), (6.1.6), and (6.2.7) we have

$$G(t) \leq C(c_0)M^2\exp(-2c_0t),$$

where

$$C(c_0) = \frac{1}{c_0}\left(\frac{2}{c_0^3} + \frac{1}{C_W c_0^4} + C_W\right).$$

Thus

$$\int_0^t \exp(\hat{\rho}_1/c_1)G(t)\,d\hat{\rho}_1 \leq C(c_0)M^2\int_0^t \exp((1/c_1 - 2c_0)\hat{\rho}_1)\,d\hat{\rho}_1.$$

Using (6.2.12) we see that $2c_0 > 1/c_1$ and then the above integral is uniformly bounded. Therefore, in view of (6.2.13), we conclude that

$$E(t) \leq (CM^2 + E(0))\exp(-t/c_1). \qquad (6.2.14)$$

The combination of (6.2.7) and (6.2.9) yield

$$\int_{\Sigma(t,\infty)} U^2\,d\hat{\underset{\sim}{\rho}} \leq (C_W E(t) + CM^2\exp(-2c_0t)). \qquad (6.2.15)$$

Using again that $2c_0 > 1/c_1$, the result follows from (6.2.14), (6.2.15). $\square$

Using the previous theorem, we can decompose a general solution as a constant term plus a exponentially decaying function, as the result below shows.

Theorem 6.2.6. *Assume that* (6.1.6) *holds and that* $U \in V(\Sigma)$, $\underset{\sim}{\sigma} \in L^2(\Sigma)$ *satisfy (C1)–(C4). Defining* $c_\infty(U)$ *as in* (6.1.1), *we have the decomposition*

$$U = \frac{1}{2}c_\infty(U) + U^*, \qquad (6.2.16)$$

where U^*, $\underset{\sim}{\sigma}$ decay to zero exponentially as in Theorem 6.2.5, i.e., (6.2.6) *is satisfied with U replaced by U^*.*

Proof. Using (6.2.16), we see that in fact $c_\infty(U^*) = 0$. Next note that if U, $\underset{\sim}{\sigma}$ satisfy (C1)–(C4), then U^*, $\underset{\sim}{\sigma}$ also satisfy (C1)–(C4). Therefore we can use the Theorem 6.2.5 to infer the exponential decay of U^*, $\underset{\sim}{\sigma}$. $\square$

As previously mentioned, the theory just developed works not only for the solution of (6.1) but for some of its approximations as well. Indeed, rewrite $V(\Sigma, p) = \{v \in V(\Sigma) : \deg_2 v \leq p\}$ and $V_0(\Sigma, p) = \{v \in V_0(\Sigma) : \deg_2 v \leq p\}$ (these spaces are defined in a equivalent form in (3.2.6)). Now, let $U(p) \in V(\Sigma, p)$ satisfy

$$\begin{aligned} \int_\Sigma \underset{\sim}{\nabla} U(p) \cdot \underset{\sim}{\nabla} v \, d\underset{\sim}{\hat{\rho}} = 0 \qquad & \text{for all } v \in V_0(\Sigma, p), \\ U(p) = U_0(p) \qquad & \text{on } \gamma_0. \end{aligned} \tag{6.2.17}$$

We show below that the problem given by (6.2.17) is well posed and that its solution decays exponentially if $U_0(p) \in \hat{\mathbb{P}}_p(-1, 1)$.

Theorem 6.2.7. *Assume that $U_0(p) \in \mathbb{P}_p(-1, 1)$. Then there exists a unique solution $U(p) \in V(\Sigma, p)$ to* (6.2.17) *and a universal constant C such that*

$$\|U(p)\|_{V(\Sigma)} \leq C \|U_0(p)\|_{H^{1/2}(\gamma_0)}.$$

Also, if $U_0(p) \in \hat{\mathbb{P}}_p(-1, 1)$, then

$$\begin{aligned} \int_{\Sigma(t,\infty)} [U(p)]^2 \, d\underset{\sim}{\hat{\rho}} &\leq \frac{4}{\pi^2} |U(p)|^2_{H^1(\Sigma)} \exp(-t\pi^2/(\pi^2+4)), \\ \int_{\Sigma(t,\infty)} |\underset{\sim}{\nabla} U(p)|^2 \, d\underset{\sim}{\hat{\rho}} &\leq |U(p)|^2_{H^1(\Sigma)} \exp(-t\pi^2/(\pi^2+4)). \end{aligned}$$

Proof. The existence and uniqueness result for (6.2.17) is completely analogous to Theorem 6.1.6. To show that the solutions decay exponentially, we prove that (C1)–(C3) hold. To prove (C1), i.e., that

$$\int_{\Sigma(t,s)} |\underset{\sim}{\nabla} U(p)|^2 \, d\underset{\sim}{\hat{\rho}} = \int_{\Sigma(t,s)} fU(p) \, d\underset{\sim}{\hat{\rho}} - \int_{\gamma_t} \partial_1 U(p) U(p) \, d\hat{\rho}_2 + \int_{\gamma_s} \partial_1 U(p) U(p) \, d\hat{\rho}_2$$

for $0 \leq t < s < \infty$, we assume that $t \neq 0$ (the case $t = 0$ follows from a simple modification in the argument below). Let $\chi_\delta(\hat{\rho}_1)$ be a smooth cut-off function such that χ_δ vanishes in $[0, t-\delta] \cup [s+\delta, \infty)$, and equals to one in $[t, s]$. Then, from (6.2.17) and Green's identity,

$$\begin{aligned} 0 = \int_\Sigma \underset{\sim}{\nabla} U(p) \cdot \underset{\sim}{\nabla}(\chi_\delta U(p)) \, d\underset{\sim}{\hat{\rho}} &= \int_\Sigma \chi_\delta |\underset{\sim}{\nabla} U(p)|^2 \, d\underset{\sim}{\hat{\rho}} + \int_\Sigma \partial_1 \chi_\delta \partial_1 U(p) U(p) \, d\underset{\sim}{\hat{\rho}} \\ &= \int_\Sigma \chi_\delta |\underset{\sim}{\nabla} U(p)|^2 \, d\underset{\sim}{\hat{\rho}} - \int_\Sigma \chi_\delta \partial_1 [\partial_1 U(p) U(p)] \, d\underset{\sim}{\hat{\rho}}. \end{aligned}$$

So, taking the limit $\delta \to 0$,

$$\begin{aligned} \int_{\Sigma(t,s)} |\underset{\sim}{\nabla} U(p)|^2 \, d\underset{\sim}{\hat{\rho}} &= \int_{\Sigma(t,s)} \partial_1 [\partial_1 U(p) U(p)] \, d\underset{\sim}{\hat{\rho}}. \\ &= -\int_{\gamma_t} \partial_1 U(p) U(p) \, d\hat{\rho}_2 + \int_{\gamma_s} \partial_1 U(p) U(p) \, d\hat{\rho}_2. \end{aligned}$$

Conditions (C2) and (C3) follow from similar arguments and (C4) follows from the Wirtinger's inequality, as in the proof of Lemma 6.2.2. We can apply then Theorem 6.2.5. $\square$

Section 6.3 – A Galerkin approximation. We start this section by explicitly showing the influence of the corners of the semi-infinite strip on the solution of the Laplace's equation. This is important, as the solution might lose regularity in the vicinity of these corners, and in this case the convergence of polynomial approximation degrades. Our main reference for this topic is the notes by Kellogg [34], which gives a clear account of the theory of corner singularities applied to Poisson problems. Then we present a theory developed by Dorr [29], [30] regarding approximation of functions with corner singularities. Finally we estimate the error of a Galerkin approximation.

We are concerned with the problem

$$\begin{gathered} \Delta U = 0 \qquad \text{in } \Sigma, \\ \frac{\partial U}{\partial n} = 0 \qquad \text{on } \partial\Sigma_{\pm}, \quad U = U_0 \qquad \text{on } \gamma_0, \end{gathered} \tag{6.3.1}$$

where $U_0 \in H^{r_0}(\gamma_0) \cap \hat{L}^2(\gamma_0)$ for some $r_0 > 3/2$. Using separation of variables, we can derive the solution of (6.3.1),

$$U(\hat{\rho}) = \sum_{j=1}^{\infty} b_j \cos\left(\frac{j\pi}{2}(\hat{\rho}_2 + 1)\right) \exp(-j\pi\hat{\rho}_1/2),$$

where b_j are the Fourier coefficient of U_0, i.e.

$$U_0(\hat{\rho}_2) = \sum_{j=1}^{\infty} b_j \cos\left(\frac{j\pi}{2}(\hat{\rho}_2 + 1)\right). \tag{6.3.2}$$

It is easy to see that $\{\cos(j\pi(\hat{\rho}_2+1)/2)\}_{j=1}^{\infty}$ is a basis of $\hat{L}^2(-1,1)$ by making the change of coordinates $y = \pi(\hat{\rho}_2+1)/2$ and checking that $\cos(jy)$ form a basis of $\hat{L}^2(0,\pi)$. Consult [37] for the details regarding the above expansion. Using the explicit formula for U, it is not hard to show its smoothness.

Lemma 6.3.1. *Assume that $U_0 \in H^{1/2}(\gamma_0) \cap \hat{L}^2(\gamma_0)$. Then there exists a unique solution $U \in H^1(\Sigma)$ to (6.3.1). Also, for any nonnegative integer k there exists a constant C such that*

$$\|U\|_{H^k(\Sigma(1,\infty))} \leq C\|U_0\|_{L^2(\gamma_0)}.$$

Proof. Note that for any integer k and any real number s, both nonnegative, there exists a constant C such that

$$\begin{aligned} \|\partial_1^k U(\hat{\rho}_1, \cdot)\|^2_{H^s(-1,1)} &\leq C \sum_{j=1}^{\infty} (1+j^2)^s j^{2k} b_j^2 \exp(-j\pi\hat{\rho}_1) \\ &\leq C \sum_{j=1}^{\infty} (1+j^2)^{s+k} b_j^2 \exp(-j\pi\hat{\rho}_1), \end{aligned} \tag{6.3.3}$$

and the result comes from integrating (6.3.3) in $(1,\infty)$:

$$\|\partial_1^k U\|^2_{L^2((1,\infty);H^s(-1,1))} \leq C\sum_{j=1}^{\infty}\frac{(1+j^2)^{s+k}}{j}b_j^2\exp(-j\pi) \leq C\sum_{j=1}^{\infty} b_j^2$$
$$\leq C\|U_0\|^2_{L^2(\gamma_0)}.$$

□

Remark. It is clear that the choice of the domain $\Sigma(1,\infty)$ in the above lemma plays no particular role, and the solution is smooth in $\Sigma(t,\infty)$ for any fixed $t>0$.

To describe the singular behavior of the solution for (6.3.1), it is enough to consider the rectangle $Q=\Sigma(0,1)$. Let $P_1=(0,1)$ and $P_2=(0,-1)$. We introduce two polar coordinate systems (r_l,θ_l) relative to P_l, $l=1,2$. The convention is that r_l gives the distance to P_l and the angle $\theta_l\in[0,\pi/2]$ increases counterclockwise, so points lying on γ_0 have $\theta_1=0$ and $\theta_2=\pi/2$.

In the next theorem [34], we show a decomposition of the solution U in singular and smooth parts and it is of paramount importance in future estimates.

Theorem 6.3.2. *Let U be the solution of* (6.3.1) *with $r_0>3/2$ such that $r_0+1/2$ is not an even integer. Let U_Q be the restriction of U to Q. Then there exist constants C_j, C such that*

$$U_Q = U_S + W, \quad U_S=\sum_{l=1}^{2}\sum_{j=1}^{J(r_0+1/2)} C_j\partial_2^{(2j-1)}U_0\big((-1)^{l+1}\big)v_l^j \tag{6.3.4,}$$

where J is defined in (3.3.3) *and*

$$v_1^j=[\theta_1\cos\big((2j-1)\theta_1\big)+\log r_1\sin((2j-1)\theta_1\big)]r_1^{(2j-1)},$$
$$v_2^j=[(\frac{\pi}{2}-\theta_2)\sin\big((2j-1)\theta_2\big)+\log r_2\cos((2j-1)\theta_2\big)]r_2^{(2j-1)},$$

and $\|W\|_{H^{r_0+1/2}(Q)} \leq C\|U_0\|_{H^{r_0}(\gamma_0)}$.

Remark. Note that $v_1^j = v_2^j = 0$ when $\hat{\rho}_1 = 0$, and therefore U_S is identically zero at $\hat{\rho}_1 = 0$.

Remark. Since $U_0 \in H^{r_0}(-1,1)$ and $2J(r_0+1/2)-1 < r_0-1/2$, then U_S is well defined. Also, note that the singular behavior of U_Q depends not only on the regularity of the Dirichlet data U_0 but also on how many derivatives of U_0 vanish at the endpoints -1, 1. For instance, $U_0(y) = y$ is analytic but gives raise to a singular solution.

Our next goal now is to estimate the error of the approximation by polynomials for functions that present corner singularities of the type $r^j \log r\xi(\theta)$ or $r^j\xi(\theta)$, where ξ is a smooth function. We follow here the ideas presented by Dorr [29], [30] and also Remark 6.3 of [14].

Consider operators of Sturm–Liouville type

$$A_j\varphi = -\partial_{\hat{\rho}_j}[(1-\hat{\rho}_j^2)\partial_{\hat{\rho}_j}\varphi], \qquad j = 1,2.$$

Since A_j is positive and self-adjoint in $L^2(Q)$, we can define the differential operator

$$A^s = A_1^s + A_2^s,$$

where s is a nonnegative real number. It is not hard to see that A^s itself is a positive self-adjoint operator in $L^2(Q)$, and that its eigenfunctions are $L_m(\hat{\rho}_1)L_n(\hat{\rho}_2)$, products of Legendre polynomials. We define $D(A^s)$ as the subspace of functions φ in $L^2(Q)$ such that $A^s\varphi$ is in $L^2(Q)$, and denote by $\|\cdot\|_{D(A^s)}$ the associated graph norm.

Remark 6.3.3. For any nonnegative integer k, we have that $D(A^{k/2}) = Z^k(Q)$, where

$$Z^k(Q) = \big\{v \in \mathcal{D}'(Q) \,:\, \|v\|_{Z^k(Q)} < \infty\big\},$$

and

$$\|v\|_{Z^k(Q)} = \left(\int_Q |v|^2 \, d\hat{\rho} + \sum_{j=1}^{2} \int_Q (1-\hat{\rho}_j^2)^k |\partial_{\hat{\rho}_j}^k v|^2 \, d\hat{\rho} \right)^{1/2}.$$

In fact, it is possible to define $Z^s(Q)$ for any nonnegative real number s. For instance, if $2s$ is not odd and $s = (1-\theta)k_1 + \theta k_2$ for some nonnegative integers k_1 and k_2 and $0 < \theta < 1$, then using interpolation by the K-method [12], [13], we can define

$$Z^s(Q) = \big(Z^{k_1}(Q), Z^{k_2}(Q)\big)_{\theta,2},$$

and $D(A^{s/2}) = Z^s(Q)$ still holds. See [29] for details.

Using the fact that $D(A^s)$ is continuously embedded in $H^s(Q)$, the lemma below follows from standard arguments [29].

Lemma 6.3.4. *Let $s > s' > 0$ be two real numbers. Then there exists a constant C such that*

$$\|v - \pi_p^{(\hat{\rho}_2)} v\|_{H^{s'}(Q)} \leq C\|v - \pi_p^{(\hat{\rho}_2)} v\|_{D(A^{s'})} \leq C p^{2s'-2s} \|v\|_{D(A^s)}.$$

Remark 6.3.5. Note that if $v = 0$ on γ_0, then $\pi_p^{(\hat{\rho}_2)} v = 0$ on γ_0 as well.

We need one extra technical result that is worked out in the proof of Lemma 3.2 of [30].

Lemma 6.3.6. *Let (r, θ) denote the polar coordinates with respect to one of the corners of Q. If $\xi_1, \xi_2 \in C^\infty([0, \pi/2])$, and γ is a nonnegative real number, then*

$$v(r, \theta) = r^\gamma (\log r \xi_1(\theta) + \xi_2(\theta)) \in Z^{2\gamma+2-\delta}(Q),$$

for arbitrarily small $\delta > 0$.

Remark 6.3.7. From Remark 6.3.3 and the continuous embedding of $\mathcal{D}(A^s)$ into $H^s(Q)$, we have that if v is defined as in Lemma 6.3.6, then $v \in \mathcal{D}(A^{\gamma+1-\delta}) \cap H^{\gamma+1-\delta}(Q)$, for arbitrarily small $\delta > 0$.

We now combine Remark 6.3.3, Lemmas 6.3.4, and 6.3.6 to conclude the following result.

Lemma 6.3.8. *Assume that* $U_0 \in H^{r_0}(-1,1)$ *with* $r_0 > 3/2$ *and that* U_S *in* (6.3.4) *is not the zero function. Then for arbitrarily small* $\delta > 0$, *there exists* C *such that*

$$\|U_S - \pi_p^{(\hat{\rho}_2)} U_S\|_{H^1(Q)} \leq C p^{-4m+2+\delta} \|U_0\|_{H^{r_0}(\gamma_0)},$$

where $m \in \{1, \ldots, J(r_0 + \frac{1}{2})\}$ *is the minimum integer such that* $|\partial_2^{(2m-1)} U_0(-1)| + |\partial_2^{(2m-1)} U_0(1)| \neq 0$.

Remark. By using the work of Babuška and Suri [10], it is possible to achieve a slight improvement on the estimate of Lemma 6.3.8, replacing $p^{-4m+2+\delta}$ by $p^{-4m+2}(\log p)$, at the expense of some technicalities.

We can approximate the smoother part of U in a standard fashion, as the next result shows. It is a direct application of Lemma A.4, combined with the regularity estimates of Lemma 6.3.1 and Theorem 6.3.2.

Lemma 6.3.9. *Under the assumptions of Theorem 6.3.2,*

$$\|U - \hat{\pi}_p^{1(\hat{\rho}_2)} U\|_{H^1(\Sigma(1,\infty))} + \|W - \hat{\pi}_p^{1(\hat{\rho}_2)} W\|_{H^1(Q)} \leq C p^{1/2 - r_0} \|U_0\|_{H^{r_0}(\gamma_0)}.$$

We define the rate of convergence of our approximation result below.

Definition 6.3.10. For $U_0 \in H^{r_0}(-1,1)$, and J as in (3.3.3), if there exists an minimum integer $m \in \{1, \ldots, J(r_0 + \frac{1}{2})\}$ such that $|\partial_2^{2m-1} U_0(-1)| + |\partial_2^{2m-1} U_0(1)| \neq 0$, let $\mu(r_0, \delta) = \min\{4m - 2 - \delta, r_0 - 1/2\}$, otherwise let $\mu(r_0, \delta) = r_0 - 1/2$.

We conclude now the following approximation result for U.

Theorem 6.3.11. *Assume that U solves (6.3.1) and that $U_0 \in H^{r_0}(\gamma_0)$, for $r_0 > 3/2$ such that $r_0 + 1/2$ is not an even integer. Then there exists $U_p \in V(\Sigma, p)$ such that $U_p = \hat{\pi}_p^1 U_0$ on γ_0 and*

$$\|U - U_p\|_{H^1(\Sigma)} \leq Cp^{-\mu(r_0,\delta)} \|U_0\|_{H^{r_0}(\gamma_0)},$$

where μ is as in Definition 6.3.10. The constant C depends on r_0 and $\delta > 0$ only.

Proof. Using Lemmas 6.3.8 and 6.3.9, we have that

$$\begin{aligned}\|U - U_p\|_{H^1(\Sigma)} \leq \|U_S - \hat{\pi}_p^{1(\hat{\rho}_2)} U_S\|_{H^1(Q)} + \|W - \hat{\pi}_p^{1(\hat{\rho}_2)} W\|_{H^1(Q)} + \|U - \hat{\pi}_p^{1(\hat{\rho}_2)} U\|_{H^1(\Sigma(1,\infty))} \\ \leq Cp^{-\mu(r_0,\delta)} \|U_0\|_{H^{r_0}(\gamma_0)}.\end{aligned}$$

Also, $U_p = \hat{\pi}_p^1 U_0$ on γ_0 since $\hat{\pi}_p^{1(\hat{\rho}_2)} U_S = U_S = 0$ on γ_0. $\square$

Now we use the above result to estimate the errors due to the Galerkin projections.

Theorem 6.3.12. *For any real number $r_0 > 3/2$ such that $r_0 + 1/2$ is not an even integer, and any arbitrarily small $\delta > 0$, there exists a constant C such that if $U \in V(\Sigma)$ solves (6.3.1) with $U_0 \in H^{r_0}(\gamma_0) \cap \hat{L}^2(\gamma_0)$, and if $U(p) \in V(\Sigma, p)$ solves*

$$\begin{aligned}\int_\Sigma \underset{\sim}{\nabla} U(p) \cdot \underset{\sim}{\nabla} v \, d\underset{\sim}{\hat{\rho}} = 0 \qquad &\text{for all } v \in V_0(\Sigma, p), \\ U(p) = \hat{\pi}_p^1 U_0 \qquad &\text{on } \gamma_0,\end{aligned}$$

then

$$|U - U(p)|_{H^1(\Sigma)} \leq Cp^{-\mu(r_0,\delta)} \|U_0\|_{H^{r_0}(\gamma_0)},$$

where μ is as in Definition 6.3.10.

Proof. Introduce $\hat{U} \in V(\Sigma)$ such that

$$\begin{aligned}\int_\Sigma \underset{\sim}{\nabla} \hat{U} \cdot \underset{\sim}{\nabla} v \, d\underset{\sim}{\hat{\rho}} = 0 \qquad &\text{for all } v \in V_0(\Sigma), \\ \hat{U} = \hat{\pi}_p^1 U_0 \qquad &\text{on } \gamma_0.\end{aligned}$$

Then,

$$|U-\hat{U}|_{H^1(\Sigma)} \le C\|U_0 - \hat{\pi}_p^1 U_0\|_{H^{1/2}(\gamma_0)} \le Cp^{1/2-r_0}\|U_0\|_{H^{r_0}(\gamma_0)}. \tag{6.3.5}$$

Now we advance to estimate $|\hat{U} - U(p)|_{H^1(\Sigma)}$. Let $U_p \in V(\Sigma, p)$ be as in Theorem 6.3.11. Then, as $U(p) - U_p \in V_0(\Sigma, p)$,

$$|U(p)-U_p|^2_{H^1(\Sigma)} = \int_\Sigma \underset{\sim}{\nabla}(U(p)-U_p)\cdot \underset{\sim}{\nabla}(\hat{U}-U_p)\, d\underset{\sim}{\hat{\rho}} \le |U(p)-U_p|_{H^1(\Sigma)}|\hat{U}-U_p|_{H^1(\Sigma)},$$

and therefore, $|U(p) - U_p|_{H^1(\Sigma)} \le |\hat{U} - U_p|_{H^1(\Sigma)}$. So, using the triangle inequality

$$\begin{aligned}|\hat{U}-U(p)|_{H^1(\Sigma)} &\le |\hat{U}-U_p|_{H^1(\Sigma)} + |U(p)-U_p|_{H^1(\Sigma)} \le 2|\hat{U}-U_p|_{H^1(\Sigma)} \\ &\le 2|\hat{U}-U|_{H^1(\Sigma)} + 2|U-U_p|_{H^1(\Sigma)}.\end{aligned}$$

The result follows from (6.3.5) and from Theorem 6.3.11. □

Remark. It is interesting to see how the corner singularities spoils an otherwise good convergence rate. For example, if $U_0(y) = y$, the Galerkin projection converges as $p^{-2+\delta}$ in $H^1(\Sigma)$, while if U_0 is still smooth but has all derivatives vanishing at the endpoints, then the convergence is faster than polynomial.

Section 6.4 – A mixed approximation. In this section, we investigate a mixed approximation for problem (6.3.1). We prove stability and exponential decay of the approximation, and finally estimate its convergence rates. Again we have to take into account corner singularities present in the exact solution.

We start by introducing

$$\|\underset{\sim}{\tau}\|^2_{\underset{\sim}{S}'_0(\Sigma)} = \|\operatorname{div}\underset{\sim}{\tau}\|^2_{L^2_{w^{-1}}(\Sigma)} + \|\underset{\sim}{\tau}\|^2_{L^2(\Sigma)},$$

$$\underset{\sim}{S}'_0(\Sigma) = \left\{\underset{\sim}{\tau} \in \underset{\sim}{\mathcal{D}}'(\Sigma) \,:\, \|\underset{\sim}{\tau}\|_{\underset{\sim}{S}'_0(\Sigma)} < \infty,\ \underset{\sim}{\tau}\cdot\underset{\sim}{n} = 0 \text{ on } \partial\Sigma_\pm\right\},$$

and rewriting the following spaces

$$V'(\Sigma, p) = \{v \in L^2_w(\Sigma) \,:\, \deg_2 v \le p\},$$

$$\underset{\sim}{S}'_0(\Sigma, p) = \{\underset{\sim}{\tau} \in \underset{\sim}{S}'_0(\Sigma) \,:\, \deg_2 \tau_1 \le p,\, \deg_2 \tau_2 \le p+1\},$$

in a more compact form. An alternative way to solve (6.3.1) is to find $U \in L^2_w(\Sigma)$ and $\underset{\sim}{\Xi} \in \underset{\sim}{S}'_0(\Sigma)$ such that

$$\begin{aligned} \int_\Sigma \underset{\sim}{\Xi} \cdot \underset{\sim}{\tau}\, d\underset{\sim}{\hat\rho} + \int_\Sigma U \operatorname{div} \underset{\sim}{\tau}\, d\underset{\sim}{\hat\rho} &= -\int_{\gamma_0} U_0 \tau_1\, d\hat\rho_2 \qquad \text{for all } \underset{\sim}{\tau} \in \underset{\sim}{S}'_0(\Sigma), \\ \int_\Sigma \operatorname{div} \underset{\sim}{\Xi} v\, d\underset{\sim}{\hat\rho} &= 0 \qquad \text{for all } v \in L^2_w(\Sigma). \end{aligned} \tag{6.4.1}$$

The approximate solutions $U(p) \in V'(\Sigma, p)$ and $\underset{\sim}{\Xi}(p) \in \underset{\sim}{S}'_0(\Sigma, p)$ satisfy

$$\begin{aligned} \int_\Sigma \underset{\sim}{\Xi}(p) \cdot \underset{\sim}{\tau}\, d\underset{\sim}{\hat\rho} + \int_\Sigma U(p) \operatorname{div} \underset{\sim}{\tau}\, d\underset{\sim}{\hat\rho} &= -\int_{\gamma_0} U_0(p) \tau_1\, d\hat\rho_2 \qquad \text{for all } \underset{\sim}{\tau} \in \underset{\sim}{S}'_0(\Sigma, p), \\ \int_\Sigma \operatorname{div} \underset{\sim}{\Xi}(p) v\, d\underset{\sim}{\hat\rho} &= 0 \qquad \text{for all } v \in V'(\Sigma, p), \end{aligned} \tag{6.4.2}$$

The stability and decaying properties of the mixed approximations are as follows.

Theorem 6.4.1. *For any $U_0(p) \in \hat{\mathbb{P}}_p(-1, 1)$, there exists unique $U(p) \in V'(\Sigma, p)$ and $\underset{\sim}{\Xi}(p) \in \underset{\sim}{S}'_0(\Sigma, p)$ such that (6.4.2) holds. Moreover, there exists a universal constant C such that*

$$\begin{aligned} \|U(p)\|_{L^2_w(\Sigma)} + \|\underset{\sim}{\Xi}(p)\|_{\underset{\sim}{S}'_0(\Sigma)} &\le C \|U_0(p)\|_{H^{1/2}(\gamma_0)}, \\ \int_{\Sigma(t,\infty)} [U(p)]^2\, d\underset{\sim}{\hat\rho} &\le 2 \|\underset{\sim}{\Xi}(p)\|^2_{L^2(\Sigma)} \exp(-t/5), \\ \int_{\Sigma(t,\infty)} |\underset{\sim}{\Xi}(p)|^2\, d\underset{\sim}{\hat\rho} &\le \|\underset{\sim}{\Xi}(p)\|^2_{L^2(\Sigma)} \exp(-t/5). \end{aligned} \tag{6.4.3}$$

Proof. We want to apply Lemma 4.2.1. As $\operatorname{div} \underset{\sim}{S}'_0(\Sigma, p) \subset V'(\Sigma, p)$, then the coercivity hypothesis holds. To show the inf-sup hypothesis also holds, let

$$\bar v(p)(\hat\rho_1, \hat\rho_2) = \sum_{j=0}^{p} \bar v_j(p)(\hat\rho_1) L_j(\hat\rho_2) \in V'(\Sigma, p),$$

and define $\bar{u}_j(p) \in L^2_w(\mathbb{R}^+)$ such that

$$\begin{aligned} \bar{u}_j''(p)(\hat{\rho}_1) &= w^2(\hat{\rho}_1)\bar{v}_j(p)(\hat{\rho}_1) \qquad \text{in } \mathbb{R}^+, \\ \bar{u}_j(p)(0) &= 0. \end{aligned} \tag{6.4.4}$$

Adapting the theory developed in Section 6.1, it is possible to prove existence and uniqueness of solution for problem (6.4.4). Also, defining $\bar{\sigma}^j(p) = \bar{u}_j'(p)$, we have that $\|\bar{\sigma}^j(p)\|_{L^2(\mathbb{R}^+)} \le C\|\bar{v}_j(p)\|_{L^2_w(\mathbb{R}^+)}$. Let

$$\underset{\sim}{\bar{\sigma}}(p) = (\bar{\sigma}_1(p), 0) \text{ with } \bar{\sigma}_1(p)(\underset{\sim}{\hat{\rho}}) = \sum_{j=0}^{p} \bar{\sigma}^j(p)(\hat{\rho}_1)L_j(\hat{\rho}_2).$$

Then

$$\|\underset{\sim}{\bar{\sigma}}\|^2_{\underset{\sim}{S}'_0(\Sigma)} = \|\underset{\sim}{\bar{\sigma}}(p)\|^2_{L^2(\mathbb{R}^+)} + \|w^{-1}\operatorname{div}\underset{\sim}{\bar{\sigma}}(p)\|^2_{L^2(\mathbb{R}^+)} \le C\|\bar{v}(p)\|^2_{L^2_w(\mathbb{R})},$$

and the inf-sup condition follows since

$$\int_\Sigma \bar{v}(p)\operatorname{div}\underset{\sim}{\bar{\sigma}}(p)\,d\underset{\sim}{\hat{\rho}} = \sum_{j=0}^{p}\int_\Sigma \bar{v}_j(p)\bar{\sigma}^j(p)L_j^2\,d\underset{\sim}{\hat{\rho}} = \|\bar{v}\|^2_{L^2_w(\Sigma)} \ge C\|\bar{v}\|_{L^2_w(\Sigma)}\|\underset{\sim}{\bar{\sigma}}\|_{\underset{\sim}{S}'_0(\Sigma)}.$$

Hence the hypotheses of Lemma 4.2.1 are satisfied and the first inequality of the lemma follows. Next we prove the exponential decay. Arguing as in the proof of Lemma 6.2.7, we have that properties (C1)–(C3) are satisfied. We show now that (C4) holds with $C_W = 4$. From (6.4.2) with $\tau_1 = 0$, we see that $\int_{\gamma_t}[\Xi_2(p)\tau_2 + U(p)\partial_2\tau_2]\,d\hat{\rho}_2 = 0$ for all $\tau_2 \in \mathring{\mathbb{P}}_{p+1}(-1,1)$ and for almost every $t \in \mathbb{R}^+$. Assuming first that $\int_{\gamma_t} U(p)\,d\hat{\rho}_2 = 0$, let $\tau_2(\hat{\rho}_2) = \int_{-1}^{\hat{\rho}_2} U(p)(t,s)\,ds$ and then

$$\begin{aligned} \int_{\gamma_t} [U(p)(t,\hat{\rho}_2)]^2\,d\hat{\rho}_2 &= -\int_{\gamma_t} \Xi_2(p)(t,\hat{\rho}_2)\int_{-1}^{\hat{\rho}_2} U(p)(t,s)\,ds\,d\hat{\rho}_2 \\ &\le 2\|\Xi_2(p)\|_{L^2(\gamma_t)}\|U(p)\|_{L^2(\gamma_t)}. \end{aligned}$$

The general case follows by considering $U(p) - (1/2)\int_{\gamma_t} U(p)$, and proceeding as in the proof of Lemma 6.2.2. We can then use Lemma 6.2.5 and the exponential decay follows. $\square$

To analyze the mixed model, we need a technical result which shows that the div and some projection operators commute. Defining $\underset{\sim}{\Pi}_p = (\pi_p^{(\hat{\rho}_2)}, \mathring{\pi}_{p+1}^{1(\hat{\rho}_2)})^T : \underset{\sim}{S}'_0(\Sigma) \to \underset{\sim}{S}'_0(\Sigma, p)$, we have the result below.

Lemma 6.4.2. *If $\underset{\sim}{\tau} \in \underset{\sim}{S}'_0(\Sigma)$, then $\pi_p^{(\hat{\rho}_2)} \operatorname{div} \underset{\sim}{\tau} = \operatorname{div} \underset{\sim}{\Pi}_p \underset{\sim}{\tau}$.*

Proof. It is enough to show that $\int_\Sigma \operatorname{div} \underset{\sim}{\Pi}_p \underset{\sim}{\tau} v \, d\underset{\sim}{\hat{\rho}} = \int_\Sigma \operatorname{div} \underset{\sim}{\tau} v \, d\underset{\sim}{\hat{\rho}}$ for all $v \in V'(\Sigma, p)$. Assuming that v is sufficiently smooth (the general case follows by density), we indeed have

$$\begin{aligned}\int_\Sigma \operatorname{div} \underset{\sim}{\Pi}_p \underset{\sim}{\tau} v \, d\underset{\sim}{\hat{\rho}} &= \int_\Sigma (-\pi_p^{(\hat{\rho}_2)} \tau_1 \partial_1 v + \partial_2 \mathring{\pi}_{p+1}^{1(\hat{\rho}_2)} \tau_2 v) \, d\underset{\sim}{\hat{\rho}} + \int_{\gamma_0} \pi_p^{(\hat{\rho}_2)} \tau_1 v \, d\hat{\rho}_2 \\ &= \int_\Sigma (-\tau_1 \partial_1 v + \partial_2 \mathring{\pi}_{p+1}^{1(\hat{\rho}_2)} \tau_2 v) \, d\underset{\sim}{\hat{\rho}} + \int_{\gamma_0} \tau_1 v \, d\hat{\rho}_2 = \int_\Sigma \partial_1 \tau_1 v + \partial_2 \tau_2 v \, d\underset{\sim}{\hat{\rho}}.\end{aligned}$$

The last step uses Lemma 4.2.7 and an integration by parts. $\square$

It is important to estimate

$$\|\underset{\sim}{\Xi} - \underset{\sim}{\Pi}_p \underset{\sim}{\Xi}\|_{L^2(\Sigma)}^2 = \|\Xi_1 - \pi_p^{(\hat{\rho}_2)} \Xi_1\|_{L^2(\Sigma)}^2 + \|\Xi_2 - \mathring{\pi}_{p+1}^{1(\hat{\rho}_2)} \Xi_2\|_{L^2(\Sigma)}^2, \tag{6.4.5}$$

where $\underset{\sim}{\Xi} = \underset{\sim}{\nabla} U$ solves (6.4.1). The approximation result of Theorem 6.3.11 makes the estimate regarding $\Xi_1 = \partial_1 U$ straightforward:

$$\|\partial_1 U - \pi_p^{(\hat{\rho}_2)} \partial_1 U\|_{L^2(\Sigma)} \le \|\partial_1 U - \partial_1 U_p\|_{L^2(\Sigma)} \le C p^{-\mu(r_0, \delta)} \|U_0\|_{H^{r_0}(-1,1)}. \tag{6.4.6}$$

Unfortunately, the estimates for $\Xi_2 = \partial_2 U$ does not come so easily, since $\mathring{\pi}_{p+1}^{1(\hat{\rho}_2)}$ does not necessarily yield the best approximation in the L^2 norm. We divide the error analysis in two cases. Assume first that $|\partial_2 U_0(-1)| + |\partial_2 U_0(1)| \neq 0$. Then, using Lemma 6.3.1, Theorem 6.3.2 and Remark 6.3.7, we have that $U \in H^{2-\delta}(\Sigma)$. Then, a straightforward application of Lemma A.4 yields the following result.

Lemma 6.4.3. *For any arbitrarily small positive real number δ, there exists a constant C such that*

$$\|\partial_2 U - \mathring{\pi}_{p+1}^{1(\hat{\rho}_2)} \partial_2 U\|_{L^2(\Sigma)} \leq Cp^{-1+\delta} \|U\|_{L^2(\mathbb{R}^+;H^{2-\delta}(-1,1))} \leq Cp^{-1+\delta} \|U_0\|_{H^{3/2}(-1,1)}. \tag{6.4.7}$$

Assume now that $|\partial_2 U_0(-1)| + |\partial_2 U_0(1)| = 0$. If $r_0 \in (3/2, 5/2)$, then

$$\|\partial_2 U - \mathring{\pi}_{p+1}^{1(\hat{\rho}_2)} \partial_2 U\|_{L^2(\Sigma)} \leq Cp^{-r_0-1/2} \|U_0\|_{H^{r_0}(-1,1)}. \tag{6.4.8}$$

Otherwise, using a duality argument [14], and defining $U_2 = \partial_2 U$,

$$\begin{aligned} \|U_2 - \mathring{\pi}_{p+1}^{1(\hat{\rho}_2)} U_2\|_{L^2(\Sigma)} &\leq Cp^{-1} \|\partial_2 U_2 - \partial_2 \mathring{\pi}_{p+1}^{1(\hat{\rho}_2)} U_2\|_{L^2(\Sigma)} \\ &= Cp^{-1} \|\partial_2 U_2 - \pi_p^{(\hat{\rho}_2)} \partial_2 U_2\|_{L^2(\Sigma)}, \end{aligned} \tag{6.4.9}$$

by Lemma 4.2.7. Note that U_2 solve the following Dirichlet problem:

$$\begin{gathered} \Delta U_2 = 0 \qquad \text{in } Q, \\ U_2 = 0 \qquad \text{on } \partial Q_\pm, \quad U_2 = \partial_2 U_0 \qquad \text{on } \gamma_0, \quad U_2 = \partial_2 U \qquad \text{on } \gamma_1, \end{gathered} \tag{6.4.10}$$

where $U_0 \in H^{r_0}(-1,1)$.

The way to obtain a good approximation for U_2 is, as in Section 6.3, by splitting U_2 in singular and smooth parts and seeking approximations for both. In the next two results we do exactly that. The following theorem comes from [34].

Theorem 6.4.4. *Let U_2 be the solution of* (6.4.10) *with $r_0 > 5/2$ such that $r_0 + 1/2$ is not an even integer. Let U_{2_Q} be the restriction of U_2 to Q. Then there exist constants C_j, C such that*

$$U_{2_Q} = U_{2_S} + W_2, \quad U_{2_S} = \sum_{l=1}^{2} \sum_{j=2}^{J(r_0+1/2)} C_j \partial_2^{2j-1} U_0\big((-1)^{l+1}\big) \bar{v}_l^{j-1}, \tag{6.4.11}$$

where

$$\bar{v}_1^j = [\theta_1 \cos(2j\theta_1) + \log r_1 \sin(2j\theta_1)] r_1^{2j},$$

$$\bar{v}_2^j = [\theta_2 \cos(2j\theta_2) + \log r_2 \sin(2j\theta_2)] r_2^{2j},$$

and $\|W_2\|_{H^{r_0 - 1/2}(Q)} \leq C\|U_0\|_{H^{r_0}(\gamma_0)}$.

The next lemma is analogous to Lemmas 6.3.8 and 6.3.9.

Lemma 6.4.5. *Assume that the hypotheses of Theorem 6.4.4 hold. Then*

$$|U_2 - \hat{\pi}_p^{(\hat{\rho}_2)} U_2|_{H^1(\Sigma(1,\infty))} + |W_2 - \hat{\pi}_p^{(\hat{\rho}_2)} W_2|_{H^1(Q)} \leq C p^{3/2 - r_0} \|U_0\|_{H^{r_0}(\gamma_0)}.$$

Also, if U_{2_S} *is not the zero function, then for each arbitrarily small* $\delta > 0$ *there exists a constant* C *such that*

$$\|U_{2_S} - \pi_p^{(\hat{\rho}_2)} U_{2_S}\|_{H^1(Q)} \leq C p^{-4m+4+\delta} \|U_0\|_{H^{r_0}(\gamma_0)},$$

where $m \in \{2, \ldots, J(r_0 + \frac{1}{2})\}$ *is the minimum integer such that* $|\partial_2^{(2m-1)} U_0(-1)| + |\partial_2^{(2m-1)} U_0(1)| \neq 0$.

Below we define the rate of convergence of our approximation result.

Definition 6.4.6. For $U_0 \in H^{r_0}(-1,1)$ with $r_0 > 3/2$, if there exists an minimum integer $m \in \{1, \ldots, J(r_0 + \frac{1}{2})\}$ such that $|\partial_2^{2m-1} U_0(-1)| + |\partial_2^{2m-1} U_0(1)| \neq 0$, let $\bar{\mu}(r_0, \delta) = \min\{4m - 3 - \delta, r_0 - 1/2\}$, otherwise let $\bar{\mu}(r_0, \delta) = r_0 - 1/2$.

Now we are ready to estimate $\|\underset{\sim}{\Xi} - \underset{\sim}{\Pi}_p \underset{\sim}{\Xi}\|_{L^2(\Sigma)}$.

Lemma 6.4.7. *For any* $r_0 > 3/2$ *such that* $r_0 + 1/2$ *is not an even integer, and any arbitrarily small* $\delta > 0$*, there exists a constant* C *such that*

$$\|\partial_1 U - \mathring{\pi}_{p+1}^{1(\hat{\rho}_2)} \partial_1 U\|_{L^2(\Sigma)} \leq C p^{-\mu(r_0,\delta)} \|U_0\|_{H^{r_0}(-1,1)},$$

$$\|\partial_2 U - \mathring{\pi}_{p+1}^{1(\hat{\rho}_2)} \partial_2 U\|_{L^2(\Sigma)} \leq C p^{-\bar{\mu}(r_0,\delta)} \|U_0\|_{H^{r_0}(-1,1)},$$

$$\|\underset{\sim}{\Xi} - \underset{\sim}{\Pi}_p \underset{\sim}{\Xi}\|_{L^2(\Sigma)} \leq C p^{-\bar{\mu}(r_0,\delta)} \|U_0\|_{H^{r_0}(-1,1)},$$

where μ is as in Definition 6.3.10 and $\bar{\mu}$ is as in Definition 6.4.6.

Proof. The first bound follows immediately from (6.4.6). The second estimate follows, for $r_0 \in (3/2, 5/2]$, from (6.4.7) and (6.4.8). For $r_0 > 5/2$, it follows from (6.4.9) and Lemma 6.4.5. Finally, the third estimate of this lemma follows from (6.4.5) and the fact that $\bar{\mu} \le \mu$. $\square$

The next theorem estimates the mixed approximation.

Theorem 6.4.8. *Assume that $U \in V'(\Sigma)$, $\underset{\sim}{\Xi} \in \underset{\sim}{S}'_0(\Sigma)$ solve* (6.4.1) *and $U(p) \in V'(\Sigma, p)$, $\underset{\sim}{\Xi}(p) \in \underset{\sim}{S}'_0(\Sigma, p)$ solve* (6.4.2). *Then for any nonnegative real number $r_0 > 3/2$ such that $r_0 + 1/2$ is not an even integer, and any arbitrarily small $\delta > 0$, there exists a constant C such that*

$$\|\underset{\sim}{\Xi} - \underset{\sim}{\Xi}(p)\|_{\underset{\sim}{L}^2(\Sigma)} \le C(\|U_0 - U_0(p)\|_{H^{1/2}(\gamma_0)} + p^{-\bar{\mu}(r_0,\delta)}\|U_0(p)\|_{H^{r_0}(\gamma_0)}).$$

Proof. Let $\tilde{U} \in L^2_w(\Sigma)$ and $\tilde{\underset{\sim}{\Xi}} \in \underset{\sim}{S}'_0(\Sigma)$ be such that

$$\int_\Sigma \tilde{\underset{\sim}{\Xi}} \cdot \underset{\sim}{\tau}\, d\underset{\sim}{\hat{\rho}} + \int_\Sigma \tilde{U} \operatorname{div} \underset{\sim}{\tau}\, d\underset{\sim}{\hat{\rho}} = \int_{\gamma_0} U_0(p)\tau_1\, d\hat{\rho}_2 \qquad \text{for all } \underset{\sim}{\tau} \in \underset{\sim}{S}'_0(\Sigma),$$
$$\int_\Sigma \operatorname{div} \tilde{\underset{\sim}{\Xi}} v\, d\underset{\sim}{\hat{\rho}} = 0 \qquad \text{for all } v \in L^2_w(\Sigma),$$

and then, from Lemma 6.3.1,

$$\|\underset{\sim}{\Xi} - \tilde{\underset{\sim}{\Xi}}\|_{\underset{\sim}{L}^2(\Sigma)} \le C\|U_0 - U_0(p)\|_{H^{1/2}(\gamma_0)}. \tag{6.4.12}$$

To conclude the estimate, we use Lemma 6.4.2 as follows:

$$\begin{aligned}\int_\Sigma [\tilde{\underset{\sim}{\Xi}} - \underset{\sim}{\Xi}(p)]\underset{\sim}{\Pi}_p\tilde{\underset{\sim}{\Xi}}\, d\underset{\sim}{\hat{\rho}} &= -\int_\Sigma [\tilde{U} - U(p)] \operatorname{div} \underset{\sim}{\Pi}_p\tilde{\underset{\sim}{\Xi}}\, d\underset{\sim}{\hat{\rho}} = -\int_\Sigma [\tilde{U} - U(p)]\pi_p^{(\hat{\rho}_2)} \operatorname{div} \tilde{\underset{\sim}{\Xi}}\, d\underset{\sim}{\hat{\rho}} \\ &= -\int_\Sigma [\tilde{U} - U(p)] \operatorname{div} \underset{\sim}{\Xi}(p)\, d\underset{\sim}{\hat{\rho}} = \int_\Sigma [\tilde{\underset{\sim}{\Xi}} - \underset{\sim}{\Xi}(p)]\underset{\sim}{\Xi}(p)\, d\underset{\sim}{\hat{\rho}},\end{aligned}$$

since $\pi_p^{(\hat{\rho}_2)} \operatorname{div} \tilde{\underset{\sim}{\Xi}} = \operatorname{div} \underset{\sim}{\Xi}(p)$, and then

$$\|\tilde{\underset{\sim}{\Xi}} - \underset{\sim}{\Xi}(p)\|^2_{L^2(\Sigma)} = \int_\Sigma [\tilde{\underset{\sim}{\Xi}} - \underset{\sim}{\Xi}(p)][\tilde{\underset{\sim}{\Xi}} - \underset{\sim}{\Pi}_p \tilde{\underset{\sim}{\Xi}}]\, d\underset{\sim}{\hat{\rho}} \leq \|\tilde{\underset{\sim}{\Xi}} - \underset{\sim}{\Xi}(p)\|_{L^2(\Sigma)} \|\tilde{\underset{\sim}{\Xi}} - \underset{\sim}{\Pi}_p \tilde{\underset{\sim}{\Xi}}\|_{L^2(\Sigma)}.$$

Next, since Π_p is a bounded operator in $L^2(\Sigma)$,

$$\begin{aligned}
\|\tilde{\underset{\sim}{\Xi}} - \underset{\sim}{\Xi}(p)\|_{L^2(\Sigma)} &\leq \|\tilde{\underset{\sim}{\Xi}} - \underset{\sim}{\Pi}_p \tilde{\underset{\sim}{\Xi}}\|_{L^2(\Sigma)} \\
&\leq \|\tilde{\underset{\sim}{\Xi}} - \underset{\sim}{\Xi}\|_{L^2(\Sigma)} + \|\underset{\sim}{\Xi} - \underset{\sim}{\Pi}_p \underset{\sim}{\Xi}\|_{L^2(\Sigma)} + \|\underset{\sim}{\Pi}_p \underset{\sim}{\Xi} - \underset{\sim}{\Pi}_p \tilde{\underset{\sim}{\Xi}}\|_{L^2(\Sigma)} \\
&\leq C\|\tilde{\underset{\sim}{\Xi}} - \underset{\sim}{\Xi}\|_{L^2(\Sigma)} + \|\underset{\sim}{\Xi} - \underset{\sim}{\Pi}_p \underset{\sim}{\Xi}\|_{L^2(\Sigma)} \leq C\|U_0 - U_0(p)\|_{H^{1/2}(\gamma_0)} + \|\underset{\sim}{\Xi} - \underset{\sim}{\Pi}_p \underset{\sim}{\Xi}\|_{L^2(\Sigma)} \\
&\leq C\|U_0 - U_0(p)\|_{H^{1/2}(\gamma_0)} + p^{-\bar{\mu}(r_0,\delta)}\|U_0(p)\|_{H^{r_0}(\gamma_0)},
\end{aligned}$$

where we used Lemma 6.4.7 to obtain the last inequality. The theorem follows from (6.4.12) and the inequality above. $\square$

Comparing Theorem 6.4.8 with Theorem 6.3.12, we see that the estimates for the mixed approximations are worse than the ones for the Galerkin approximation. For instance, if $U_0(\hat{\rho}_2) = U_0(p)(\hat{\rho}_2) = \hat{\rho}_2$, then we can bound the error coming from the mixed methods as

$$\|\underset{\sim}{\Xi} - \underset{\sim}{\Xi}(p)\|_{L^2(\Sigma)} \leq C p^{-1+\delta} \|U_0\|_{H^{r_0}(\gamma_0)},$$

while we bound the error from the Galerkin methods as

$$\|\underset{\sim}{\nabla} U - \underset{\sim}{\nabla} U(p)\|_{L^2(\Sigma)} \leq C p^{-2+\delta} \|U_0\|_{H^{r_0}(\gamma_0)}.$$

It is not clear whether the upper bound of Theorem 6.4.8 is sharp or not, and, to the best of our knowledge, there is no numerical evidence to support either case. The culprit for this possible loss of accuracy is the use of a duality argument. In fact, Eriksson [31] worked out a one-dimensional example and showed that the duality argument does not yield the best possible error estimate for the p-method.

Chapter 7

Variational approaches for modeling elastic plates

This chapter presents various classes of models for a linearly elastic isotropic clamped plate problem. The models that we consider here are based on two variational principles which are the analogues of SP and for SP′ for elasticity. They are the *Hellinger–Reissner principles*. This approach appeared in a joint work with Alessandrini, Arnold and Falk [2], which included an error analysis for one of the models, based on the two energy principles (or Prager–Synge theorem).

In what follows we describe the resulting equations for some "low order" models. Most of these equations never appeared before in the literature.

Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain and let $\varepsilon \in (0,1]$ represent the plate thickness. The plate occupies the set $P^\varepsilon = \Omega \times (-\varepsilon, \varepsilon)$. We denote its lateral side by $\partial P^\varepsilon_L = \partial\Omega \times (-\varepsilon,\varepsilon)$, and the union of its top and bottom by $\partial P^\varepsilon_\pm = \Omega \times \{-\varepsilon, \varepsilon\}$. We are concerned with the problem of finding the displacement $\underline{u}\colon P^\varepsilon \to \mathbb{R}^3$ and stress $\underline{\sigma}\colon P^\varepsilon \to \mathbb{R}^{3\times 3}_{\text{sym}}$ (the space of 3×3 symmetric matrices) such that

$$
\begin{aligned}
\underline{\underline{A}}\,\underline{\sigma}^\varepsilon = \underline{e}(\underline{u}^\varepsilon), \quad \underline{\operatorname{div}}\,\underline{\sigma}^\varepsilon = -\underline{f}^\varepsilon \qquad &\text{in } P^\varepsilon,\\
\underline{\sigma}^\varepsilon \underline{n} = \underline{g}^\varepsilon \qquad \text{on } \partial P^\varepsilon_\pm, \quad \underline{u}^\varepsilon = 0 \qquad &\text{on } \partial P^\varepsilon_L,
\end{aligned}
\tag{7.1}
$$

where $\underline{f}^\varepsilon\colon P^\varepsilon \to \mathbb{R}^3$ and $\underline{g}^\varepsilon\colon \partial P^\varepsilon_\pm \to \mathbb{R}^3$ represent the volume and traction loads. We denote the symmetric part of the gradient of $\underline{u}$ by

$$
\underline{e}(\underline{u}^\varepsilon) = \frac{1}{2}(\underline{\nabla}\,\underline{u}^\varepsilon + \underline{\nabla}^T\,\underline{u}^\varepsilon),
$$

i.e., $e_{ij}(\underline{u}^\varepsilon) = (\partial_i u^\varepsilon_j + \partial_j u^\varepsilon_i)/2$. Also, $(\underline{\operatorname{div}}\,\underline{\sigma}^\varepsilon)_i = \sum_{j=1}^3 \partial_j \sigma^\varepsilon_{ij}$. The compliance tensor $\underline{\underline{A}}$ is such that $\underline{\underline{A}}\,\underline{\tau} = (1+\nu)\underline{\tau}/E - \nu\operatorname{tr}(\underline{\tau})\underline{\delta}/E$, where $E > 0$ is the Young's modulus, $\nu \in [0, 1/2)$ is the Poisson's ratio, and $\underline{\delta}$ is the 3×3 identity matrix.

Extending the notation previously employed, we use one underbar for first order tensors in three variables, two underbars for second order tensors in three variables, etc. Similar notation holds with undertildes for tensors in two variables. We can then decompose 3-vectors and 3×3 matrices as follows:

$$\underline{u} = \begin{pmatrix} \underset{\sim}{u} \\ u_3 \end{pmatrix}, \quad \underline{\underline{\sigma}} = \begin{pmatrix} \underset{\approx}{\sigma} & \underset{\sim}{\sigma} \\ \underset{\sim}{\sigma}^T & \sigma_{33} \end{pmatrix}.$$

The three-dimensional elasticity problem decouples into two problems, one related to the *stretching* of the plate, another related to the *bending*. For a function k defined on P^ε or $\partial P^\varepsilon_\pm$, there is a unique decomposition into its even and odd parts with respect to x_3^ε, i.e., $k = k^{\mathrm{even}} + k^{\mathrm{odd}}$ where

$$k^{\mathrm{even}}(\underline{x}^\varepsilon) = \frac{k(\underset{\sim}{x}^\varepsilon, x_3^\varepsilon) + k(\underset{\sim}{x}^\varepsilon, -x_3^\varepsilon)}{2}, \quad k^{\mathrm{odd}}(\underline{x}^\varepsilon) = \frac{k(\underset{\sim}{x}^\varepsilon, x_3^\varepsilon) - k(\underset{\sim}{x}, -x_3^\varepsilon)}{2}.$$

We decompose then

$$\underline{u}^\varepsilon = \underline{u}^{\varepsilon^{\mathrm{s}}} + \underline{u}^{\varepsilon^{\mathrm{b}}}, \quad \underline{\underline{\sigma}}^\varepsilon = \underline{\underline{\sigma}}^{\varepsilon^{\mathrm{s}}} + \underline{u}^{\varepsilon^{\mathrm{b}}}, \quad \underline{g}^\varepsilon = \underline{g}^{\varepsilon^{\mathrm{s}}} + \underline{g}^{\varepsilon^{\mathrm{b}}}, \quad \underline{f}^\varepsilon = \underline{f}^{\varepsilon^{\mathrm{s}}} + \underline{f}^{\varepsilon^{\mathrm{b}}},$$

where

$$\underline{u}^{\varepsilon^{\mathrm{s}}} = \begin{pmatrix} \underset{\sim}{u}^{\varepsilon^{\mathrm{even}}} \\ u_3^{\varepsilon^{\mathrm{odd}}} \end{pmatrix}, \quad \underline{\underline{\sigma}}^{\varepsilon^{\mathrm{s}}} = \begin{pmatrix} \underset{\approx}{\sigma}^{\varepsilon^{\mathrm{even}}} & \underset{\sim}{\sigma}^{\varepsilon^{\mathrm{odd}}} \\ (\underset{\sim}{\sigma}^{\varepsilon^{\mathrm{odd}}})^T & \sigma_{33}^{\varepsilon^{\mathrm{even}}} \end{pmatrix},$$
$$\underline{u}^{\varepsilon^{\mathrm{b}}} = \begin{pmatrix} \underset{\sim}{u}^{\varepsilon^{\mathrm{odd}}} \\ u_3^{\varepsilon^{\mathrm{even}}} \end{pmatrix}, \quad \underline{\underline{\sigma}}^{\varepsilon^{\mathrm{b}}} = \begin{pmatrix} \underset{\approx}{\sigma}^{\varepsilon^{\mathrm{odd}}} & \underset{\sim}{\sigma}^{\varepsilon^{\mathrm{even}}} \\ (\underset{\sim}{\sigma}^{\varepsilon^{\mathrm{even}}})^T & \sigma_{33}^{\varepsilon^{\mathrm{odd}}} \end{pmatrix},$$
$$\underline{g}^{\varepsilon^{\mathrm{s}}} = \begin{pmatrix} \underset{\sim}{g}^{\varepsilon^{\mathrm{even}}} \\ g_3^{\varepsilon^{\mathrm{odd}}} \end{pmatrix}, \quad \underline{g}^{\varepsilon^{\mathrm{b}}} = \begin{pmatrix} \underset{\sim}{g}^{\varepsilon^{\mathrm{odd}}} \\ g_3^{\varepsilon^{\mathrm{even}}} \end{pmatrix}, \quad \underline{f}^{\varepsilon^{\mathrm{s}}} = \begin{pmatrix} \underset{\sim}{f}^{\varepsilon^{\mathrm{even}}} \\ f_3^{\varepsilon^{\mathrm{odd}}} \end{pmatrix}, \quad \underline{f}^{\varepsilon^{\mathrm{b}}} = \begin{pmatrix} \underset{\sim}{f}^{\varepsilon^{\mathrm{odd}}} \\ f_3^{\varepsilon^{\mathrm{even}}} \end{pmatrix}.$$

It is easy to see that the stretching part $\underline{u}^{\varepsilon^{\mathrm{s}}}$, $\underline{\underline{\sigma}}^{\varepsilon^{\mathrm{s}}}$ is the solution of (7.1) with $\underline{g}^\varepsilon$ replaced by $\underline{g}^{\varepsilon^{\mathrm{s}}}$ and $\underline{f}^\varepsilon$ replaced by $\underline{f}^{\varepsilon^{\mathrm{s}}}$. Similarly for the bending part $\underline{u}^{\varepsilon^{\mathrm{b}}}$, $\underline{\underline{\sigma}}^{\varepsilon^{\mathrm{b}}}$.

Section 7.1 – The HR *models.* It is possible to characterize the solution of (7.1) in an alternative manner. Indeed, let

$$\underline{V}(P^\varepsilon) = \{\, \underline{v} \in \underline{H}^1(P^\varepsilon) \,:\, \underline{v} = 0 \text{ on } \partial P^\varepsilon_L \,\}, \quad \underline{\underline{S}}(P^\varepsilon) = \underline{\underline{L}}^2(P^\varepsilon).$$

Then the *first Hellinger-Reissner principle*, or HR for short, holds.

HR: $(\underline{u}^\varepsilon, \underline{\underline{\sigma}}^\varepsilon)$ is the unique critical point of

$$L(\underline{v}, \underline{\underline{\tau}}) = \frac{1}{2}\int_{P^\varepsilon} \underline{\underline{A}}\,\underline{\underline{\tau}} : \underline{\underline{\tau}}\, d\underline{x}^\varepsilon - \int_{P^\varepsilon} \underline{\underline{\tau}} : \underline{\underline{e}}(\underline{v})\, d\underline{x}^\varepsilon + \int_{P^\varepsilon} \underline{f}^\varepsilon \cdot \underline{v}\, d\underline{x}^\varepsilon + \int_{\partial P^\varepsilon_\pm} \underline{g}^\varepsilon \cdot \underline{v}\, d\underset{\sim}{x}^\varepsilon$$

on $\underline{V}(P^\varepsilon) \times \underline{\underline{S}}(P^\varepsilon)$.

Finding the critical point of L is equivalent to find $\underline{u}^\varepsilon \in \underline{V}(P^\varepsilon)$ and $\sigma^\varepsilon \in \underline{\underline{S}}(P^\varepsilon)$ such that

$$\begin{aligned} \int_{P^\varepsilon} \underline{\underline{A}}\,\underline{\underline{\sigma}}^\varepsilon : \underline{\underline{\tau}}\, d\underline{x} - \int_{P^\varepsilon} \underline{\underline{e}}(\underline{u}^\varepsilon) : \underline{\underline{\tau}}\, d\underline{x}^\varepsilon = 0 \qquad & \text{for all } \underline{\underline{\tau}} \in \underline{\underline{S}}(P^\varepsilon), \\ \int_{P^\varepsilon} \underline{\underline{\sigma}}^\varepsilon : \underline{\underline{e}}(\underline{v})\, d\underline{x}^\varepsilon = \int_{P^\varepsilon} \underline{f} \cdot \underline{v}\, d\underline{x}^\varepsilon + \int_{\partial P^\varepsilon_\pm} \underline{g} \cdot \underline{v}\, d\underset{\sim}{x}^\varepsilon \qquad & \text{for all } \underline{v} \in \underline{V}(P^\varepsilon). \end{aligned} \tag{7.1.1}$$

A first type of models appears when we look for critical points of L in subspaces of $\underline{V}(P^\varepsilon)$ and $\underline{\underline{S}}(P^\varepsilon)$ that have polynomial dependence in the transverse direction. For instance, let p be a positive integer and let

$$\begin{aligned} \underline{V}(P^\varepsilon, p) &= \big\{ \underline{v} \in \underline{V}(P^\varepsilon) : \deg_3 \underset{\sim}{v} \le p,\ \deg_3 v_3 \le p-1 \big\}, \\ \underline{\underline{S}}(P^\varepsilon, p) &= \big\{ \underline{\underline{\tau}} \in \underline{\underline{S}}(P^\varepsilon) : \deg_3 \underset{\approx}{\tau} \le p,\ \deg_3 \underset{\sim}{\tau} \le p-1,\ \deg_3 \tau_{33} \le p-2 \big\}. \end{aligned} \tag{7.1.2}$$

Then a critical point $(\underline{u}^\varepsilon(p), \underset{\approx}{\sigma}^\varepsilon(p)) \in \underline{V}(P^\varepsilon, p) \times \underline{\underline{S}}(P^\varepsilon, p)$ of L characterizes the $\mathrm{HR}_1(p)$ model. Carefully varying the polynomial degrees of the components for displacements and stress yields different subspaces and models. We summarize some of them in the table below. Besides the already defined $\mathrm{HR}_1(p)$, we also present the $\mathrm{HR}_2(p)$ and $\mathrm{HR}_3(p)$ models.

Table 7.1. HR Plate models.

model	$\deg_3 \underset{\approx}{\sigma}$	$\deg_3 \underset{\sim}{\sigma}$	$\deg_3 \sigma_{33}$	$\deg_3 \underset{\sim}{u}$	$\deg_3 u_3$
$\mathrm{HR}_1(p)$	p	$p-1$	$p-2$	p	$p-1$
$\mathrm{HR}_2(p)$	p	$p-1$	p	p	$p-1$
$\mathrm{HR}_3(p)$	p	$p+1$	p	p	$p+1$

In the case of a plate under bending, for p odd, $\mathrm{HR}_2(p)$ was called MPp in Alessandrini's thesis [1]. For $p = 3$ it yields the model of Lo, Christensen and Wu [38] (for both bending and stretching). Still considering the bending situation, the $\mathrm{HR}_3(p)$ models were denoted by MP$(p + 1)$ by Alessandrini [1], and for $p = 1$ it is also referred as the $(1, 1, 2)$ model [6].

Analogously to the original three-dimensional problem, the models can be equivalently characterized by a weak formulation, i.e., we shall seek $\underline{u}^\varepsilon(p) \in \underline{V}(P^\varepsilon, p)$ and $\underline{\underline{\sigma}}^\varepsilon(p) \in \underline{\underline{S}}(P^\varepsilon, p)$ such that

$$\int_{P^\varepsilon} \underline{\underline{A}}\,\underline{\underline{\sigma}}^\varepsilon(p) : \underline{\underline{\tau}}\, d\underline{x} - \int_{P^\varepsilon} \underline{\underline{e}}(\underline{u}^\varepsilon(p)) : \underline{\underline{\tau}}\, d\underline{x}^\varepsilon = 0 \quad \text{for all } \underline{\underline{\tau}} \in \underline{\underline{S}}(P^\varepsilon, p), \tag{7.1.3}$$

$$\int_{P^\varepsilon} \underline{\underline{\sigma}}^\varepsilon(p) : \underline{\underline{e}}(\underline{v})\, d\underline{x}^\varepsilon = \int_{P^\varepsilon} \underline{f} \cdot \underline{v}\, d\underline{x}^\varepsilon + \int_{\partial P^\varepsilon_\pm} \underline{g} \cdot \underline{v}\, d\underset{\sim}{x}^\varepsilon \quad \text{for all } \underline{v} \in \underline{V}(P^\varepsilon, p). \tag{7.1.4}$$

With the choices of spaces presented in Table 7.1, the existence and uniqueness of solutions for (7.1.3) and (7.1.4) follows from $\underline{\underline{e}}(\underline{V}(P^\varepsilon, p)) \subset \underline{\underline{S}}(P^\varepsilon, p)$. Note also that for both the $\mathrm{HR}_2(p)$ and $\mathrm{HR}_3(p)$ models, $\underline{\underline{A}}^{-1}\underline{\underline{e}}(\underline{V}(P^\varepsilon, p)) \subset \underline{\underline{S}}(P^\varepsilon, p)$ and it follows that the constitutive equation $\underline{\underline{A}}\,\underline{\underline{\sigma}}^\varepsilon(p) = \underline{\underline{e}}(\underline{u}^\varepsilon(p))$ is satisfied exactly. As a consequence, $\underline{u}^\varepsilon(p)$ is the minimizer (in $\underline{V}(P^\varepsilon, p)$) of the potential energy

$$J(\underline{v}) = \frac{1}{2} \int_{P^\varepsilon} \underline{\underline{A}}^{-1}\underline{\underline{e}}(\underline{v}) : \underline{\underline{e}}(\underline{v})\, d\underline{x}^\varepsilon - \int_{P^\varepsilon} \underline{f}^\varepsilon \cdot \underline{v}\, d\underline{x}^\varepsilon - \int_{\partial P^\varepsilon_\pm} \underline{g}^\varepsilon \cdot \underline{v}\, d\underset{\sim}{x}^\varepsilon,$$

i.e., $\mathrm{HR}_2(p)$ and $\mathrm{HR}_3(p)$ are minimum energy models. This sort of model is quite widespread in the literature, but a very upsetting characteristic is that its simplest version,

$\mathrm{HR}_2(1)$, is worthless. In fact, Paumier and Raoult [47] showed that for a minimum energy model to be *consistent*, i.e. to be asymptotically convergent to the biharmonic model as $\varepsilon \to 0$, $u_3(p)$ must be at least a quadratic polynomial. $\mathrm{HR}_3(1)$ is the simplest consistent minimum energy model, but its final form is more complicated than the membrane and the Resissner–Mindlin models, having one extra equation (and unknown) in each case. The $\mathrm{HR}_1(p)$ is *not* a minimum energy model. It is convergent for $p = 1$ and it yields a membrane problem for the stretching part and a problem of Resissner–Mindlin type with shear correction factor 1 for the bending part.

Before presenting details regarding the lowest order example for each of the HR models, some notation is necessary. If we define $\underset{\approx}{A}\underset{\sim}{\tau} = (1+\nu)\underset{\sim}{\tau}/E - \nu\,\mathrm{tr}(\underset{\sim}{\tau})\underset{\sim}{\delta}/E$, then

$$\underline{\underline{A}}\,\underline{\tau} = \begin{pmatrix} \underset{\approx}{A}\underset{\sim}{\tau} - \frac{\nu}{E}\tau_{33}\underset{\sim}{\delta} & \frac{1+\nu}{E}\underset{\sim}{\tau} \\ \frac{1+\nu}{E}\underset{\sim}{\tau}^T & \frac{\tau_{33}}{E} - \frac{\nu}{E}\,\mathrm{tr}(\underset{\sim}{\tau}) \end{pmatrix}.$$

It is useful to know (and straightforward to check) that

$$\underset{\approx}{A}^{-1}\underset{\sim}{\tau} = \frac{E}{1+\nu}\left(\underset{\sim}{\tau} + \frac{\nu}{1-\nu}\,\mathrm{tr}(\underset{\sim}{\tau})\underset{\sim}{\delta}\right).$$

Let

$$\underline{f}^k(\underset{\sim}{x}^\varepsilon) = \varepsilon^{-1}\int_{-\varepsilon}^{\varepsilon} \underline{f}^\varepsilon(\underset{\sim}{x}^\varepsilon, x_3^\varepsilon) Q_k(x_3^\varepsilon)\, dx_3^\varepsilon,$$

$$\underline{g}^0(\underset{\sim}{x}^\varepsilon) = \frac{1}{2}\big[\underline{g}^\varepsilon(\underset{\sim}{x}^\varepsilon, \varepsilon) + \underline{g}^\varepsilon(\underset{\sim}{x}^\varepsilon, -\varepsilon)\big], \quad \underline{g}^1(\underset{\sim}{x}^\varepsilon) = \frac{1}{2}\big[\underline{g}^\varepsilon(\underset{\sim}{x}^\varepsilon, \varepsilon) - \underline{g}^\varepsilon(\underset{\sim}{x}^\varepsilon, -\varepsilon)\big].$$

The constants

$$\lambda = \frac{E}{2(1+\nu)}, \quad c_1 = \frac{-E\nu^2}{12(1-\nu^2)(2\nu-1)}, \quad c_2 = \frac{2(1-\nu)}{\nu},$$

will appear in what follows.

The $\mathrm{HR}_1(1)$ **model**. We first present its final form and then show how to derive it. Writing the model solution as

$$\begin{aligned} \underline{u}(\underline{x}^\varepsilon) &= \begin{pmatrix} \underset{\sim}{\eta}(\underset{\sim}{x}^\varepsilon) \\ 0 \end{pmatrix} + \begin{pmatrix} -\underset{\sim}{\varphi}(\underset{\sim}{x}^\varepsilon)x_3^\varepsilon \\ \omega(\underset{\sim}{x}) \end{pmatrix}, \\ \underline{\underline{\sigma}}(p)(\underline{x}^\varepsilon) &= \begin{pmatrix} \underset{\approx}{\sigma}^0(\underset{\sim}{x}^\varepsilon) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \underset{\approx}{\sigma}^1(\underset{\sim}{x}^\varepsilon)x_3^\varepsilon & \underset{\sim}{\sigma}^0(\underset{\sim}{x}^\varepsilon) \\ (\underset{\sim}{\sigma}^0)^T(\underset{\sim}{x}^\varepsilon) & 0 \end{pmatrix}, \end{aligned} \tag{7.1.5.}$$

Then for the stretching part we have that $\underset{\sim}{\eta}$ satisfies the membrane equation

$$\begin{aligned} -\varepsilon \underset{\sim}{\operatorname{div}} \underset{\approx}{A}^{-1} \underset{\approx}{e}(\underset{\sim}{\eta}) &= \frac{\varepsilon}{2}\underset{\sim}{f}^0 + \underset{\sim}{g}^0 \qquad \text{in } \Omega, \\ \underset{\sim}{\eta} &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{7.1.6}$$

After determining $\underset{\sim}{\eta}$, we are able to find the in-plane components of the stress from

$$\underset{\approx}{\sigma}^0 = \underset{\approx}{A}^{-1}\underset{\approx}{e}(\underset{\sim}{\eta}). \tag{7.1.7}$$

Concerning the bending part, $\underset{\sim}{\varphi}$ and ω solve the Reissner–Mindlin equation with shear correction factor 1:

$$-\frac{\varepsilon^3}{3}\underset{\sim}{\operatorname{div}} \underset{\approx}{A}^{-1}\underset{\approx}{e}(\underset{\sim}{\varphi}) + \varepsilon\lambda(\underset{\sim}{\varphi} - \underset{\sim}{\nabla}\,\omega) = -\varepsilon\big(\frac{1}{2}\underset{\sim}{f}^1 + \underset{\sim}{g}^1\big) \qquad \text{in } \Omega, \tag{7.1.8}$$

$$\varepsilon\lambda \operatorname{div}(\underset{\sim}{\varphi} - \underset{\sim}{\nabla}\,\omega) = \frac{\varepsilon}{2}f_3^0 + g_3^0 \qquad \text{in } \Omega, \tag{7.1.9}$$

$$\underset{\sim}{\varphi} = 0, \quad \omega = 0 \qquad \text{on } \partial\Omega.$$

The in-plane and shear stress components are found from

$$\underset{\approx}{\sigma}^1 = -\underset{\approx}{A}^{-1}\underset{\approx}{e}(\underset{\sim}{\varphi}), \quad \underset{\sim}{\sigma}^0 = \lambda(-\underset{\sim}{\varphi} + \underset{\sim}{\nabla}\,\omega). \tag{7.1.10}$$

We deduce the above equations by assuming (7.1.5) and using (7.1.3), (7.1.4). Considering the stretching problem first, we use test functions of the form

$$\underline{\underline{\tau}}(\underline{x}^\varepsilon) = \begin{pmatrix} \underset{\approx}{\tau}(\underset{\sim}{x}^\varepsilon) & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{where } \underset{\approx}{\tau} \in \underset{\approx}{L}^2(\Omega),$$

in (7.1.3). From

$$\underline{\underline{A\tau}} = \begin{pmatrix} \underset{\approx}{A}\underset{\sim}{\tau} & 0 \\ \times & -\frac{\nu}{E}\,\mathrm{tr}(\underset{\approx}{\tau}) \end{pmatrix},$$

and an integration in the vertical direction, equation (7.1.7) holds. Similarly, if we substitute $\underline{v}^T(\underline{x}^\varepsilon) = (\underset{\sim}{v}^T, 0)(\underset{\sim}{x}^\varepsilon)$, where $\underset{\sim}{v} \in \underset{\sim}{\mathring{H}}^1(\Omega)$, in (7.1.4) and integrate in the vertical direction, then

$$\int_\Omega \underset{\approx}{\sigma}^0 : \underset{\approx}{e}(\underset{\sim}{v})\, d\underset{\sim}{x}^\varepsilon = \int_\Omega (\frac{1}{2}\underset{\sim}{f}^0 + \varepsilon^{-1}\underset{\sim}{g}^0) \cdot \underset{\sim}{v}\, d\underset{\sim}{x}^\varepsilon \qquad \text{for all } \underset{\sim}{v} \in \underset{\sim}{\mathring{H}}^1(\Omega). \tag{7.1.11}$$

Equation (7.1.6) follows from (7.1.11) after an integration by parts and from (7.1.7).

The procedure to realize the bending part of $\mathrm{HR}_1(1)$ is basically the same. Assuming the test functions in (7.1.3) to be of the form

$$\underline{\underline{\tau}}(\underline{x}^\varepsilon) = \begin{pmatrix} \underset{\approx}{\tau}(\underset{\sim}{x}^\varepsilon)x_3^\varepsilon & \underset{\sim}{\tau}(\underset{\sim}{x}^\varepsilon) \\ \underset{\sim}{\tau}^T(\underset{\sim}{x}^\varepsilon) & 0 \end{pmatrix}, \qquad \text{where } \underset{\approx}{\tau} \in \underset{\approx}{L}^2(\Omega) \text{ and } \underset{\sim}{\tau} \in \underset{\sim}{L}^2(\Omega),$$

and using that

$$\underline{\underline{A\tau}} = \begin{pmatrix} \underset{\approx}{A}\underset{\sim}{\tau}x_3^\varepsilon & \frac{1+\nu}{E}\underset{\sim}{\tau} \\ \frac{1+\nu}{E}\underset{\sim}{\tau}^T & -\frac{\nu}{E}\,\mathrm{tr}(\underset{\approx}{\tau})x_3^\varepsilon \end{pmatrix},$$

we see that (7.1.10) follows. Next we use $\underline{v}^T(\underline{x}^\varepsilon) = (\underset{\sim}{v}^T(\underset{\sim}{x}^\varepsilon)x_3^\varepsilon, 0)$ with $\underset{\sim}{v} \in \underset{\sim}{\mathring{H}}^1(\Omega)$ as test functions in (7.1.4), and then

$$\int_\Omega \frac{2\varepsilon^3}{3}\underset{\approx}{\sigma}^1 : \underset{\approx}{e}(\underset{\sim}{v}) + 2\varepsilon\underset{\sim}{\sigma}^0 \cdot \underset{\sim}{v}\, d\underset{\sim}{x}^\varepsilon = \int_\Omega (\varepsilon\underset{\sim}{f}^1 + 2\varepsilon\underset{\sim}{g}^1) \cdot \underset{\sim}{v}\, d\underset{\sim}{x}^\varepsilon \qquad \text{for all } \underset{\sim}{v} \in \underset{\sim}{\mathring{H}}^1(\Omega). \tag{7.1.12}$$

Substituting (7.1.10) and integrating by parts yields (7.1.8). We assume then that $\underline{v}^T(\underline{x}^\varepsilon) = (0, v(\underset{\sim}{x}^\varepsilon))$ where $v \in \mathring{H}^1(\Omega)$ and (7.1.4) yields

$$\int_\Omega 2\varepsilon\underset{\sim}{\sigma}^0 \cdot \underset{\sim}{\nabla}\, v\, d\underset{\sim}{x}^\varepsilon = \int_\Omega (\varepsilon f_3^0 + 2g_3^0)v\, d\underset{\sim}{x}^\varepsilon \qquad \text{for all } v \in \mathring{H}^1(\Omega). \tag{7.1.13}$$

Finally, using (7.1.10), we see that (7.1.9) holds.

The derivation for other models is analogous and become tedious as p increases. We present then the final equations for few of them.

Since the $\mathrm{HR}_2(1)$ model is not consistent, we choose to not present the final equations. See [2] instead. Nonetheless, it is worthwhile to mention that the same spurious mode that appears in the bending part also shows up in the stretching situation.

The $\mathrm{HR}_3(1)$ **Model**. It is the simplest consistent minimum energy model. Writing the solutions as

$$\underline{u}(\underline{x}^\varepsilon) = \begin{pmatrix} \underset{\sim}{\eta}(\underset{\sim}{x}^\varepsilon) \\ \omega(\underset{\sim}{x}^\varepsilon)x_3^\varepsilon \end{pmatrix} + \begin{pmatrix} -\underset{\sim}{\varphi}(\underset{\sim}{x}^\varepsilon)x_3^\varepsilon \\ \omega^0(\underset{\sim}{x}^\varepsilon) + \omega^2(\underset{\sim}{x}^\varepsilon)Q_2(x_3^\varepsilon) \end{pmatrix},$$

$$\underline{\sigma}(\underline{x}^\varepsilon) = \begin{pmatrix} \underset{\approx}{\sigma}^0(\underset{\sim}{x}^\varepsilon) & \underset{\sim}{\sigma}^1(\underset{\sim}{x}^\varepsilon)x_3^\varepsilon \\ (\underset{\sim}{\sigma}^1)^T(\underset{\sim}{x}^\varepsilon)x_3^\varepsilon & \sigma_{33}^0(\underset{\sim}{x}^\varepsilon) \end{pmatrix}$$
$$+ \begin{pmatrix} \underset{\approx}{\sigma}^1(\underset{\sim}{x}^\varepsilon)\underset{\sim}{x}^\varepsilon & \underset{\sim}{\sigma}^0(\underset{\sim}{x}^\varepsilon) + \underset{\sim}{\sigma}^2(\underset{\sim}{x}^\varepsilon)Q_2(x_3^\varepsilon) \\ (\underset{\sim}{\sigma}^0)^T(\underset{\sim}{x}^\varepsilon) + (\underset{\sim}{\sigma}^2)^T(\underset{\sim}{x}^\varepsilon)Q_2(x_3^\varepsilon) & \sigma_{33}^1(\underset{\sim}{x}^\varepsilon)x_3^\varepsilon \end{pmatrix},$$

then we have for the stretching part that

$$-\varepsilon \,\underset{\sim}{\operatorname{div}}\, \underset{\approx}{A}^{-1}\underset{\approx}{e}(\underset{\sim}{\eta}) - 12\varepsilon c_1\, \underset{\sim}{\nabla}(\operatorname{div}\underset{\sim}{\eta} + \frac{c_2}{2}\omega) = \frac{\varepsilon}{2}\underset{\sim}{f}^0 + \underset{\sim}{g}^0 \qquad \text{in } \Omega,$$
$$6c_1c_2(\operatorname{div}\underset{\sim}{\eta} + \frac{c_2}{2}\omega) - \frac{\varepsilon^2}{3}\lambda\,\Delta\,\omega = \frac{1}{2}f_3^1 + g_3^1 \qquad \text{in } \Omega,$$
$$\underset{\sim}{\eta} = 0, \quad \omega = 0 \qquad \text{on } \partial\Omega.$$

Note that this model takes into account the transverse components of the load that also contribute for the stretching. These terms are not present in the $\mathrm{HR}_1(1)$ model or in the membrane equation coming from asymptotic methods. The stress components for stretching come from substituting

$$\underset{\approx}{\sigma}^0 = \underset{\approx}{A}^{-1}\underset{\approx}{e}(\underset{\sim}{\eta}) + 12c_1(\operatorname{div}\underset{\sim}{\eta} + \frac{c_2}{2}\omega)\underset{\approx}{\delta},$$
$$\underset{\sim}{\sigma}^1 = \lambda\,\underset{\sim}{\nabla}\,\omega, \quad \sigma_{33}^0 = 6c_1c_2(\operatorname{div}\underset{\sim}{\eta} + \frac{c_2}{2}\omega).$$

The displacement components under bending solve

$$-\frac{\varepsilon^3}{3}\,\underset{\sim}{\operatorname{div}}\,\underset{\approx}{A}^{-1}\underset{\approx}{e}(\underset{\sim}{\varphi}) - 4\varepsilon^3\,\underset{\sim}{\nabla}\,c_1(\operatorname{div}\underset{\sim}{\varphi} - \frac{3}{2}c_2\omega^2) + \varepsilon\lambda(\underset{\sim}{\varphi} - \underset{\sim}{\nabla}\,\omega^0) = -\varepsilon(\frac{1}{2}\underset{\sim}{f}^1 + \underset{\sim}{g}^1),$$

$$\varepsilon\lambda\operatorname{div}(\underset{\sim}{\varphi} - \underset{\sim}{\nabla}\,\omega^0) = \frac{\varepsilon}{2}f_3^0 + g_3^0,$$

$$\frac{\varepsilon^2}{30}\lambda\operatorname{div}\underset{\sim}{\nabla}\,\omega^2 + c_1c_2(\operatorname{div}\underset{\sim}{\varphi} - \frac{3}{2}c_2\omega^2) = \frac{\varepsilon^{-2}}{6}(-f_3^2 - \varepsilon g_3^0).$$

Here we have a set of equations that are more complex than $\mathrm{HR}_1(1)$ but also include the second moment of f_3.

The equations below yield the stress components:

$$\underset{\approx}{\sigma}^1 = -\underset{\approx}{A}^{-1}\underset{\approx}{e}(\underset{\sim}{\varphi}) - 12c_1(\operatorname{div}\underset{\sim}{\varphi} - \frac{3}{2}c_2\omega^2)\underset{\approx}{\delta},$$

$$\underset{\sim}{\sigma}^0 = \lambda(-\underset{\sim}{\varphi} + \underset{\sim}{\nabla}\,\omega^0), \quad \underset{\sim}{\sigma}^2 = \lambda\,\underset{\sim}{\nabla}\,\omega^2,$$

$$\sigma_{33}^1 = -6c_1c_2(\operatorname{div}\underset{\sim}{\varphi} - \frac{3}{2}c_2\omega^2).$$

Section 7.2 – The HR$'$ *models.* Another way to characterize the solution of (7.1) is by the *second Hellinger–Reissner principle*, or HR$'$ for short.

Define

$$\underline{V}'(P^\varepsilon) = \underline{L}^2(P^\varepsilon), \quad \underline{\underline{S}}'_{\underline{g}}(P^\varepsilon) = \{\,\underline{\underline{\tau}} \in \underline{\underline{H}}(\operatorname{div}, P^\varepsilon) \,:\, \underline{\underline{\tau}}\,\underline{n} = \underline{g} \text{ on } \partial P^\varepsilon_\pm\,\}.$$

Then we have

HR$'$: $(\underline{u}^\varepsilon, \underline{\underline{\sigma}}^\varepsilon)$ is the unique critical point of

$$L'(\underline{v}, \underline{\underline{\tau}}) = \frac{1}{2}\int_{P^\varepsilon} \underline{\underline{\underline{\underline{A}}}}\,\underline{\underline{\tau}} \,:\, \underline{\underline{\tau}}\,d\underline{x}^\varepsilon + \int_{P^\varepsilon} \underline{\operatorname{div}}\,\underline{\underline{\tau}} \cdot \underline{v}\,d\underline{x}^\varepsilon + \int_{P^\varepsilon} \underline{f}^\varepsilon \cdot \underline{v}\,d\underline{x}^\varepsilon$$

on $\underline{V}'(P^\varepsilon) \times \underline{\underline{S}}'_{\underline{g}}(P^\varepsilon)$.

An equivalent statement is that $\underline{u}^\varepsilon \in \underline{V}'(P^\varepsilon)$ and $\underline{\underline{\tau}} \in \underline{\underline{S}}'_{\underline{g}}(P^\varepsilon)$ satisfy

$$\int_{P^\varepsilon} \underline{\underline{\underline{\underline{A}}}}\,\underline{\underline{\sigma}}^\varepsilon : \underline{\underline{\tau}}\,d\underline{x} + \int_{P^\varepsilon} \underline{u}^\varepsilon \cdot \underline{\operatorname{div}}\,\underline{\underline{\tau}}\,d\underline{x}^\varepsilon = 0 \qquad \text{for all } \underline{\underline{\tau}} \in \underline{\underline{S}}'_0(P^\varepsilon),$$

$$\int_{P^\varepsilon} \underline{\operatorname{div}}\,\underline{\underline{\sigma}}^\varepsilon \cdot \underline{v}\,d\underline{x}^\varepsilon = \int_{P^\varepsilon} -\underline{f} \cdot \underline{v}\,d\underline{x}^\varepsilon \qquad \text{for all } \underline{v} \in \underline{V}'(P^\varepsilon).$$

By seeking a critical point for L' on subspaces $\underline{V}'(P^\varepsilon, p) \times \underline{\underline{S}}'_g(P^\varepsilon, p) \subset \underline{V}'(P^\varepsilon) \times \underline{\underline{S}}'_g(P^\varepsilon)$, we define classes of HR$'$ models. The elements of these subspaces will have certain polynomial dependence in the transverse direction, and we will specify four differents classes of HR$'$ models in the table below.

TABLE 7.2. HR$'$ Plate models.

model	$\deg_3 \underset{\approx}{\sigma}$	$\deg_3 \underset{\sim}{\sigma}$	$\deg_3 \sigma_{33}$	$\deg_3 \underset{\sim}{u}$	$\deg_3 u_3$
$\mathrm{HR}'_1(p)$	p	$p-1$	p	p	$p-1$
$\mathrm{HR}'_2(p)$	p	$p+1$	p	p	$p-1$
$\mathrm{HR}'_3(p)$	p	$p+1$	p	p	$p+1$
$\mathrm{HR}'_4(p)$	p	$p+1$	$p+2$	p	$p+1$

For pure bending and p odd, Alessandrini [1] denoted $\mathrm{HR}'_1(p)$ by HR p.0, $\mathrm{HR}'_3(p)$ by HR $p+1.0$, and $\mathrm{HR}'_4(p)$ by HR $p+1.1$. A nice feature of some of the above models is that $\underline{\operatorname{div}}\, \underline{\underline{S}}'_0(P^\varepsilon, p) = \underline{V}'(P^\varepsilon, p)$ and therefore, not only

$$\underline{\operatorname{div}}\, \underline{\underline{\sigma}}^\varepsilon(p) = -\pi_{\underline{V}'} \underline{f}^\varepsilon,$$

where $\pi_{\underline{V}'} \underline{f}^\varepsilon$ is the orthogonal L^2 projection of $\underline{f}^\varepsilon$ into $\underline{V}'(P^\varepsilon, p)$, but also $\underline{\underline{\sigma}}^\varepsilon(p)$ minimizes the complementary energy

$$J_c(\underline{\underline{\tau}}) = \frac{1}{2} \int_{P^\varepsilon} \underline{\underline{\underline{\underline{A}}}}\,\underline{\underline{\tau}} : \underline{\underline{\tau}}\, d\underline{x}^\varepsilon$$

over all $\underline{\underline{\tau}} \in S'_g(P^\varepsilon, p)$ such that $\underline{\operatorname{div}}\, \underline{\underline{\tau}} = -\pi_{\underline{V}'} \underline{f}^\varepsilon$.

We summarize next, for $p = 1$, some of the HR$'$ models. To derive the equations of a particular model, we proceed as in Section 7.1. See also [2], where the equations for $\mathrm{HR}'_4(1)$ are found explicitly. As the $\mathrm{HR}'_1(1)$, and $\mathrm{HR}'_3(1)$ models are not consistent we do not show them here.

The $\mathrm{HR}'_2(1)$ **Model**. Assume that the displacement

$$\underline{u}(\underline{x}^\varepsilon) = \begin{pmatrix} \underset{\sim}{\eta}(\underset{\sim}{x}^\varepsilon) \\ 0 \end{pmatrix} + \begin{pmatrix} -\underset{\sim}{\varphi}(\underset{\sim}{x}^\varepsilon) x_3^\varepsilon \\ \omega(\underset{\sim}{x}^\varepsilon) \end{pmatrix},$$

and the stress

$$\underline{\sigma}(\underline{x}^{\varepsilon}) = \begin{pmatrix} \underset{\approx}{\sigma}^0(\underset{\sim}{x}^{\varepsilon}) & \varepsilon^{-1}\underset{\sim}{g}^0 x_3^{\varepsilon} \\ \varepsilon^{-1}(\underset{\sim}{g}^0)^T x_3^{\varepsilon} & g_3^1 \end{pmatrix} + \begin{pmatrix} \underset{\approx}{\sigma}^1(\underset{\sim}{x}^{\varepsilon}) x_3^{\varepsilon} & \underset{\sim}{\sigma}^0(\underset{\sim}{x}^{\varepsilon})[1-\varepsilon^{-2}Q_2(x_3^{\varepsilon})] + \underset{\sim}{g}^1 \\ (\underset{\sim}{\sigma}^0)^T(\underset{\sim}{x}^{\varepsilon})[1-\varepsilon^{-2}Q_2(x_3^{\varepsilon})] + (\underset{\sim}{g}^1)^T & \varepsilon^{-1} g_3^0 x_3^{\varepsilon} \end{pmatrix}.$$

Then, the equations defining the first components of the displacement for the stretching part are

$$-\varepsilon \underset{\sim}{\operatorname{div}} \underset{\approx}{A}^{-1} \underset{\approx}{e}(\underset{\sim}{\eta}) = \frac{\varepsilon}{2}\underset{\sim}{f}^0 + \underset{\sim}{g}^0 + \varepsilon \frac{\nu}{1-\nu} \underset{\sim}{\nabla}\, g_3^1 \qquad \text{in } \Omega,$$
$$\underset{\sim}{\eta} = 0 \text{ on } \partial\Omega.$$

Note that here we have basically the same equations as in $\mathrm{HR}_1(1)$ plus a term taking into account the contributions of g_3^1. It is easy to compute the in-plane stress components by substituting

$$\underset{\approx}{\sigma}^0 = \underset{\approx}{A}^{-1} \underset{\approx}{e}(\underset{\sim}{\eta}) + \frac{\nu}{1-\nu} g_3^1 \underset{\approx}{\delta}.$$

For the bending part we have that

$$-\frac{\varepsilon^3}{3} \underset{\sim}{\operatorname{div}} \underset{\approx}{A}^{-1} \underset{\approx}{e}(\underset{\sim}{\varphi}) + \frac{5}{6}\varepsilon\lambda(\underset{\sim}{\varphi} - \underset{\sim}{\nabla}\,\omega) = -\varepsilon(\frac{1}{2}\underset{\sim}{f}^1 + \frac{5}{6}\underset{\sim}{g}^1) - \frac{\nu}{3(1-\nu)}\varepsilon^2 \underset{\sim}{\nabla}\, g_3^0 \qquad \text{in } \Omega,$$
$$\frac{5}{6}\varepsilon\lambda \operatorname{div}(\underset{\sim}{\varphi} - \underset{\sim}{\nabla}\,\omega) = \frac{\varepsilon}{2} f_3^0 + g_3^0 + \frac{\varepsilon}{6}\operatorname{div} \underset{\sim}{g}^1 \qquad \text{in } \Omega,$$
$$\underset{\sim}{\varphi} = 0, \quad \omega = 0 \qquad \text{on } \partial\Omega.$$

This time we find the Reissner–Mindlin model with shear correction factor 5/6.

The stress components can be found by substituting

$$\underset{\approx}{\sigma}^1 = -\underset{\approx}{A}^{-1} \underset{\approx}{e}(\underset{\sim}{\varphi}) + \frac{\nu}{1-\nu}\varepsilon^{-1} g_3^0 \underset{\approx}{\delta}, \quad \underset{\sim}{\sigma}^0 = \frac{5}{6}[\lambda(-\underset{\sim}{\varphi} + \underset{\sim}{\nabla}\,\omega) - \underset{\sim}{g}^1].$$

The $\mathrm{HR}_4'(1)$ **Model**. We look for displacement solution in the form:

$$\underline{u}(\underline{x}^{\varepsilon}) = \begin{pmatrix} \underset{\sim}{\eta}(\underset{\sim}{x}^{\varepsilon}) \\ \omega(\underset{\sim}{x}^{\varepsilon}) x_3^{\varepsilon} \end{pmatrix} + \begin{pmatrix} -\underset{\sim}{\varphi}(\underset{\sim}{x}^{\varepsilon}) x_3^{\varepsilon} \\ \omega^0(\underset{\sim}{x}^{\varepsilon}) + \omega^2(\underset{\sim}{x}^{\varepsilon})[Q_2(\underset{\sim}{x}^{\varepsilon}) + \frac{\varepsilon^2}{5}] \end{pmatrix},$$

and the stress

$$\underline{\sigma}(\underline{x}^\varepsilon) = \begin{pmatrix} \underset{\approx}{\sigma}^0(\underset{\sim}{x}^\varepsilon) & \varepsilon^{-1}\underset{\sim}{g}^0 x_3^\varepsilon \\ \varepsilon^{-1}(\underset{\sim}{g}^0)^T x_3^\varepsilon & (1-\varepsilon^{-2}Q_2(x_3^\varepsilon))\sigma_{33}^0(\underset{\sim}{x}^\varepsilon) + g_3^1 \end{pmatrix}$$
$$+ \begin{pmatrix} \underset{\approx}{\sigma}^1(\underset{\sim}{x}^\varepsilon)x_3^\varepsilon & \underset{\sim}{\sigma}^0(\underset{\sim}{x}^\varepsilon)(1-\varepsilon^{-2}Q_2(x_3^\varepsilon)) + \underset{\sim}{g}^1 \\ (\underset{\sim}{\sigma}^0)^T(\underset{\sim}{x}^\varepsilon)(1-\varepsilon^{-2}Q_2(x_3^\varepsilon)) + (\underset{\sim}{g}^1)^T & \sigma_{33}^1(\underset{\sim}{x}^\varepsilon)(x_3^\varepsilon - \varepsilon^{-2}Q_3(x_3^\varepsilon)) + \varepsilon^{-1}g_3^0 x_3^\varepsilon \end{pmatrix}.$$

The equations defining the first two displacement components for the pure stretching case are

$$-\varepsilon \underset{\sim}{\operatorname{div}} \underset{\approx}{A}^{-1} \underset{\approx}{e}(\underset{\sim}{\eta}) = \frac{\varepsilon}{2}\underset{\sim}{f}^0 + \underset{\sim}{g}^0 + \varepsilon\frac{\nu}{1-\nu}\underset{\sim}{\nabla}\Big(\frac{\varepsilon}{3}\operatorname{div}\underset{\sim}{g}^0 + \frac{1}{2}f_3^1 + g_3^1\Big) \qquad \text{in } \Omega,$$
$$\underset{\sim}{\eta} = 0 \qquad \text{on } \partial\Omega.$$

Next we can compute the other unknowns by substitution.

$$\underset{\approx}{\sigma}^0 = \underset{\approx}{A}^{-1}\underset{\approx}{e}(\underset{\sim}{\eta}) + \frac{\nu}{1-\nu}\left(\frac{\varepsilon}{3}\operatorname{div}\underset{\sim}{g}^0 + \frac{1}{2}f_3^1 + g_3^1\right)\underset{\approx}{\delta},$$
$$\sigma_{33}^0 = \frac{\varepsilon}{3}\operatorname{div}\underset{\sim}{g}^0 + \frac{1}{2}f_3^1,$$
$$\omega = \frac{1}{E}[-\nu\operatorname{tr}(\underset{\approx}{\sigma}^0) + \frac{6}{5}\sigma_{33}^0 + g_3^1].$$

For pure bending,

$$-\frac{\varepsilon^3}{3}\underset{\sim}{\operatorname{div}}\underset{\approx}{A}^{-1}\underset{\approx}{e}(\underset{\sim}{\varphi}) + \frac{5}{6}\varepsilon\lambda(\underset{\sim}{\varphi} - \underset{\sim}{\nabla}\omega) = -\varepsilon\Big(\frac{1}{2}\underset{\sim}{f}^1 + \frac{5}{6}\underset{\sim}{g}^1\Big)$$
$$- \frac{\nu}{15(1-\nu)}\varepsilon^3\underset{\sim}{\nabla}\Big(\operatorname{div}\underset{\sim}{g}^1 + 6\varepsilon^{-1}g_3^0 + \frac{1}{2}f_3^0 + \frac{5}{2}\varepsilon^{-2}f_3^2\Big) \qquad \text{in } \Omega,$$
$$\frac{5}{6}\varepsilon\lambda\operatorname{div}(\underset{\sim}{\varphi} - \underset{\sim}{\nabla}\omega) = \frac{\varepsilon}{2}f_3^0 + g_3^0 + \frac{\varepsilon}{6}\operatorname{div}\underset{\sim}{g}^1 \qquad \text{in } \Omega,$$
$$\underset{\sim}{\varphi} = 0 \quad \omega = 0 \qquad \text{on } \partial\Omega.$$

This is again a Reissner–Mindlin model, with shear correction factor of 5/6. Additional moments of the load are taken into account. Compare with the $\mathrm{HR}_1(1)$ and $\mathrm{HR}_2'(1)$ models.

For the other unknowns,

$$
\begin{gathered}
\underset{\approx}{\sigma}^1 = -A^{-1}\underset{\approx}{e}(\underset{\sim}{\varphi}) + \frac{\nu}{5(1-\nu)}(\operatorname{div}\underset{\sim}{g}^1 + 6\varepsilon^{-1}g_3^0 + \frac{1}{2}f_3^0 + \frac{5}{2}\varepsilon^{-2}f_3^2)\underset{\approx}{\delta}, \\
\underset{\sim}{\sigma}^0 = \frac{5}{6}\lambda(-\underset{\sim}{\varphi} + \underset{\sim}{\nabla}\,\omega) - \frac{5}{6}\underset{\sim}{g}^1, \quad \sigma_{33}^1 = \frac{1}{5}(\operatorname{div}\underset{\sim}{g}^1 + \varepsilon^{-1}g_3^0 + \frac{1}{2}f_3^0 + \frac{5}{2}\varepsilon^{-2}f_3^2), \\
\omega_2 = -\frac{\nu}{3E}\operatorname{tr}(\underset{\approx}{\sigma}^1) + \frac{10}{21E}\sigma_{33}^1 + \frac{\varepsilon^{-1}}{3E}g_3^0.
\end{gathered}
$$

Chapter 8

Concluding discussion

In this dissertation we proposed and applied a technique to estimate approximation properties of various variational models. The results obtained here are more general than previous ones, and take into account, simultaneously, both the thickness of the domain and the order of the model. Using our own notation, we describe next in more details some works that are closely related to ours.

In the first paper of a series of three, Vogelius and Babuška [56] analyze the convergence of minimum energy models for scalar elliptic problems in a multidimensional plate P^ε with thickness ε. They consider the homogeneous problem ($f^\varepsilon = 0$) with Neumann boundary condition ($g^\varepsilon \neq 0$) on the top and bottom of the plate and zero Dirichlet condition on the lateral side $\partial P^\varepsilon_\pm$, and project the exact solution into a subspace V_N that is not always polynomial, but depends on the coefficients of the problem. If the boundary layer is disregarded, a N-term truncated asymptotic expansion $S_N(u^\varepsilon)$ belongs to V_{2N}. So, using the fact that the model minimizes energy, they estimate the modeling error as follows:

$$\|u^\varepsilon - u^\varepsilon_{2N}\|_E \leq \inf_{v \in V_{2N}} \|u^\varepsilon - v\|_E \leq \|u^\varepsilon - S_N(u^\varepsilon)\|_E, \tag{8.1}$$

where $\|\cdot\|_E$ denotes the energy norm and $u^\varepsilon_{2N} \in V_{2N}$ denotes the model solution. They then bound the right hand side of (8.1).

There are some drawbacks in this approach. First of all, if we were to consider a general function $f^\varepsilon \neq 0$, then the subspace V_N would depend on f^ε as well. This, as we point out in the introduction, occurs in a generalization for linearly elastic plates by Miara [42], where the subspaces depend on the loads. Secondly, it is not clear how to derive sharp bounds in norms other than the energy norm, or how to estimate variational models that are not of minimum energy type. Finally, (8.1) disregards the influence of N on the rate of convergence. In a second paper [57], they study in details the approximation properties of the space V_{2N} (with $\varepsilon = 1$) and are able to show that $\|u^\varepsilon - u^\varepsilon_{2N}\|_E$ converges to zero with N. Note however that the convergence in ε and in N have to considered separately.

The same difficulties arise in the Ph.D. dissertation of Schwab [49] and in a paper of Ovaskainen and Pitkäranta [46]. Schwab starts his thesis by studying minimum energy models for the Laplace problem in a multidimensional plate. He first analyzes the problem in the semi-infinite "plate" $\mathbb{R} \times (-\varepsilon, \varepsilon)$, thus avoiding the boundary layers. The way he obtained his estimates is essentially the same as described above. He considers next a bounded plate and estimates the modeling error by comparing the difference between the solutions of the unbounded and bounded problems. Finally he investigates linear elasticity problems in the unbounded domains $\mathbb{R} \times (-\varepsilon, \varepsilon)$ and $\mathbb{R}^2 \times (-\varepsilon, \varepsilon)$.

Ovaskainen and Pitkäranta analyze the minimum energy models for a thin linearly elastic strip under traction. In this work, they also use a truncated asymptotic expansion to derive upper bounds for the modeling error.

In our work with Alessandrini et al. [2] we proceeded in a completely different way. We analyzed the $\mathrm{HR}'_4(1)$ modeling error for a linearly elastic clamped plate. The analysis was based on the two energies principle (or Prager–Synge theorem). The HR' models are amenable to this kind of approach, as the stress tensor satisfies the equilibrium equation (for simple loads) and the traction boundary conditions, by construction. To apply the two energies principle, one has only to correct the trace of the displacement at the lateral boundary. The final result is a $O(\varepsilon^{1/2})$ convergence rate in the relative energy norm, for a variety of loads and tractions. It is not clear how to extend these ideas to the HR models nor how to take into account the influence of p in the convergence. And again bounds in norms other than the energy norm seem out of reach.

We consider the strategy presented in this thesis quite flexible, and free of the above mentioned disadvantages. Previous approaches could only estimate minimum energy and HR$'$ models in the energy norm. In contrast, we are able to analyze models that are *not* of minimum energy or HR$'$ type, and we can bound the modeling error in various norms. Also, we obtain estimates which simultaneously show the effect of ε and p. The method requires a detailed and rigorous asymptotic analysis, but when this is known, as it is for the Poisson and the linearly elastic plate problems, the method is relatively simple to apply and leads to sharp, transparent estimates.

Appendix A

Projection operators

We discuss here some properties of polynomial approximation, and the results of this appendix are used in Chapters 3 and 6, and also in Appendix B. Our main reference is the article by Bernardi and Maday [14]. We start by recalling the inverse inequality for polynomials.

Lemma A.1 (Inverse inequality). *For any integer n and real number r such that $0 \leq n \leq r$, there exists a constant C such that*

$$\|v\|_{H^r(-1,1)} \leq Cp^{2(r-n)}\|v\|_{H^n(-1,1)}$$

for every $v \in \mathbb{P}_p$.

Recall that π_p denotes the L^2 orthogonal projection operator from $L^2(-1,1)$ to $\mathbb{P}_p(-1,1)$. Recall also that $\hat{\pi}_p^1$ denotes the orthogonal projection operator from $H^1(-1,1) \cap \hat{L}^2(-1,1)$ to $\hat{\mathbb{P}}_p(-1,1)$, and that $\mathring{\pi}^1$ is the orthogonal projection operator from $\mathring{H}^1(-1,1)$ to $\mathring{\mathbb{P}}_p(-1,1)$, both with respect to the inner product that induces the norm $|\cdot|_{H^1(-1,1)}$.

The following result holds, [14], [17], [18].

Lemma A.2. *For any nonnegative real number s, there exists a constant C such that if $r \leq s$, then*

$$\|\varphi - \pi_p\varphi\|_{H^r(-1,1)} \leq \begin{cases} Cp^{(3r/2)-s}\|\varphi\|_{H^s(-1,1)} & \text{if } r < 1, \\ Cp^{2r-(1/2)-s}\|\varphi\|_{H^s(-1,1)} & \text{if } 1 \leq r, \end{cases}$$

for any $\varphi \in \|\varphi\|_{H^s(-1,1)}$. Also, for $0 \leq r \leq 1 \leq s$, there exists a constant C such that

$$\|\varphi - \hat{\pi}_p^1\varphi\|_{H^r(-1,1)} \leq Cp^{r-s}\|\varphi\|_{H^s(-1,1)} \qquad \text{for } \varphi \in H^s(-1,1) \cap \hat{L}^2(-1,1),$$

$$\|\varphi - \mathring{\pi}_p^1\varphi\|_{H^r(-1,1)} \leq Cp^{r-s}\|\varphi\|_{H^s(-1,1)} \qquad \text{for } \varphi \in H^s(-1,1) \cap \mathring{H}^1(-1,1).$$

We prove now the boundedness of some projection operators.

Lemma A.3. *For any nonnegative real number r, there exists a constant C such that for $\varphi \in L^2(-1,1)$,*

$$\|\pi_p \varphi\|_{H^r(-1,1)} \le \begin{cases} C\|\varphi\|_{H^{3r/2}(-1,1)} & \text{if } r < 1, \\ C\|\varphi\|_{H^{2r-1/2}(-1,1)} & \text{if } 1 \le r, \end{cases}$$

and for $\varphi \in H^1(-1,1) \cap \hat{L}^2(-1,1)$,

$$\|\hat{\pi}_p^1 \varphi\|_{H^r(-1,1)} \le \begin{cases} C\|\varphi\|_{H^1(-1,1)} & \text{if } 0 \le r < 1, \\ C\|\varphi\|_{H^{3r/2-1/2}(-1,1)} & \text{if } 1 \le r < 2, \\ C\|\varphi\|_{H^{2r-3/2}(-1,1)} & \text{if } 2 \le r. \end{cases}$$

Proof. The bounds for π_p follow immediately from Lemma A.2, and we use these to find the bounds for $\hat{\pi}_p^1$. Indeed, if $2 \le r$, then

$$\begin{aligned} \|\hat{\pi}_p^1 \varphi\|_{H^r(-1,1)} \le C\|(\hat{\pi}_p^1 \varphi)'\|_{H^{r-1}(-1,1)} = C\|\pi_{p-1}\varphi'\|_{H^{r-1}(-1,1)} \\ \le C\|\varphi'\|_{H^{2r-5/2}(-1,1)} \le C\|\varphi\|_{H^{2r-3/2}(-1,1)}. \end{aligned}$$

The case $1 \le r < 2$ follows from similar arguments, and for $0 \le r < 1$, we employ Lemma A.2. $\square$

We again use the upper index $(\hat{\rho}_2)$ on the projector operators to indicate the action on the variable $\hat{\rho}_2$ only. The following error estimate holds.

Lemma A.4. *For any real number $s \ge 1$, there exists a constant C such that for any real numbers $0 \le a < b$,*

$$\|\varphi - \hat{\pi}_p^{1(\hat{\rho}_2)} \varphi\|_{L^2(\Sigma(a,b))} + \|\varphi - \mathring{\pi}_p^{(\hat{\rho}_2)} \varphi\|_{L^2(\Sigma(a,b))} \le C p^{-s} \|\varphi\|_{L^2((a,b);H^s(-1,1))}.$$

Proof. Note first from Lemma A.2 that for almost every $\hat{\rho}_1 \in \mathbb{R}^+$,

$$\|\varphi - \hat{\pi}_p^{1(\hat{\rho}_2)}\varphi\|_{L^2(\gamma_{\hat{\rho}_1})}\|\varphi - \mathring{\hat{\pi}}_p^{(\hat{\rho}_2)}\varphi\|_{L^2(\gamma_{\hat{\rho}_1})} \leq Cp^{-s}\|\varphi\|_{H^s(\gamma_{\hat{\rho}_1})},$$

and then we conclude the estimates by integrating in $\mathbb{R}^+$. $\square$

Appendix B

One-dimensional mixed approximations

We discuss here mixed approximations for some one-dimensional equations. The results of this appendix are used in Chapter 4.

Lemma B.1. *Given $u \in H^2(-1,1) \cap \hat{L}^2(-1,1)$ and $\sigma = u'$, there exists unique $u(p) \in \hat{\mathbb{P}}_p(-1,1)$ and $\sigma(p) \in \mathbb{P}_{p+1}(-1,1)$ with $\sigma(p)(-1) = \sigma(-1)$, and $\sigma(p)(1) = \sigma(1)$, such that*

$$\begin{aligned} \int_{-1}^{1} [\sigma - \sigma(p)]\tau + [u - u(p)]\tau' \, d\hat{\rho}_2 = 0 \qquad &\text{for all } \tau \in \mathring{\mathbb{P}}_{p+1}(-1,1), \\ \int_{-1}^{1} [\sigma - \sigma(p)]' v \, d\hat{\rho}_2 = 0 \qquad &\text{for all } v \in \hat{\mathbb{P}}_p(-1,1). \end{aligned} \tag{B.1}$$

Moreover, for any nonnegative real number s, there exists a constant C such that

$$\|u(p)\|_{L^2(-1,1)} + \|\sigma(p)\|_{H^1(-1,1)} \le C\|u\|_{H^2(-1,1)},$$

$$\|u - u(p)\|_{L^2(-1,1)} \le Cp^{-2-s}\|u\|_{H^{s+2}(-1,1)},$$

$$\|u - u(p)\|_{H^{1/2}(-1,1)} \le Cp^{-1-s}\|u\|_{H^{s+2}(-1,1)},$$

$$\|\sigma - \sigma(p)\|_{L^2(-1,1)} \le Cp^{-1-s}\|u\|_{H^{s+2}(-1,1)},$$

$$\|\sigma - \sigma(p)\|_{H^1(-1,1)} \le Cp^{-s}\|u\|_{H^{s+2}(-1,1)}.$$

Proof. Let $\tilde{\sigma}(\hat{\rho}_2) = (1/2)(\sigma(1) + \sigma(-1)) + (\hat{\rho}_2/2)(\sigma(1) - \sigma(-1))$, and define $\sigma_0(p) = \sigma(p) - \tilde{\sigma} \in \mathring{\mathbb{P}}_{p+1}$. Using the framework of the mixed methods, it is easy to show the well-posedness of problem (B.1) with $\sigma(p)$ replaced by $\sigma_0(p) + \tilde{\sigma}$. In fact, with the notation of Lemma 4.2.1, let

$$X = \mathring{\mathbb{P}}_{p+1}(-1,1), \quad M = \hat{\mathbb{P}}_p(-1,1),$$
$$a(\sigma,\tau) = \int_{-1}^{1} \sigma\tau \, d\hat{\rho}_2 \quad b(\tau, v) = \int_{-1}^{1} \tau' v \, d\hat{\rho}_2.$$

Since $\partial_2 X := \left\{\tau' \,:\, \tau \in X\right\} = M$, both the coercivity of $a(\cdot,\cdot)$ and the inf-sup condition are satisfied, so the existence, uniqueness and stability results hold for $u(p)$, $\sigma_0(p)$, and therefore for $\sigma(p)$ as well. We prove next the error estimates. From the second equation in (B.1), we see that $\sigma'(p) = \pi_p\sigma'$, and then, for $s \geq 0$, $\|\sigma' - \sigma'(p)\|_{L^2(-1,1)} \leq Cp^{-s}\|\sigma\|_{H^{s+1}(-1,1)}$. We proceed to prove the $L^2(-1,1)$ estimate of $\sigma - \sigma(p)$. First note that using (B.1) and Lemma 4.2.7,

$$\begin{aligned}
\int_{-1}^{1} [\sigma - \sigma(p)]\mathring{\pi}^1_{p+1}\sigma_0 \, d\hat{\rho}_2 &= -\int_{-1}^{1} [u - u(p)](\mathring{\pi}^1_{p+1}\sigma_0)' \, d\hat{\rho}_2 \\
= -\int_{-1}^{1} [u - u(p)]\pi_p\sigma_0' \, d\hat{\rho}_2 &= -\int_{-1}^{1} [u - u(p)]\sigma_0'(p) \, d\hat{\rho}_2 = \int_{-1}^{1} [\sigma - \sigma(p)]\sigma_0(p) \, d\hat{\rho}_2,
\end{aligned}$$

where $\sigma_0 = \sigma - \tilde{\sigma}$. Then

$$\begin{aligned}
\|\sigma - \sigma(p)\|^2_{L^2(-1,1)} &= \int_{-1}^{1} [\sigma - \sigma(p)](\sigma_0 - \mathring{\pi}^1_{p+1}\sigma_0) \, d\hat{\rho}_2 \\
&\leq \|\sigma - \sigma(p)\|_{L^2(-1,1)}\|\sigma_0 - \mathring{\pi}^1_{p+1}\sigma_0\|_{L^2(-1,1)} \\
&\leq Cp^{-1-s}\|\sigma - \sigma(p)\|_{L^2(-1,1)}\|\sigma_0\|_{H^{s+1}(-1,1)},
\end{aligned}$$

from Lemma A.2. We estimate next $u - u(p)$ in $L^2(-1,1)$ by a duality argument. Set $\hat{u} \in H^2(-1,1) \cap \hat{L}^2(-1,1)$ and $\hat{\sigma} = \hat{u}' \in \mathring{H}^1(-1,1)$ as the solution of $\hat{\sigma}' = \pi_p u - u(p)$. Then $\|\hat{u}\|_{H^2(-1,1)} + \|\hat{\sigma}\|_{H^1(-1,1)} \leq C\|\pi_p u - u(p)\|_{L^2(-1,1)}$, and from Lemma 4.2.7,

$$\begin{aligned}
\|\pi_p u - u(p)\|^2_{L^2(-1,1)} &= \int_{-1}^{1} [\pi_p u - u(p)]\hat{\sigma}' \, d\hat{\rho}_2 = \int_{-1}^{1} [\pi_p u - u(p)](\mathring{\pi}^1_{p+1}\hat{\sigma})' \, d\hat{\rho}_2 \\
&= \int_{-1}^{1} [u - u(p)](\mathring{\pi}^1_{p+1}\hat{\sigma})' \, d\hat{\rho}_2 = -\int_{-1}^{1} [\sigma - \sigma(p)]\mathring{\pi}^1_{p+1}\hat{\sigma} \, d\hat{\rho}_2 \\
&= \int_{-1}^{1} [\sigma - \sigma(p)](\hat{\sigma} - \mathring{\pi}^1_{p+1}\hat{\sigma}) \, d\hat{\rho}_2 - \int_{-1}^{1} [\sigma - \sigma(p)]\hat{\sigma} \, d\hat{\rho}_2 \\
&= \int_{-1}^{1} [\sigma - \sigma(p)](\hat{\sigma} - \mathring{\pi}^1_{p+1}\hat{\sigma}) \, d\hat{\rho}_2 + \int_{-1}^{1} [\sigma - \sigma(p)]'(\hat{u} - \pi_p\hat{u}) \, d\hat{\rho}_2 \\
&\leq C\left(p^{-1}\|\sigma - \sigma(p)\|_{L^2(-1,1)}\|\hat{\sigma}\|_{H^1(-1,1)} + p^{-2}\|\sigma' - \sigma'(p)\|_{L^2(-1,1)}\|\hat{u}\|_{H^2(-1,1)}\right).
\end{aligned}$$

So, $\|\pi_p u - u(p)\|_{L^2(-1,1)} \leq Cp^{-2-s}\|u\|_{H^{s+2}(-1,1)}$, and we conclude the result using the triangle inequality

$$\|u - u(p)\|_{L^2(-1,1)} \leq \|u - \pi_p u\|_{L^2(-1,1)} + \|\pi_p u - u(p)\|_{L^2(-1,1)},$$

and Lemma A.2. Finally, the $H^{1/2}(-1,1)$ estimate comes from an application of Lemmas A.1 and A.2:

$$\begin{aligned}\|u - u(p)\|_{H^{1/2}(-1,1)} &\leq \|u - \pi_p u\|_{H^{1/2}(-1,1)} + \|\pi_p u - u(p)\|_{H^{1/2}(-1,1)} \\ &\leq C(p^{-5/4-s} + p^{-1-s})\|u\|_{H^{s+2}(-1,1)} \leq Cp^{-1-s}\|u\|_{H^{s+2}(-1,1)}.\end{aligned}$$

□

Lemma B.2. *Under the same hypotheses of Lemma B.1, for any nonnegative real number s there exists a constant C such that*

$$\|u(p)\|_{H^s(-1,1)} \leq \begin{cases} C\|u\|_{H^2(-1,1)} & \text{if } 0 \leq s \leq 5/4, \\ C\|u\|_{H^{3s-7/4}(-1,1)} & \text{if } 5/4 \leq s < 7/4, \\ C\|u\|_{H^{4s-7/2}(-1,1)} & \text{if } 7/4 \leq s. \end{cases} \tag{B.2}$$

Proof. Define $\hat{u}(p)(\hat{\rho}_2) = \int_{-1}^{\hat{\rho}_2} \sigma(p)\,dz - d \in \hat{\mathbb{P}}_{p+2}(-1,1)$, where d is a constant that enforces the zero average. Since $\hat{u}'(p) = \sigma(p)$, then (B.1) implies that

$$\int_{-1}^{1} [\hat{u}'(p)\tau + u(p)\tau']\,d\hat{\rho}_2 = 0.$$

Integrating the first term by parts, and using that $\hat{u}(p)$ and $u(p)$ have zero average, we have $u(p) = \pi_p\hat{u}(p)$. We can bound then arbitrarily high norms of $u(p)$. Let $s \geq 5/4$. Using Lemma A.3, we have that

$$\begin{aligned}\|u(p)\|_{H^s(-1,1)} &= \|\pi_p\hat{u}(p)\|_{H^s(-1,1)} \leq C\|\hat{u}(p)\|_{H^{2s-1/2}(-1,1)} \\ &\leq C\|\sigma(p)\|_{H^{2s-3/2}(-1,1)} \leq C\|\sigma'(p)\|_{H^{2s-5/2}(-1,1)} = C\|\pi_p\sigma'\|_{H^{2s-5/2}(-1,1)} \\ &\leq \begin{cases} C\|\sigma'\|_{H^{3s-15/4}(-1,1)} & \text{if } 5/4 \leq s < 7/4, \\ C\|\sigma'\|_{H^{4s-11/2}(-1,1)} & \text{if } 7/4 \leq s. \end{cases}\end{aligned}$$

Using that $\sigma = u'$ the result holds for $s \geq 5/4$. For $0 \leq s < 5/4$ follows immediately since, in this case, $\|u(p)\|_{H^s} \leq \|u(p)\|_{H^{5/4}} \leq \|u(p)\|_{H^2}$. $\square$

Appendix C

Notation index

Rectangle

Domains.

$R^\varepsilon = (-1,1) \times (-\varepsilon, \varepsilon)$ – Page 17

$\partial R^\varepsilon_L = \{-1,1\} \times (-\varepsilon, \varepsilon)$ – Page 17

$\partial R^\varepsilon_\pm = (-1,1) \times \{-\varepsilon, \varepsilon\}$ – Page 17

$R = (-1,1) \times (-1,1)$ – Page 17

$\partial R_L = \{-1,1\} \times (-1,1)$ – Page 17

$\partial R_\pm = (-1,1) \times \{-1,1\}$ – Page 17

R^ε_0 – Page 40

Function Spaces and Norms.

$\underset{\sim}{S}(R^\varepsilon) = \underset{\sim}{L}^2(R^\varepsilon)$ – Page 31

$\underset{\sim}{S}(R^\varepsilon, p) = \left\{ \underset{\sim}{\tau} \in \underset{\sim}{S}(R^\varepsilon) \,:\, \deg_2 \tau_1 \leq p,\, \deg_2 \tau_2 \leq p-1 \right\}$ – Page 32

$\underset{\sim}{S}'_{g^\varepsilon}(R^\varepsilon) = \left\{ \underset{\sim}{\sigma} \in \underset{\sim}{H}(\mathrm{div}, R^\varepsilon) \,:\, \underset{\sim}{\sigma} \cdot \underset{\sim}{n} = g^\varepsilon \text{ on } \partial R^\varepsilon_\pm \right\}$ – Page 44

$\underset{\sim}{S}'_{g^\varepsilon}(R^\varepsilon, p) = \left\{ \underset{\sim}{\tau} \in \underset{\sim}{S}'_{g^\varepsilon}(R^\varepsilon) \,:\, \deg_2 \tau_1 \leq p,\, \deg_2 \tau_2 \leq p-1 \right\}$ – Page 45

$V(R^\varepsilon) = \left\{ v \in H^1(R^\varepsilon) \,:\, v = 0 \text{ on } \partial R^\varepsilon_L \right\}$ – Page 31

$V(R^\varepsilon, p) = \left\{ v \in V(R^\varepsilon) \,:\, \deg_2 v \leq p \right\}$ – Page 31

$V'(R^\varepsilon) = L^2(R^\varepsilon)$ – Page 44

$V'(R^\varepsilon, p) = \left\{ v \in V'(R^\varepsilon) \,:\, \deg_2 v \leq p \right\}$ – Page 45

$|g|_{C(\partial R_L)} = |g(-1,-1)| + |g(-1,1)| + |g(1,-1)| + |g(1,1)|$ – Page 21

$|||(f,g)|||_{(\partial R_L, N)} = \sum_{k=0}^N \|\partial_1^{2k} f\|_{L^2(\partial R_L)} + \|\partial_1^{2k} g\|_{C(\partial R_L)}$ – Page 21

$\|v\|_{(m,s,R)} = \|v\|_{H^m((-1,1);H^s(-1,1))}$ – Page 21

$|||(f,g)|||_{N,R} = \|f\|_{(N,0,R)} + \|g\|_{H^N(\partial R_\pm)}$ – Page 21

$a_s = \|f\|_{(0,s)} + \|g\|_{L^2(\partial R_\pm)}$ – Page 38

$a_s^b = \|f\|_{H^s(\partial R_L)} + |g|_{C(\partial R_L)}$ – Page 38

$a_s^1 = \|f\|_{(1,s,R)} + \|g\|_{H^1(\partial R_\pm)}$ – Page 38

$a = |||(f,g)|||_{4,R} + |||(f,g)|||_{2,\partial R_L}$ – Page 38

Other definitions.

$\deg_2 \to$ polynomial degree in the x_2 direction – Page 32

e_N – Page 24

$\hat{\rho}_- = \varepsilon^{-1}(1 + x_1)$ – Page 20

$\hat{\rho}_+ = \varepsilon^{-1}(1 - x_1)$ – Page 20

Plate

Domains.

Ω – Page 5

$P^\varepsilon = \Omega \times (-\varepsilon, \varepsilon)$ – Page 5

$\partial P_L^\varepsilon = \partial\Omega \times (-\varepsilon, \varepsilon)$ – Page 5

$\partial P_\pm^\varepsilon = \Omega \times \{-\varepsilon, \varepsilon\}$ – Page 5

$P = \Omega \times (-1, 1)$ – Page 8

$\partial P_L = \partial\Omega \times (-1, 1)$ – Page 63

$\partial P_\pm = \Omega \times \{-1, 1\}$ – Page 63

$\partial P_0^\varepsilon = \Omega \times \{-1, 1\}$ – Page 72

$\hat{Q} = \mathbb{R}^+ \times (0, 2\pi) \times (-1, 1)$ – Page 76

$\partial\hat{Q}_\pm = \mathbb{R}^+ \times (0, 2\pi) \times \{-1, 1\}$ – Page 76

Function Spaces and Norms.

$\|u\|_{(m,s,P^\varepsilon)} = \|u\|_{H^m(\Omega;H^s(-\varepsilon,\varepsilon))}$

$\|u\|_{(m,s,P)} = \|u\|_{H^m(\Omega;H^s(-1,1))}$

$\mathring{H}^1(\Omega;\mathbb{P}_p(-a,a)) \to$ space of polynomials with coefficients in $\mathring{H}^1(\Omega)$ – Page 6

$\underset{\sim}{S}(P^\varepsilon) = \underset{\sim}{L}^2(P^\varepsilon)$ – Page 69

$\underset{\sim}{S}(P^\varepsilon, p) = \left\{\underset{\sim}{\tau} \in \underset{\sim}{S}(P^\varepsilon) \,:\, \deg_3 \underset{\sim}{\tau} \le p,\, \deg_3 \tau_3 \le p-1\right\}$– Page 69

$\underset{\sim}{S}'_g(P^\varepsilon) = \left\{\underset{\sim}{\sigma} \in \underset{\sim}{H}(\mathrm{div}, P^\varepsilon) \,:\, \underset{\sim}{\sigma} \cdot \underset{\sim}{n} = g^\varepsilon \text{ on } \partial P^\varepsilon_\pm\right\}$ – Page 73

$V(P^\varepsilon) = \left\{v \in H^1(P^\varepsilon) \,:\, v = 0 \text{ on } \partial P^\varepsilon_L\right\}$ – Page 6

$V(P^\varepsilon, p) = \left\{v \in V(P^\varepsilon) \,:\, \deg_3 v \le p\right\}$ – Page 69

$V'(P^\varepsilon) = L^2(P^\varepsilon)$ – Page 73

Other definitions.

$a_1^j = -(\kappa(\theta))^{j+1}$ – Page 66

$a_2^j = (j+1)(\kappa(\theta))^j$ – Page 66

$a_3^j = (j/2)(j+1)(\kappa(\theta))^{j-1}\kappa'(\theta)$ – Page 66

$\tilde{a}_s = \|f\|_{L^2(\Omega;H^s(-1,1))} + \|g\|_{L^2(\partial P_\pm)}$ – Page 71

$\tilde{a}_s^1 = \|f\|_{H^1(\Omega;H^s(-1,1))} + \|g\|_{H^1(\partial P_\pm)}$ – Page 71

$\tilde{a}_s^b$ – Page 71

$\deg_3 \to$ polynomial degree in the x_3 direction – Page 69

$\tilde{e}_N^\varepsilon$ – Page 68

$\hat{J}(\rho,\theta) = 1 - \rho\kappa(\theta)$ – Page 66

$\kappa \to$ curvature of $\partial\Omega$ – Page 66

$\rho \to$ variable in the normal direction of Ω – Pages 8, 65

$\hat{\rho} = \varepsilon^{-1}\rho$ – Page 66

$\theta \to$ arclength of $\partial\Omega$ – Pages 8, 65

Semi-infinite Strip

Domains.

$\gamma_t = \big\{ \underset{\sim}{\rho} \in \Sigma \,:\, \hat{\rho}_1 = t \big\}$ – Page 79

$\partial\Sigma_{\pm} = \mathbb{R}^+ \times \{-1, 1\}$ – Page 20

$\Sigma = \mathbb{R}^+ \times (-1, 1)$ – Page 20

$\Sigma(t, s) = \big\{ \underset{\sim}{\rho} \in \Sigma \,:\, t < \hat{\rho}_1 < s \big\}$ – Page 79

Function Spaces and Norms.

$L^2_w(\Sigma) = \big\{ v \in \mathcal{D}'(\Sigma) \,:\, wv \in L^2(\Sigma) \big\}$ – Page 80

$\|v\|_{L^2_w(\Sigma)} = \|wv\|_{L^2(\Sigma)}$ – Page 80

$\underset{\sim}{S}'_0(\Sigma) = \big\{ \underset{\sim}{\tau} \in \underset{\sim}{\mathcal{D}}'(\Sigma) \,:\, \|\underset{\sim}{\tau}\|_{\underset{\sim}{S}'_0(\Sigma)} < \infty,\, \underset{\sim}{\tau} \cdot \underset{\sim}{n} = 0 \text{ on } \partial\Sigma_{\pm} \big\}$ – Page 97

$\|\underset{\sim}{\tau}\|_{\underset{\sim}{S}'_0(\Sigma)} = \big(\|\operatorname{div} \underset{\sim}{\tau}\|^2_{L^2_{w^{-1}}(\Sigma)} + \|\underset{\sim}{\tau}\|^2_{L^2(\Sigma)} \big)^{1/2}$ – Page 97

$\underset{\sim}{S}'_0(\Sigma, p) = \big\{ \underset{\sim}{\tau} \in \underset{\sim}{S}'_0(\Sigma) \,:\, \deg_2 \tau_1 \le p,\, \deg_2 \tau_3 \le p + 1 \big\}$ – Pages 49, 98

$V(\Sigma) = \big\{ v \in \mathcal{D}'(\Sigma) \,:\, v \in L^2_w(\Sigma),\, \underset{\sim}{\nabla} v \in \underset{\sim}{L}^2(\Sigma) \big\}$ – Page 80

$\|v\|_{V_w(\Sigma)} = \big(\|v\|^2_{L^2_w(\Sigma)} + \|\underset{\sim}{\nabla} v\|^2_{L^2(\Sigma)} \big)^{1/2}$ – Page 80

$\|v\|_{V(\Sigma)} = \big(\int_\Sigma |\underset{\sim}{\nabla} v|^2 \, d\underset{\sim}{\hat{\rho}} + \int_{\gamma_0} v^2 \, d\hat{\rho}_2 \big)^{1/2}$ – Page 81

$V_0(\Sigma) = \big\{ v \in V(\Sigma) \,:\, v = 0 \text{ on } \gamma_0 \big\}$ – Page 82

$V^*(\Sigma) \rightarrow$ Dual space of $V_0(\Sigma)$ – Page 82

$\|\cdot\|_{V^*(\Sigma)} \rightarrow$ dual norm – Page 82

$V(\Sigma, p) = \big\{ v \in V(\Sigma) \,:\, \deg_2 v \le p \big\}$ – Pages 35, 89

$V_0(\Sigma, p) = \big\{ v \in V_0(\Sigma) \,:\, \deg_2 v \le p \big\}$ – Pages 35, 89

$V'(\Sigma, p) = \big\{ v \in L^2_w(\Sigma) \,:\, \deg_2 v \le p \big\}$ – Pages 49, 98

Other definitions.

$c_\infty(u) = \int_\Sigma \hat{\rho}_1 f(\underset{\sim}{\hat{\rho}}) \, d\underset{\sim}{\hat{\rho}} + \int_{\gamma_0} u \, d\hat{\rho}_2$ – Page 79

$C_W \rightarrow$ Wirtinger constant – Page 84

$w(\underset{\sim}{\hat{\rho}}) = (1 + \hat{\rho}_1)^{-1}$ – Page 80

Projection operators

$\pi_p : L^2(-1,1) \to \mathbb{P}_p(-1,1)$ – Page 54

$\mathring{\pi}^1_p : \mathring{H}^1(-1,1) \to \mathring{\mathbb{P}}_p(-1,1)$ – Page 54

$\hat{\pi}^1_p : H^1(-1,1) \cap \hat{L}^2(-1,1) \to \hat{\mathbb{P}}_p(-1,1)$ – Page 37

$\underset{\sim}{\Pi}_p = (\pi_p^{(\hat{\rho}_2)}, \mathring{\pi}_{p+1}^{1(\hat{\rho}_2)})^T : \underset{\sim}{S}'_0(\Sigma) \to \underset{\sim}{S}'_0(\Sigma, p)$ – Page 100

$\pi_{V'}$: L^2 projection into $V'(R^\varepsilon, p)$ or into $V'(P^\varepsilon, p)$ – Pages 45, 73

Other definitions

$H^s(D)$, $\mathring{H}^s(D)$, $H^m(D, E) \to$ Sobolev spaces – Page 15

$\hat{L}^2(a,b) \to$ functions in $L^2(a,b)$ with zero average – Page 15

$\mathbb{P}_p(a,b) \to$ space of polynomials of degree p defined in (a,b) – Page 6

$\hat{\mathbb{P}}_p(a,b) = \mathbb{P}_p(a,b) \cap \hat{L}^2(a,b)$ – Page 10

$\mathring{\mathbb{P}}_p(a,b) = \mathbb{P}_p(a,b) \cap \mathring{H}^1(a,b)$ – Page 48

$[x] \to$ greatest integer not greater than x

∂_j, ∂_{ij}, $\partial_j^k \to$ derivatives – Page 5

$\partial_y \to$ derivative in the y direction – Page 15

f^k, g^0, g^1 – Pages 32, 69

$J(s) = \max\{j \in \mathbb{Z} : 2j < s\}$ – Page 39

$\underset{\sim}{n}$, $\underline{n} \to$ outward normal – Page 15

ν – Pages 9, 25, 68

$L_j \to$ Legendre Polynomials in $(-1,1)$ – Page 32

$Q_j(z) = \varepsilon^j L_i(\varepsilon^{-1} z)$ – Page 32

χ_-, χ_+, $\chi \to$ cut-off functions – Pages 8, 27, 68

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