Robust Model Reduction Discretizations based on Adaptive BDDC Techniques

Alexandre Madureira and Marcus Sarkis

Abstract We consider a reduced model technique of multiscale type to approximate the exact discrete solution of a two-dimensional symmetric elliptic partial differential equation of a finite element discretization with heterogeneous coefficients. The method is of Galerkin type and follows the Variational Multiscale–VMS and Localized Orthogonal Decomposition–LOD approaches in the sense that it decouples the underlying discrete space into *multiscale* and *fine* subspaces. The multiscale basis functions are obtained by mapping BDDC coarse basis modes and adaptive edges modes (based on special local generalized eigenvalue problems) to functions of global minimal energy. As a result, given a desired target threshold, we can select enough edges modes to built a model reduction with desired a priori energy error with respect to the exact discrete solution. These global minimal energy modes do not depend on the right-hand side, they depend on the discrete operator and on the target threshold.

1 Introduction

Consider the problem of finding the weak solution $u : \Omega \to \mathbb{R}$ of

$$\begin{aligned} -\div \mathscr{A}\nabla u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Here $\Omega \subset \mathbb{R}^2$ is an open bounded domain with polygonal boundary $\partial\Omega$, the symmetric tensor $\mathscr{A} \in [L^\infty(\Omega)]^{2\times 2}_{\text{sym}}$ is uniformly positive definite and bounded.

Alexandre Madureira
Laboratório Nacional Computação Científica, Petrópolis, Brazil, e-mail: alm@lncc.br

Marcus Sarkis
Worcester Polytechnic Institute, e-mail: msarkis@wpi.edu

For almost all $x \in \Omega$ let the positive constants c_1 and c_2 be such that

$$c_1|v|^2 \leq a_{\min}(x)|v|^2 \leq \mathscr{A}(x)\, v \cdot v \leq a_{\max}(x)|v|^2 \leq c_2|v|^2 \quad \text{for all } v \in \mathbb{R}^2, \text{ a.e. } x \in \Omega.$$

The associated variational formulation is given by: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) := \int_\Omega \mathscr{A}\nabla u \cdot \nabla v\, dx = \int_\Omega f v\, dx =: (f,v) \quad \forall v \in H_0^1(\Omega).$$

Recently, methods that do not rely on the regularity of the solution were introduced: generalized finite element methods [1], the rough polyharmonic splines [22], the variational multiscale method (VMS) [13], and the Localized Orthogonal Decomposition (LOD) [16, 10]. These methods are based on splitting approximation spaces into *fine* and *multiscale* subspaces, and the numerical solution of (1) is sought in the latter. We note that these works were designed for the low-contrast case, that is, c_2/c_1 not large. We note that for a class of coefficients $\mathscr{A}$, that is, when local Poincaré inequality constants are not large, the LOD methodology works [24].

On the other side, there exist several domain decomposition solvers which are optimal with respect to mesh and contrast. All of them are based on extracting coarse basis functions from local generalized eigenvalue problems. For non-overlapping domain decomposition based on the technique named *adaptive choice of primal constraints* was introduced in [19], revisited in [23, 15]; see also [6, 21] and references therein. We note that earlier ideas were also introduced in [3]. This robustness also was developed for overlapping domain decomposition methods and we refer the earlier works in [7, 20].

In this paper we consider *Approximate Component Mode Synthesis*–ACMS methods [4, 5, 2, 9, 12, 11, 14]; these methods require extra solution regularity and do not work for high contrast. The goal here is to develop a discretization that has optimal energy a priori error approximation, assuming no regularity on the solution and on $\mathscr{A}$. To do that we combine adaptive BDDC and LOD techniques; see also [17] for a similar combination however for mixed finite element discretizations.

The remainder of the this paper is organized as follows. Section 2 describes the substructuring decomposition into interior and interface unknowns and in Section 3 we present the goal of the paper. In Section 4 the model reduction method via adaptive BDDC is proposed and the results are discussed. In Section 5 we consider how to deal with elementwise problems. In Section 6 numerical results are presented.

2 Discrete Substructuring Formulation

We start by defining a partition of Ω by a triangular finite element regular mesh $\mathscr{T}_H$ with elements of characteristic length $H > 0$. Let $\partial\mathscr{T}_h$ be the mesh skeleton, and $\mathscr{N}_H$ the set of nodes on $\partial\mathscr{T}_h \backslash \partial\Omega$. Consider $\mathscr{T}_h$, a refinement of $\mathscr{T}_H$, in the sense that every (coarse) edge of the elements in $\mathscr{T}_H$ can be written as a union of edges

of $\mathscr{T}_h$. We assume that $h < H$. Let $\mathscr{N}_h$ be the set of nodes of $\mathscr{T}_h$ on the skeleton $\partial\mathscr{T}_h\backslash\partial\Omega$; thus all nodes in $\mathscr{N}_h$ belong to edges of elements in $\mathscr{T}_H$.

For $v \in H^1(\Omega)$ let

$$|v|^2_{H^1_{\mathscr{A}}(\Omega)} = \|\mathscr{A}^{1/2}\nabla v\|^2_{L^2(\Omega)}, \qquad |v|^2_{H^1_{\mathscr{A}}(\mathscr{T})} = \sum_{\tau\in\mathscr{T}} \|\mathscr{A}^{1/2}\nabla v\|^2_{L^2(\tau)},$$

where $\mathscr{T} \subset \mathscr{T}_H$ denotes a given set of elements. Let $V_h \subset H^1_0(\Omega)$ be the space of continuous piecewise linear functions related to $\mathscr{T}_h$. Let $u_h \in V_h$ such that

$$a(u_h, v_h) = (f, v_h) \quad \text{for all } v_h \in V_h.$$

We assume that u_h approximates u well, but we remark that u_h is never computed; the goal here is to develop numerical schemes which yield good approximations for u_h, therefore, the schemes proposed can be viewed as a model reduction method.

We can decompose $u_h = u_h^{\mathbb{B}} \oplus u_h^{\mathbb{H}}$ in its bubble (belonging to $V_h^{\mathbb{B}}$) and a-discrete harmonic components (belonging to $V_h^{\mathbb{H}}$), respectively, where

$$V_h^{\mathbb{B}} = \{v_h \in V_h : v_h = 0 \text{ on } \partial\tau,\ \tau \in \mathscr{T}_H\},$$
$$V_h^{\mathbb{H}} = \{u_h^{\mathbb{H}} \in V_h : a(u_h^{\mathbb{H}}, v_h^{\mathbb{B}}) = 0 \text{ for all } v_h^{\mathbb{B}} \in V_h^{\mathbb{B}}\},$$

i.e., $V_h^{\mathbb{H}} = (V_h^{\mathbb{B}})^{\perp_a}$. It follows immediately from the definitions that

$$a(u_h^{\mathbb{H}}, v_h^{\mathbb{H}}) = (f, v_h^{\mathbb{H}}) \quad \text{for all } v_h^{\mathbb{H}} \in V_h^{\mathbb{H}}, \qquad a(u_h^{\mathbb{B}}, v_h^{\mathbb{B}}) = (f, v_h^{\mathbb{B}}) \quad \text{for all } v_h^{\mathbb{B}} \in V_h^{\mathbb{B}}.$$

Although the problem related to $u_h^{\mathbb{B}}$ is global, it can be decomposed in local uncoupled problems, as discussed in Section 5.

Note that any function in $V_h^{\mathbb{H}}$ is uniquely determined by its trace on the boundary of elements in $\mathscr{T}_H$. Let us define

$$\Lambda_h = \{v_h|_{\partial\mathscr{T}_h} : v_h \in V_h^{\mathbb{H}}\} \subset H^{1/2}(\partial\mathscr{T}_h),$$

and $T : \Lambda_h \to V_h^{\mathbb{H}}$ be the *local* discrete-harmonic extension operator given by

$$(T\mu_h)|_{\partial\mathscr{T}_h} = \mu_h, \qquad \text{and} \qquad a(T\mu_h, v_h^{\mathbb{B}}) = 0 \quad \text{for all } v_h^{\mathbb{B}} \in V_h^{\mathbb{B}}.$$

For $\tau \in \mathscr{T}_H$, let $\Lambda_h^\tau = \Lambda_h|_{\partial\tau}$, that is, the restriction of functions on Λ_h to $\partial\tau$. Define the bilinear forms $s : \Lambda_h \times \Lambda_h \to \mathbb{R}$ and $s^\tau : \Lambda_h^\tau \times \Lambda_h^\tau \to \mathbb{R}$ such that, for $\mu_h, \nu_h \in \Lambda_h$,

$$s(\mu_h, \nu_h) = \sum_{\tau\in\mathscr{T}_H} s_\tau(\mu_h^\tau, \nu_h^\tau) \quad \text{where} \quad s_\tau(\mu_h^\tau, \nu_h^\tau) = \int_\tau \mathscr{A}\nabla T^\tau\mu_h^\tau \cdot \nabla T^\tau\nu_h^\tau\, dx$$

where T^τ is the restriction of T to τ. Let $\lambda_h = u_h|_{\partial\mathscr{T}_h}$. Then $u_h^{\mathbb{H}} = T\lambda_h$ and

$$s(\lambda_h, \mu_h) = (f, T\mu_h) \quad \text{for all } \mu_h \in \Lambda_h. \tag{2}$$

3 Main Goal of the Paper

Let us introduce $\rho \in L^\infty(\Omega)$ such that $\rho(x) \in [\rho_{\min}, \rho_{\max}]$ almost everywhere for some positive constants $\rho_{\min}$ and $\rho_{\max}$, and define $g = f/\rho$ and the spaces for g or f such that

$$\|g\|_{L^2_\rho(\Omega)} = \|\rho^{1/2} g\|_{L^2(\Omega)} = \|f\|_{L^2_{1/\rho}(\Omega)} < \infty$$

The main goal of this paper is the following: Given a threshold δ, construct a lower-dimensional subspace $\Lambda_h^{ms} \subset \Lambda_h$, such that for any $g \in L^2_\rho(\Omega)$ (or equivalently $f \in L^2_{1/\rho}(\Omega)$) the multiscale solution $\lambda_h^{ms}(g) \in \Lambda_h^{ms}$ of

$$s(\lambda_h^{ms}, \mu_h^{ms}) = (\rho g, T\mu_h^{ms}) \quad \text{for all } \mu_h^{ms} \in \Lambda_h^{ms}$$

satisfies

$$|u_h^{\mathbb{H}} - T\lambda_h^{ms}|^2_{H^1_{\mathscr{A}}(\Omega)} = s(\lambda_h - \lambda_h^{ms}, \lambda_h - \lambda_h^{ms}) \leq C\delta^2 \|g\|^2_{L_\rho(\Omega)}. \tag{3}$$

where the constant C does not depend on g, $\mathscr{A}$ or ρ.

The reason for introducing the weight function ρ is to normalize the equation (1). For example, assume that $\mathscr{A}(x) = 10^{-6}$. Then the solution of (1) satisfies $-\Delta u = f/10^{-6}$. This means that if we want to obtain an approximation like (3) with $\rho = 1$ and with C independently of $\mathscr{A}$, it would require a large space Λ_h^{ms}, maybe as large as the fine space Λ_h. So, it is natural for this case to choose $\rho = 10^{-6}$. And vice-versa, if $\mathscr{A}(x) = 10^6$, an estimate like (3) with $\rho = 1$ would be too easy, it would not give a good relative energy approximation. We think that a judicious choice is $\rho(x) = a_{\min}(x)$ since the approximation also will capture the anisotropy of $\mathscr{A}(x)$. Another reason is that similarly as discussed in [8], the dimension of the space Λ_h^{ms} is related to the number of highly conductive fingerings crossing the edges of the coarse triangulation $\mathscr{T}_H$.

4 Model Reduction via BDDC

We now propose a scheme to approximate λ_h in (2) based on LOD and BDDC techniques. Decompose $\Lambda = \Lambda^0 \oplus \widetilde{\Lambda}_h$ by

$$\widetilde{\Lambda}_h = \{\lambda \in \Lambda_h : \lambda(x_i) = 0 \text{ for all } x_i \in \mathscr{N}_H\},$$
$$\Lambda^0 = \{\lambda \in \Lambda_h : \lambda(x_i) = 0 \text{ for all } x_i \in \mathscr{N}_h \backslash \mathscr{N}_H\}.$$

Let e be an edge of $\partial\mathscr{T}_h \backslash \partial\Omega$ shared by the elements τ and τ' of $\mathscr{T}_H$, and denote $\widetilde{\Lambda}_h^e = \widetilde{\Lambda}_h|_e$, that is, the restriction of functions on $\widetilde{\Lambda}_h$ to e. Note that a function $\widetilde{\mu}_h^e \in \widetilde{\Lambda}_h^e$ vanishes at the end-points of e; it is thus possible to extend continuously by zero to either $\partial\tau$ or $\partial\tau'$. Let us denote this extension by $R^T_{e,\tau} : \widetilde{\Lambda}_h^e \to \Lambda_h^\tau$.

Let us define $S^{\tau}_{ee} : \widetilde{\Lambda}^e_h \to (\widetilde{\Lambda}^e_h)'$, where $(\widetilde{\Lambda}^e_h)'$ is the dual space of $\widetilde{\Lambda}^e_h$, by

$$(\widetilde{\mu}^e_h, S^{\tau}_{ee}\widetilde{\nu}^e_h)_e = (R^T_{e,\tau}\widetilde{\mu}^e_h, S^{\tau}R^T_{e,\tau}\widetilde{\nu}^e_h)_{\partial\tau} \quad \text{for all } \widetilde{\mu}^e_h, \widetilde{\nu}^e_h \in \widetilde{\Lambda}^e_h,$$

where $(\cdot,\cdot)_e$ is the $L^2(e)$ inner product and

$$(\mu^{\tau}_h, S^{\tau}\nu^{\tau}_h)_{\partial\tau} = \int_{\tau} \mathscr{A}\nabla T^{\tau}\mu^{\tau}_h \cdot \nabla T^{\tau}\nu^{\tau}_h \, dx \qquad \text{for all } \mu^{\tau}_h, \nu^{\tau}_h \in \Lambda^{\tau}_h.$$

In a similar fashion, define $S^{\tau}_{e^c e}$, $S^{\tau}_{ee^c}$ and $S^{\tau}_{e^c e^c}$, related to the degrees of freedom on $e^c = \partial\tau\backslash e$. We remind that e is an open edge, not containing its endpoints.

Let us introduce M^{τ}_{ee} by

$$(\tilde{\mu}^e_h, M^{\tau}_{ee}\tilde{\nu}^e_h)_e = \int_{\tau} \rho \, (T^{\tau}R^T_{e,\tau}\tilde{\mu}^e_h)\,(T^{\tau}R^T_{e,\tau}\tilde{\nu}^e_h)\, dx$$

and define $\widehat{S}^{\tau}_{ee} = \delta^{-2}M^{\tau}_{ee} + S^{\tau}_{ee}$, where δ is the target precision of the method, that can be set by the user.

Define also

$$\widetilde{S}^{\tau}_{ee} = S^{\tau}_{ee} - S^{\tau}_{ee^c}(S^{\tau}_{e^c e^c})^{-1}S^{\tau}_{e^c e},$$

and it is easy to show that

$$(\widetilde{\nu}^e_h, \widetilde{S}^{\tau}_{ee}\widetilde{\nu}^e_h) \leq (\nu_h, S^{\tau}\nu_h) \quad \text{for all } \nu_h \in \Lambda^{\tau}_h \text{ so that } R_{e,\tau}\nu_h = \widetilde{\nu}^e_h, \tag{4}$$

where the restriction operator $R_{e,\tau} : \Lambda_h \to \widetilde{\Lambda}^e_h$ is so that $R_{e,\tau}\nu_h(x_i) = \widetilde{\nu}^e_h(x_i)$ for all nodes $x_i \in \mathscr{N}_e := (\mathscr{N}_h\backslash\mathscr{N}_H)\cap e$.

In what follows, to take into account high contrast coefficients, we consider the following generalized eigenvalue problem: Find eigenpairs $(\alpha^e_i, \tilde{\mu}^e_{h,i}) \in (\mathbb{R}, \widetilde{\Lambda}^e_h)$, where $\alpha^e_1 \geq \alpha^e_2 \geq \alpha^e_3 \geq \cdots \geq \alpha^e_{\mathscr{N}_e} > 1$, such that if the edge e of $\partial\mathscr{T}_h\backslash\partial\Omega$ is shared by elements τ and τ' of $\mathscr{T}_H$, we solve

$$(\widehat{S}^{\tau}_{ee} + \widehat{S}^{\tau'}_{ee})\tilde{\mu}^e_{h,i} = \alpha^e_i(\widetilde{S}^{\tau}_{ee} + \widetilde{S}^{\tau'}_{ee})\tilde{\mu}^e_{h,i}. \tag{5}$$

The eigenfunctions $\tilde{\mu}^e_{h,i}$ are chosen to be orthonormal with respect to the norm $(\cdot,(\widehat{S}^{\tau}_{ee} + \widehat{S}^{\tau'}_{ee})\cdot)_e$.

Now we decompose $\widetilde{\Lambda}^e_h := \widetilde{\Lambda}^{e,\triangle}_h \oplus \widetilde{\Lambda}^{e,\Pi}_h$ where for a given $\alpha_{\text{stab}} > 1$,

$$\widetilde{\Lambda}^{e,\triangle}_h := \text{span}\{\tilde{\mu}^e_{h,i} : \alpha^e_i < \alpha_{\text{stab}}\}, \qquad \widetilde{\Lambda}^{e,\Pi}_h := \text{span}\{\tilde{\mu}^e_{h,i} : \alpha^e_i \geq \alpha_{\text{stab}}\}.$$

The value of α_{stab} is tuned with $\mathscr{A}(x) = \rho(x) = 1$ so that the dimension of $\widetilde{\Lambda}^{e,\Pi}_h$ is small. Hence, for general $\mathscr{A}(x)$ and $\rho(x)$, the space $\widetilde{\Lambda}^{e,\Pi}_h$ will consist mostly of eigenvectors associated to the heterogeneities of $\mathscr{A}(x)$ with respect to $\rho(x)$.

For adaptive BDDC preconditioners, in general the generalized eigenvalue problem is defined by

$$(S^{\tau}_{ee} + S^{\tau'}_{ee})\tilde{\mu}^e_{h,i} = \alpha^e_i(\widetilde{S}^{\tau}_{ee} + \widetilde{S}^{\tau'}_{ee})\tilde{\mu}^e_{h,i} \tag{6}$$

We note that this generalized eigenvalue problem would be enough for establishing exponential decay for the multiscale basis functions. In (5), the term $\delta^{-2}M^{\tau}_{ee}$ was added to S^{τ}_{ee}. This is needed when dealing with approximation results such as Theorem 4 since in the proof it is required that $\|v\|_{L^2_{\tilde{\rho}}(\Omega)} \leq \delta |v|_{H^1_{\mathscr{A}}(\Omega)}$ for $v \in T\widetilde{\Lambda}^{\triangle}_h$ defined below.

To define our ACMS–NLSD (*Approximate Component Mode Synthesis Non-Localized Spectral Decomposition*) method for high-contrast coefficients, let

$$\widetilde{\Lambda}^{\Pi}_h = \{\tilde{\mu}_h \in \widetilde{\Lambda}_h : \tilde{\mu}_h|_e \in \widetilde{\Lambda}^{e,\Pi}_h \text{ for all } e \in \partial\mathscr{T}_h\},$$
$$\widetilde{\Lambda}^{\triangle}_h = \{\tilde{\mu}_h \in \widetilde{\Lambda}_h : \tilde{\mu}_h|_e \in \widetilde{\Lambda}^{e,\triangle}_h \text{ for all } e \in \partial\mathscr{T}_h\}.$$

Note that $\Lambda_h = \Lambda^{\Pi}_h \oplus \widetilde{\Lambda}^{\triangle}_h$, where

$$\Lambda^{\Pi}_h = \Lambda^0_h \oplus \widetilde{\Lambda}^{\Pi}_h$$

and Λ^0_h is the set of functions on Λ_h which vanish on all nodes of $\mathscr{N}_h \backslash \mathscr{N}_H$. Denote

$$(v_h, S\mu_h)_{\partial\mathscr{T}_h} = \sum_{\tau\in\mathscr{T}_H} (v^{\tau}_h, S^{\tau}\mu^{\tau}_h)_{\partial\tau}.$$

We now introduce the ACMS–NLSD multiscale functions. For $\tau \in \mathscr{T}_H$, consider the operator $P^{\tau,\triangle} : \Lambda_h \to \widetilde{\Lambda}^{\triangle}_h$ as follows: Given $\mu_h \in \Lambda_h$, find $P^{\tau,\triangle}\mu_h \in \widetilde{\Lambda}^{\triangle}_h$ solving

$$(\widetilde{v}^{\triangle}_h, SP^{\tau,\triangle}\mu_h)_{\partial\mathscr{T}_h} = (\widetilde{v}^{\triangle}_h, S^{\tau}\mu_h)_{\partial\tau} \quad \text{for all } \widetilde{v}^{\triangle}_h \in \widetilde{\Lambda}^{\triangle}_h \tag{7}$$

and define $P^{\triangle} : \Lambda_h \to \widetilde{\Lambda}^{\triangle}_h$ given by $P^{\triangle} = \sum_{\tau\in\mathscr{T}_H} P^{\tau,\triangle}$. It is easy to see that $P^{\triangle}$ is an orthogonal projection on $\widetilde{\Lambda}^{\triangle}_h$ with respect to S.

Consider $\Lambda^{\text{ms}}_h = (I - P^{\triangle})\Lambda^{\Pi}_h$. We note that $(I - P^{\triangle})\Lambda^{\Pi}_h \neq \Lambda^{\Pi}_h$ since Λ^{Π}_h and $\widetilde{\Lambda}^{\triangle}_h$ are not orthogonal with respect to S. What we have is that $\widetilde{\Lambda}^{e,\triangle}_h$ and $\widetilde{\Lambda}^{e,\Pi}_h$ are orthogonal with respect to $\widetilde{S}^{\tau}_{ee} + \widetilde{S}^{\tau'}_{ee}$. If $\mu^{\Pi}_h \in \Lambda^{\Pi}_h$ is a local function, $(I - P^{\triangle})\mu^{\Pi}_h$ will not be necessarily local. However, we can show its exponential decay.

The ACMS–NLSD method is defined by: Find $\lambda^{\text{ms}}_h \in \Lambda^{\text{ms}}_h$ such that

$$(v^{\text{ms}}_h, S\lambda^{\text{ms}}_h)_{\partial\mathscr{T}_h} = (\rho g, Tv^{\text{ms}}_h) \quad \text{for all } v^{\text{ms}}_h \in \Lambda^{\text{ms}}_h. \tag{8}$$

Note that

$$(v^{\text{ms}}_h, S\lambda^{\text{ms}}_h)_{\partial\mathscr{T}_h} = \int_{\Omega} \mathscr{A}\nabla Tv^{\text{ms}}_h \cdot \nabla T\lambda^{\text{ms}}_h\, dx = \int_{\Omega} \rho g Tv^{\text{ms}}_h\, dx.$$

Remark 1. In [12, 11], different but still local eigenvalue problems are introduced, aiming to build approximation spaces. Their analysis however requires extra regularity of the coefficients, and the error estimate is not robust with respect to contrast.

Below we present several results where proofs will be published in [18].

Using local arguments, the next lemma states that a weighted Poincaré inequality can be obtained on the space $\widetilde{\Lambda}_h^{\triangle}$.

Lemma 1 *[18] Let $\widetilde{\mu}_h^{\triangle} \in \widetilde{\Lambda}_h^{\triangle}$. Then*

$$\|T\widetilde{\mu}_h^{\triangle}\|_{L^2_{\tilde{\rho}}(\Omega)} \leq (L^2\alpha_{stab})^{1/2}\delta|T\widetilde{\mu}_h^{\triangle}|_{H^1_{\mathscr{A}}(\Omega)}, \tag{9}$$

where L is the maximum number of edges that an element of $\mathscr{T}_H$ can have.

The next lemma states that the energy stability of the interpolation onto the primal space Λ^{Π}.

Lemma 2 *[18] Let $\mu_h \in \Lambda_h$ and let $\mu_h = \mu_h^{\Pi} + \widetilde{\mu}_h^{\triangle}$. Then*

$$|T\mu_h^{\Pi}|_{H^1_{\mathscr{A}}(\Omega)} \leq (2+2L^2\alpha_{stab})^{1/2}|T\mu_h|_{H^1_{\mathscr{A}}(\Omega)}.$$

The next lemma follows directly from the definition of the generalized eigenvalue problem and properties of $\widetilde{\Lambda}_h^{e,\triangle}$ and (4).

Lemma 3 *[18] Let e be a common edge of τ, $\tau' \in \mathscr{T}_H$, and $\widetilde{\mu}_h^{\triangle} \in \widetilde{\Lambda}_h^{\triangle}$. Then, defining $\widetilde{\mu}_h^{e,\triangle} = \widetilde{\mu}_h^{\triangle}|_e$ and $\widetilde{\mu}_h^{\tau,\triangle} = \widetilde{\mu}_h^{\triangle}|_{\partial\tau}$ it follows that*

$$|T^{\tau}R_{e,\tau}^T\widetilde{\mu}_h^{e,\triangle}|^2_{H^1_{\mathscr{A}}(\tau)} + |T^{\tau}R_{e,\tau'}^T\widetilde{\mu}_h^{e,\triangle}|^2_{H^1_{\mathscr{A}}(\tau')} \leq \alpha_{stab}\big(|T^{\tau}\widetilde{\mu}_h^{\tau,\triangle}|^2_{H^1_{\mathscr{A}}(\tau)} + |T^{\tau}\widetilde{\mu}_h^{\tau',\triangle}|^2_{H^1_{\mathscr{A}}(\tau')}\big)$$

Our main theorem follows.

Theorem 4 *[18] Let $\lambda_h = u_h|_{\partial\mathscr{T}_h}$, and λ_h^{ms} solution of* (8)*. Then $\lambda_h - \lambda_h^{ms} \in \widetilde{\Lambda}_h^{\triangle}$ and*

$$|u_h^{\mathbb{H}} - T\lambda_h^{ms}|^2_{H^1_{\mathscr{A}}(\Omega)} \leq L^2\alpha_{stab}\delta^2\|g\|^2_{L^2_{\tilde{\rho}}(\Omega)}.$$

4.1 Decaying for the High-Contrast Case

In the next two lemmas, we show first that we can control the energy on the exterior region outside the patch of j-neighbor elements $\mathscr{T}_{j+1}(\tau)$ by the energy on the strip $\mathscr{T}_{j+2}(\tau)\backslash\mathscr{T}_j(\tau)$. Next, we state the exponential decay of $P^{\tau,\triangle}\nu_h$.

Lemma 5 *[18] Let $\mu_h \in \Lambda_h$ and let $\tilde{\phi}_h^{\triangle} = P^{\tau,\triangle}\mu_h$ for some fixed element $\tau \in \mathscr{T}_H$. Then, for any integer $j \geq 1$,*

$$|T\tilde{\phi}_h^{\triangle}|^2_{H^1_{\mathscr{A}}(\mathscr{T}_H\backslash\mathscr{T}_{j+1}(\tau))} \leq L^2\alpha_{\text{stab}}|T\tilde{\phi}_h^{\triangle}|^2_{H^1_{\mathscr{A}}(\mathscr{T}_{j+2}(\tau)\backslash\mathscr{T}_j(\tau))}.$$

The next lemma states the exponential decay of $P^{\tau,\triangle}\nu_h$.

Corollary 6 *[18] Assume that $\tau \in \mathscr{T}_H$ and $\nu_h \in \Lambda_h$ and let $\tilde{\phi}_h^{\triangle} = P^{\tau,\triangle}\nu_h \in \widetilde{\Lambda}_h^{\triangle}$. For any integer $j \geq 1$,*

$$|T\tilde{\phi}_h^{\triangle}|^2_{H^1_{\mathscr{A}}(\mathscr{T}_H\setminus\mathscr{T}_{j+1}(\tau))} \leq e^{-\frac{[(j+1)/2]}{1+L^2\alpha_{\text{stab}}}}|T\tilde{\phi}_h^{\triangle}|^2_{H^1_{\mathscr{A}}(\mathscr{T}_H)}.$$

where $[s]$ is the integer part of s.

Inspired by the exponential decay stated in Corollary 6, we define the operator $P^{\triangle,j}$ as follows. First, for a fixed $\tau \in \mathscr{T}_H$, let

$$\widetilde{\Lambda}_h^{\triangle,\tau,j} = \{\widetilde{\mu}_h \in \widetilde{\Lambda}_h^{\triangle} : T\widetilde{\mu}_h = 0 \text{ on } \mathscr{T}_H\setminus\mathscr{T}_j(\tau)\},$$

i.e., the support of $\widetilde{\Lambda}_h^{\triangle,\tau,j}$ is just a patch of size j elements around the element τ.

For $\mu_h \in \Lambda_h$, define $P^{\triangle,\tau,j}\mu_h \in \widetilde{\Lambda}_h^{\tau,j}$ such that

$$s(P^{\triangle,\tau,j}\mu_h, \widetilde{\mu}_h) = s_\tau(\mu_h, \widetilde{\mu}_h) \quad \text{for all } \widetilde{\mu}_h \in \widetilde{\Lambda}_h^{\triangle,\tau,j},$$

and let

$$P^{\triangle,j}\mu_h = \sum_{\tau\in\mathscr{T}_H} P^{\triangle,\tau,j}\mu_h. \tag{10}$$

Finally, define the approximation $\lambda_H^{\Pi,j} \in \Lambda_H^{\Pi}$ such that

$$s\big((I-P^{\triangle,j})\lambda_H^{\Pi,j}, (I-P^{\triangle,j})\mu_H^{\Pi}\big) = (\rho g, T(I-P^{\triangle,j})\mu_H^{\Pi}) \quad \text{for all } \mu_H^{\Pi} \in \Lambda_H^{\Pi}, \tag{11}$$

and then let $\lambda_h^{ms,j} = (I-P^{\triangle,j})\lambda_H^{\Pi,j}$. We name as ACMS–LSD (*Approximate Component Mode Synthesis Localized Spectral Decomposition*) method.

We now state the approximation error of the method, starting by a technical result essential to obtain the final estimate.

Lemma 7 *[18] Consider $\nu_h \in \Lambda_h$ and the operators $P^{\triangle}$ defined by* (7) *and $P^{\triangle,j}$ by* (10) *for $j > 1$. Then*

$$|T(P^{\triangle}-P^{\triangle,j})\nu_h|^2_{H^1_{\mathscr{A}}(\mathscr{T}_H)} \leq (c_\gamma j)^2(L^2\alpha_{\text{stab}})^2 e^{-\frac{[(j-1)/2]}{1+L^2\alpha_{\text{stab}}}}|T\nu_h|^2_{H^1_{\mathscr{A}}(\mathscr{T}_H)},$$

where c_γ is a constant depending only on the shape of $\mathscr{T}_H$ such that

$$\sum_{\tau\in\mathscr{T}_H} |v|^2_{H^1(\mathscr{T}_j(\tau))} \leq (c_\gamma j)^2|v|^2_{H^1(\mathscr{T}_H)} \quad \forall v \in H^1(\mathscr{T}_H). \tag{12}$$

Theorem 8 *[18] Define $u_h^{\mathbb{H}}$ by* (2) *and let $\lambda_h^{ms,j} = (I-P^{\triangle,j})\lambda_H^{\Pi,j}$, where $\lambda_H^{\Pi,j}$ is as in* (11)*. Then*

$$|u_h^{\mathbb{H}} - T\lambda^{ms,j}|_{H^1_{\mathscr{A}}(\mathscr{T}_H)} \leq \delta L(2\alpha_{\text{stab}})^{1/2}\|g\|_{L^2_{\rho}(\Omega)} + c_\gamma j L^2\alpha_{\text{stab}} e^{-\frac{[(j-1)/2]}{2(1+L^2\alpha_{\text{stab}})}}|u_h^{\mathbb{H}}|_{H^1_{\mathscr{A}}(\mathscr{T}_H)}.$$

5 Spectral Multiscale Problems inside Substructures

To approximate $u_h^{\mathbb{B}}$ on an element $\tau \in \mathscr{T}_H$, we introduce a multiscale method by first building the approximation space $V^{\mathbb{B},\text{ms}}(\tau) := \text{Span}\{\psi_{h,1}^{\tau}, \psi_{h,2}^{\tau}, \cdots, \psi_{h,N_\tau}^{\tau}\}$ generated by the following generalized eigenvalue problem: Find the eigenpairs $(\alpha_i^{\tau}, \psi_{h,i}^{\tau}) \in (\mathbb{R}, V_h^{\mathbb{B}}(\tau))$ such that

$$a_\tau(v_h, \psi_{h,i}^{\tau}) = \alpha_i^{\tau}(\rho v_h, \psi_{h,i}^{\tau}) \qquad \text{for all} \quad v_h \in V_h^{\mathbb{B}}(\tau)$$

where

$$a_\tau(v_h, \psi_{h,i}^{\tau}) = \int_\tau \mathscr{A} \nabla v_h \cdot \nabla \psi_{h,i}^{\tau} dx \qquad \text{and} \qquad (\rho v_h, \psi_{h,i}^{\tau})_\tau = \int_\tau \rho v_h \psi_{h,i}^{\tau} dx,$$

and $0 < \alpha_1^{\tau} \leq \alpha_2^{\tau} \leq \cdots \leq \alpha_{N_\tau}^{\tau} < 1/\delta^2$ and $\alpha_{N_\tau+1}^{\tau} \geq 1/\delta^2$. The local multiscale problem is defined by: Find $u_h^{\mathbb{B},\text{ms}} \in V_h^{\mathbb{B},\text{ms}}$ such that

$$a(u_h^{\mathbb{B},\text{ms}}, v_h) = (\rho g, v_h) \qquad \text{for all} \qquad v_h \in V_h^{\mathbb{B},\text{ms}}.$$

We obtain

$$|u_h^{\mathbb{B}} - u_h^{\mathbb{B},\text{ms}}|^2_{H^1_{\mathscr{A}}(\Omega)} = (\rho g, u_h^{\mathbb{B}} - u_h^{\mathbb{B},\text{ms}}) \leq \delta |u_h^{\mathbb{B}} - u_h^{\mathbb{B},\text{ms}}|_{H^1_{\mathscr{A}}(\Omega)} \|g\|_{L^2_\rho(\Omega)},$$

and therefore,

$$|u_h^{\mathbb{B}} - u^{\mathbb{B},\text{ms}}|_{H^1_{\mathscr{A}}(\Omega)} \leq \delta \|g\|_{L^2_\rho(\Omega)}.$$

6 Numerical Experiments

Let $\Omega = [0,1] \times [0,1]$. We consider a Cartesian coarse mesh made of $2^M \times 2^M$ squares subdomains. We next subdivide each square subdomain into $2^{N-M} \times 2^{N-M}$ equal fine squares and then subdivide further into two 45-45-90 triangular elements. Denote $H = 2^{-M}$ and $h = 2^{-N}$ as the sizes of the subdomains and the fine elements, respectively.

The first numerical test is to examine the exponential decay of the multiscale basis functions. We assume that $\mathscr{A}(x)$ is scalar and $\rho(x) = \mathscr{A}(x)$. The distribution of $\rho(x)$ is shown in the left Figure 1. The coefficient $\rho = 100$ inside the **H**-shape region and $\rho = 1$ outside. We assume that $N = 6$ and $M = 3$, that is, 8×8 subdomain distribution and 8×8 local mesh inside each subdomain. This distribution of the coefficients $\mathscr{A}(x)$ and subdomains has the property that $\mathscr{A}(x) = 100$ at the subdomain corner node at $x = (1/2, 1/2)$ and $\mathscr{A} = 1$ at the remaining subdomains corners nodes. Figure 1 on the right shows the decay of the multiscale basis function associated to the coarse node $x = (1/2, 1/2)$ when $\Lambda_h^{\Pi} = \Lambda_h^0$ (equivalently $\widetilde{\Lambda}_h^{\triangle} = \widetilde{\Lambda}_h$), that is, with $\alpha_{stab} = \infty$ (without edges eigenfunctions). We can see that this multiscale

basis function does not decay exponentially away from $x = (1/2, 1/2)$. The white holes you see in the picture occurs because the value of the function is closed to zero. The reason for the non-decay is because this basis function wants to have small energy, that is, this basis function wants to have value near one on the **H**-shape region since $\mathscr{A}$ is large there. We now consider the adaptive case with $\alpha_{stab} = 1.5$. On the left and right of Figure 2 we show the exponential decay (in the log-normal scale) when $\delta = \infty$ and $\delta = H$, respectively. As expected from the theory, the eigenvalue problem (6) is enough to obtain the exponential decay, however, it is not enough for approximation.

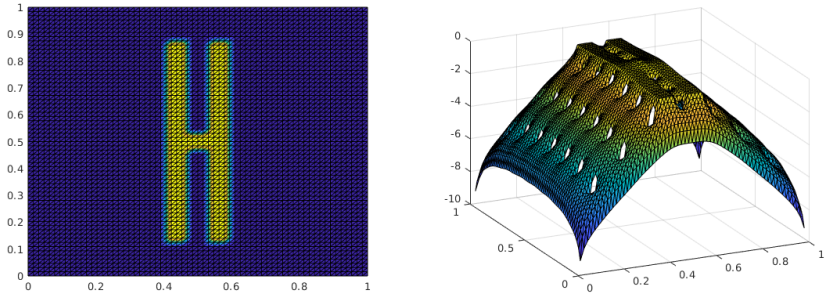

Fig. 1 On left, the distribution of the coefficient for a 8×8 subdomain decomposition. On the right, the plot of a multiscale basis functions without adaptivity. Note that there is no exponential decay whatsoever.

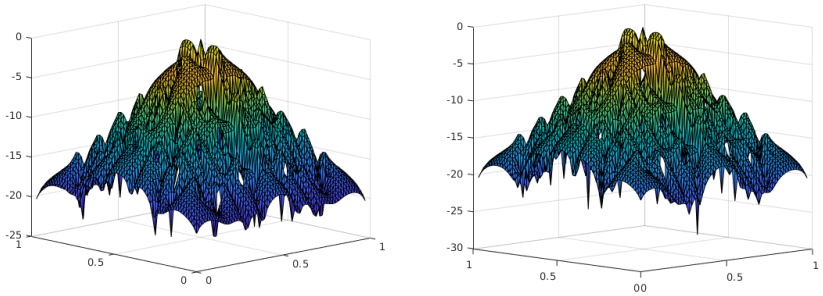

Fig. 2 Log-normal plot showing the decay of a multiscale basis functions with adaptivity, for $\delta = \infty$ (left figure) and $\delta = H$ (right figure)

In the second numerical test we keep the same distribution of coefficients in Figure 1 again choose $N = 6$ and $M = 3$. To make the problem a little more complicated, we multiply $\mathscr{A}$ and ρ in each element by independently uniformly random distributions between zero and one. Similarly, we let f to be constant in each element

given by another independently uniformly random distributions between zero and one. In Table 1 we show the energy errors for different values of δ. We also include the total number of edges functions required by the ACMS–NLSD method (without localization) for a δ tolerance. We take $\alpha_{stab} = 1.5$. Just as a reference, there are 112 interior subdomain edges; see that we can obtain a 0.22% relative energy error using an average of one eigenvector per subdomain edge.

δ	$\lvert u-u^{ms}\rvert_{H^1_a}$	$\frac{\lvert u-u^{ms}\rvert_{H^1_a}}{\lvert u\rvert_{H^1_a}}$	$\frac{\lvert u-u^{ms}\rvert_{H^1_a}}{\lVert f\rVert_{L^2_\rho}}$	Neigs
1/8	0.0095	0.0083	0.0079	78
1/16	0.0064	0.0056	0.0053	92
1/32	0.0025	0.0022	0.0021	112
1/64	0.0014	0.0012	0.0011	226

Table 1 The energy errors for different target accuracies δ. The last column shows Neigs (the total number of multiscale edges functions).

The last numerical test we investigate the dependence of the energy error $|u - u^{ms,j}|_{H^1_a}$ with respect to the localization j, that is, the ACMS–LSD method with localization j. We can see in Table 2 that the localization works really well.

$\delta\backslash j$	-1	0	1	2
1/8	0.43870	0.0095	0.0095	0.0095
1/16	0.0977	0.0064	0.0064	0.0064
1/32	0.1702	0.0025	0.0025	0.0025
1/64	0.0795	0.0014	0.0014	0.0014

Table 2 The energy errors for different target accuracies δ and localization j.

References

1. Ivo Babuska and Robert Lipton. Optimal local approximation spaces for generalized finite element methods with application to multiscale problems. *Multiscale Model. Simul.*, 9(1):373–406, 2011.
2. M. C. C. BAMPTON and R. R. CRAIG, JR. Coupling of substructures for dynamic analyses. *AIAA Journal*, 6(7):1313–1319, 1968.
3. Petter E. Bjørstad, Jacko Koster, and Piotr Krzyżanowski. Domain decomposition solvers for large scale industrial finite element problems. In Tor Sørevik, Fredrik Manne, Assefaw Hadish Gebremedhin, and Randi Moe, editors, *Applied Parallel Computing. New Paradigms for HPC in Industry and Academia*, pages 373–383, Berlin, Heidelberg, 2001. Springer Berlin Heidelberg.

4. F. Bourquin. Component mode synthesis and eigenvalues of second order operators: discretization and algorithm. *RAIRO Modél. Math. Anal. Numér.*, 26(3):385–423, 1992.
5. Frédéric Bourquin. Analysis and comparison of several component mode synthesis methods on one-dimensional domains. *Numer. Math.*, 58(1):11–33, 1990.
6. Juan G. Calvo and Olof B. Widlund. An adaptive choice of primal constraints for BDDC domain decomposition algorithms. *Electron. Trans. Numer. Anal.*, 45:524–544, 2016.
7. Juan Galvis and Yalchin Efendiev. Domain decomposition preconditioners for multiscale flows in high-contrast media. *Multiscale Model. Simul.*, 8(4):1461–1483, 2010.
8. Juan Galvis and Yalchin Efendiev. Domain decomposition preconditioners for multiscale flows in high contrast media: reduced dimension coarse spaces. *Multiscale Model. Simul.*, 8(5):1621–1644, 2010.
9. A. Heinlein, U. Hetmaniuk, A. Klawonn, and O. Rheinbach. The approximate component mode synthesis special finite element method in two dimensions: parallel implementation and numerical results. *J. Comput. Appl. Math.*, 289:116–133, 2015.
10. Fredrik Hellman, Patrick Henning, and Axel Må lqvist. Multiscale mixed finite elements. *Discrete Contin. Dyn. Syst. Ser. S*, 9(5):1269–1298, 2016.
11. Ulrich Hetmaniuk and Axel Klawonn. Error estimates for a two-dimensional special finite element method based on component mode synthesis. *Electron. Trans. Numer. Anal.*, 41:109–132, 2014.
12. Ulrich L. Hetmaniuk and Richard B. Lehoucq. A special finite element method based on component mode synthesis. *M2AN Math. Model. Numer. Anal.*, 44(3):401–420, 2010.
13. Thomas J. R. Hughes, Gonzalo R. Feijóo, Luca Mazzei, and Jean-Baptiste Quincy. The variational multiscale method—a paradigm for computational mechanics. *Comput. Methods Appl. Mech. Engrg.*, 166(1-2):3–24, 1998.
14. Walter C. Hurty. Vibrations of structural systems by component mode synthesis. *Journal of the Engineering Mechanics Division*, 86(4):51–70, 1960.
15. A. Klawonn, P. Radtke, and O. Rheinbach. FETI-DP methods with an adaptive coarse space. *SIAM J. Numer. Anal.*, 53(1):297–320, 2015.
16. Axel Må lqvist and Daniel Peterseim. Localization of elliptic multiscale problems. *Math. Comp.*, 83(290):2583–2603, 2014.
17. Alexandre Madureira and Marcus Sarkis. Adaptive deluxe bddc mixed and hybrid primal discretizations. In Petter E. Bjørstad, Susanne C. Brenner, Lawrence Halpern, Hyea Hyun Kim, Ralf Kornhuber, Talal Rahman, and Olof B. Widlund, editors, *Domain Decomposition Methods in Science and Engineering XXIV*, pages 465–473, Cham, 2018. Springer International Publishing.
18. Alexandre L. Madureira and Marcus Sarkis. Adaptive ACMS: A robust localized Approximated Component Mode Synthesis Method. *arXiv e-prints*, page arXiv:1709.04044, Sep 2017.
19. Jan Mandel and Bedřich Sousedík. Adaptive selection of face coarse degrees of freedom in the BDDC and the FETI-DP iterative substructuring methods. *Comput. Methods Appl. Mech. Engrg.*, 196(8):1389–1399, 2007.
20. Frédéric Nataf, Hua Xiang, Victorita Dolean, and Nicole Spillane. A coarse space construction based on local Dirichlet-to-Neumann maps. *SIAM J. Sci. Comput.*, 33(4):1623–1642, 2011.
21. Duk-Soon Oh, Olof B. Widlund, Stefano Zampini, and Clark R. Dohrmann. BDDC algorithms with deluxe scaling and adaptive selection of primal constraints for Raviart-Thomas vector fields. *Math. Comp.*, 87(310):659–692, 2018.
22. Houman Owhadi, Lei Zhang, and Leonid Berlyand. Polyharmonic homogenization, rough polyharmonic splines and sparse super-localization. *ESAIM Math. Model. Numer. Anal.*, 48(2):517–552, 2014.
23. Clemens Pechstein and Clark R. Dohrmann. A unified framework for adaptive BDDC. *Electron. Trans. Numer. Anal.*, 46:273–336, 2017.
24. Daniel Peterseim and Robert Scheichl. Robust numerical upscaling of elliptic multiscale problems at high contrast. *Comput. Methods Appl. Math.*, 16(4):579–603, 2016.