HIERARCHICAL MODELING OF HETEROGENEOUS PLATES

ANA C. CARIUS AND ALEXANDRE L. MADUREIRA

ABSTRACT. The use of asymptotic limits to model heterogeneous plates can be troublesome, since it requires *a priori* knowledge on the ratio between characteristic lengths of heterogeneities and thickness. Moreover, it also relies on some assumption on the inclusions, like periodicity.

We propose and analyze here *hierarchical modeling* techniques, and show that such approach not only avoids such pitfalls, but it is actually simpler to obtain, and it provably converges to the correct asymptotic limits. Its derivation does not requires any restrictive assumptions on the heterogeneities.

1. INTRODUCTION

Three-dimensional plate models involve *dimension reduction* techniques. The aim is the generation of approximate two-dimensional models from three-dimensional problems, and classical techniques consider, *a priori*, mechanical or geometrical hypothesis. Dimension reduction modeling is important in the study of three-dimensional plates since two-dimensional models are simpler than three-dimensional ones, in particular from the numerical point of view. It is necessary however to establish in what sense the two-dimensional approximation for the three-dimensional model is satisfactory.

There are different dimension reduction techniques, remarking that combinations of them are sometimes used [2, 4, 8, 9, 11, 16, 18–20]. See also [1, 3] for an interesting investigation of a similar question, related to effective boundary conditions. A classical approach is to employ geometrical and physical considerations to derive models. An alternative is to use *asymptotic techniques*, which are often used not only to justify models, but also to *obtain them*. The third way, which we explore here, is to use *hierarchical models*, based on careful choices of variational formulations.

We consider here an elliptic problem, for simplicity the Poisson equation, posed in a heterogeneous plate. The presence of two small parameters (the thickness and the inclusions) brings an extra difficulty to the modeling problem. For instance, depending on the relationship between these two parameters the problem has distinct asymptotic limits. This situation was carefully investigated by Caillerie [8], under a periodicity assumption, and he showed that the vanishing thickness limit and homogenization do not commute, leading to different plate models. It seems clear that this is not a reasonable way to obtain a good model that is convergent for all regimes.

Date: April 26, 2014.

Key words and phrases. Homogenization; Plates; Dimension reduction.

The second author was supported by the grant CNPq 308360/2010-9.

Seeking to overcome such limitations, this work explores *hierarchical modeling* as a dimensional reduction technique, obtaining a unique two-dimensional model that asymptotically converges to the exact, three-dimension solution, regardless of relative sizes of the thickness and heterogeneities.

In this work, we consider the Poisson equation in a heterogeneous plate of thickness 2δ. Let $P^\delta = \Omega \times (-\delta, \delta)$, where $\Omega \subset \mathbb{R}^2$ is a bounded open domain, with Lipschitz continuous border $\partial\Omega$. We denote the top and the bottom of the plate by $\partial P^\delta_\pm = \Omega \times \{-\delta, \delta\}$, and the lateral part of the plate by $\partial P^\delta_L = \partial\Omega \times (-\delta, \delta)$.

Consider the Poisson problem of finding $u^{\delta\epsilon}_{3D} : P^\delta \to \mathbb{R}$ solution of

$$
\begin{gathered}
-\operatorname{div} \underline{\underline{a}}^\epsilon \, \underline{\nabla} \, u^{\delta\epsilon}_{3D} = f^\delta \quad \text{in } P^\delta, \\
\big(\underline{\underline{a}}^\epsilon \, \underline{\nabla} \, u^{\delta\epsilon}_{3D}\big) \cdot \underline{n} = 0 \quad \text{on } \partial P^\delta_\pm, \qquad u^{\delta\epsilon}_{3D} = 0 \quad \text{on } \partial P^\delta_L,
\end{gathered} \tag{1}
$$

where $f^\delta \in L^2(P^\delta)$ and $\underline{\underline{a}}^\epsilon : \Omega \to \mathbb{R}^{3\times 3}_{\text{sym}}$ are given. Here, $\mathbb{R}^{3\times 3}_{\text{sym}}$ is the space of 3×3 symmetric matrices. The thermal conductivity tensor $\underline{\underline{a}}^\epsilon : P^\delta \to \mathbb{R}^{3\times 3}_{\text{sym}}$ might be quite arbitrary, but we append the symbol ϵ to indicate that small inclusions are allowed. As usual, we assume that there exist constants c_0 and c_1 that independ on ε and such that

$$
c_0 \sum_{i=1}^{3} \xi_i^2 \le \underline{\xi}^T \underline{\underline{a}}^\epsilon \underline{\xi} \le c_1 \sum_{i=1}^{3} \xi_i^2 \tag{2}
$$

for every $\underline{\xi} \in \mathbb{R}^3$ and almost every $\underline{x} \in P^\delta$. Finally, for simplicity, we do not allow $\underline{\underline{a}}^\epsilon$ to depend on x_3 [6].

We introduce the notation $\underline{x} = (\underset{\sim}{x}, x_3) \in P^\delta$ to indicate a point in the domain P^δ, where $\underset{\sim}{x} = (x_1, x_2) \in \Omega$ and $x_3 \in (-\delta, \delta)$. Analogously, a vector with three components is denoted, for instance, by $\underline{v} = (\underset{\sim}{v}, v_3)$, where $\underset{\sim}{v} = (v_1, v_2)$, and we use similar notation for the operator $\underline{\nabla} = (\underset{\sim}{\nabla}, \partial/\partial x_3)$, where $\underset{\sim}{\nabla} = (\partial/\partial x_1, \partial/\partial x_2)$. In what follows, we decompose an element $\underline{\underline{a}} \in \mathbb{R}^{3\times 3}_{\text{sym}}$ as

$$
\underline{\underline{a}} = \begin{pmatrix} \underset{\approx}{a} & \underset{\sim}{a} \\ \underset{\sim}{a}^T & a_{33} \end{pmatrix},
$$

where $\underset{\approx}{a}$ is a 2×2 symmetric matrix.

We will denote by $L^2(\Omega)$ the space of square Lebesgue integrable functions, by $H^1(\Omega)$ the subspace of $L^2(\Omega)$ of functions that have derivatives in $L^2(\Omega)$, and by $H^1_0(\Omega)$ the space of functions in $H^1(\Omega)$ with zero trace on $\partial\Omega$.

The rest of this paper is organized as follows. In Section 2 we present the hierarchical modeling method, and obtain a two-dimensional model for (1). In Section 3 we argue that the model obtained is asymptotically consistent, in a sense that we make clear. Finally some conclusions are drawn in Section 4.

2. Hierarchical Modeling

Defining

$$
V(P^\delta) = \{v \in H^1(P^\delta) : v = 0 \text{ on } \partial P^\delta_L\},
$$

we have that $u_{3D}^{\delta\epsilon} \in V(P^\delta)$ solves

$$\int_{P^\delta} \underline{\underline{a}}^\epsilon \, \underline{\nabla}\, u_{3D}^{\delta\epsilon} \cdot \underline{\nabla}\, v \, d\underline{x} = \int_{P^\delta} f^\delta v \, d\underline{x}, \quad \text{for all } v \in V(P^\delta).$$

To obtain the hierarchical models, it is enough to consider subspaces of $V(P^\delta)$ with polynomial dependence in the x_3 variable. Let

$$V_p(P^\delta) = \{v \in V(P^\delta) : v(\underset{\sim}{x}, x_3) = v_0(\underset{\sim}{x}) + v_1(\underset{\sim}{x})x_3 + v_2(\underset{\sim}{x})x_3^2 + \cdots + v_p(\underset{\sim}{x})x_3^p, \\ \text{where } v_0, v_1, v_2 \ldots, v_p \in H_0^1(\Omega)\}.$$

We investigate here the simplest asymptotically consistent model given by $\tilde{u}_{3D}^{\delta\epsilon} \in V_1(P^\delta)$ such that

$$\int_{P^\delta} \underline{\underline{a}}^\epsilon \, \underline{\nabla}\, \tilde{u}_{3D}^{\delta\epsilon} \cdot \underline{\nabla}\, \tilde{v} \, d\underline{x} = \int_{P^\delta} f^\delta \tilde{v} \, d\underline{x} \quad \text{for all } \tilde{v} \in V_1(P^\delta).$$

Writing

$$\tilde{u}_{3D}^{\delta\epsilon}(\underline{x}) = \omega_0^{\delta\epsilon}(\underset{\sim}{x}) + \omega_1^{\delta\epsilon}(\underset{\sim}{x})x_3, \tag{3}$$

using (2) and the definition of $V_1(P^\delta)$, we obtain that

$$\int_{P^\delta} (\underset{\sim}{\nabla}\, \omega_0^{\delta\epsilon} + \underset{\sim}{\nabla}\, \omega_1^{\delta\epsilon} x_3, \omega_1^{\delta\epsilon}) \cdot \underline{\underline{a}}^\epsilon (\underset{\sim}{\nabla}\, v_0, 0) \, d\underline{x} = \int_{P^\delta} f^\delta v_0 \, d\underline{x} \quad \text{for all } v_0 \in H_0^1(\Omega),$$
$$\int_{P^\delta} (\underset{\sim}{\nabla}\, \omega_0^{\delta\epsilon} + \underset{\sim}{\nabla}\, \omega_1^{\delta\epsilon} x_3, \omega_1^{\delta\epsilon}) \cdot \underline{\underline{a}}^\epsilon (\underset{\sim}{\nabla}\, v_1 x_3, v_1) \, d\underline{x} = \int_{P^\delta} f^\delta v \, d\underline{x} \quad \text{for all } v_1 \in H_0^1(\Omega),$$

Integrating with respect to x_3, we gather that $\omega_0^{\delta\epsilon}$ and $\omega_1^{\delta\epsilon}$, both in $H_0^1(\Omega)$, are the weak solution of

$$\begin{gathered} -\operatorname{div}\big(\underset{\approx}{a}^\epsilon \cdot \underset{\sim}{\nabla}\, \omega_0^{\delta\epsilon} + \underset{\sim}{a}^\epsilon \omega_1^{\delta\epsilon}\big) = f_0 \quad \text{in } \Omega. \\ -\frac{\delta^2}{3} \operatorname{div} \underset{\approx}{a}^\epsilon \, \underset{\sim}{\nabla}\, \omega_1^{\delta\epsilon} + \underset{\sim}{a}^\epsilon \, \underset{\sim}{\nabla}\, \omega_0^{\delta\epsilon} + a_{33}^\epsilon \omega_1^{\delta\epsilon} = \delta f_1 \quad \text{in } \Omega, \\ \omega_0^{\delta\epsilon} = \omega_1^{\delta\epsilon} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{4}$$

where we define

$$f_0 = \frac{1}{2\delta} \int_{-\delta}^{\delta} f^\delta \, dx_3, \qquad f_1 = \frac{1}{2\delta^2} \int_{-\delta}^{\delta} x_3 f^\delta \, dx_3.$$

Notice that no assumptions on the heterogeneities are necessary to obtain the two-dimensional model, and its solution depends non-trivially on δ and ϵ. The next step is to show that, at least in certain particular cases, the asymptotic behavior of the model mimics that of the original solution of (1).

3. Asymptotic consistency

We argue in this section that the hierarchical model just presented has the same limits as the exact three-dimensional solution, as δ and ϵ go to zero, no matter the order. The work of Caillerie [8], which presented these results for the original solution $u_{3D}^{\delta\epsilon}$ is of utmost importance here.

3.1. **The vanishing thickness asymptotic limit.** To consider what is the asymptotic limit of $\tilde{u}_{3D}^{\delta\epsilon}$ as $\delta \to 0$, it is enough to analyze (4) and use (3). Assume first that f_0 and f_1 are independent of δ. Formally taking $\delta \to 0$ in the second equation of (4), a step that we will justify latter, we gather that $\omega_0^{\delta\epsilon}$ converges to ω_0^ϵ and $\omega_1^{\delta\epsilon}$ to ω_1^ϵ, where

$$\omega_1^\epsilon = -\frac{1}{a_{33}^\epsilon} \underset{\sim}{a}^\epsilon \cdot \underset{\sim}{\nabla}\, \omega_0^\epsilon. \tag{5}$$

Substituting (5) in the first equation of (4), we have

$$\begin{gathered} -\operatorname{div} \underset{\approx}{A}^\epsilon \underset{\sim}{\nabla}\, \omega_0^\epsilon = f_0 \quad \text{in } \Omega, \\ \omega_0^\epsilon = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{6}$$

where

$$A_{\alpha\beta}^\epsilon = a_{\alpha\beta}^\epsilon - \frac{a_{\alpha 3}^\epsilon a_{3\beta}^\epsilon}{a_{33}^\epsilon} \qquad \text{for } \alpha, \beta = 1, 2.$$

Note that (6) is well-posed since $\underset{\approx}{A}^\epsilon$ is uniformly positive definite, i.e., inequalities similar to (2) hold. To see this, it is enough to check that $\underset{\sim}{\eta} \cdot \underset{\approx}{A}^\epsilon \underset{\sim}{\eta} = \underset{\sim}{\xi} \cdot \underset{=}{a}^\epsilon \underset{\sim}{\xi}$ if $\underset{\sim}{\xi} = (\underset{\sim}{\eta}, -\underset{\sim}{a}^\epsilon \cdot \underset{\sim}{\eta}/a_{33}^\varepsilon)$.

At this point, we remark that (6) is the equation satisfied by the weak limit of $u_{3D}^{\delta\epsilon}$, as shown in [8]. We note that some care has to be taken when interpreting this statement since the solutions of (1) for $\delta > 0$ are three-dimensional functions, but their limit is not. Actually, the limit is independent of the transverse variable, and thus can be identified with a function defined in Ω. The whole process of computing vanishing thickness limits is taken in a "thick" plate $\Omega \times (-1, 1)$, since such domain is δ-independent [8, 9, 15].

The justification of the above formal limit procedure could use outer asymptotic expansions [17], or estimates as in [2]. We however proceed as follows.

Let the variational formulation for (4),

$$a(\omega_0^{\delta\epsilon}, \omega_1^{\delta\epsilon}; v_0, v_1) = \int_\Omega f_0 v_0 \, d\underset{\sim}{x} + \delta \int_\Omega f_1 v_1 \, d\underset{\sim}{x}, \tag{7}$$

where

$$\begin{aligned} a(\omega_0^{\delta\epsilon}, \omega_1^{\delta\epsilon}; v_0, v_1) = \int_\Omega \underset{\approx}{a}^\epsilon \underset{\sim}{\nabla}\, \omega_0^{\delta\epsilon} \cdot \underset{\sim}{\nabla}\, v_0 \, d\underset{\sim}{x} + \int_\Omega \omega_1^{\delta\epsilon} \underset{\sim}{a}^\epsilon \cdot \underset{\sim}{\nabla}\, v_0 \, d\underset{\sim}{x} + \frac{\delta^2}{3} \int_\Omega \underset{\approx}{a}^\epsilon \underset{\sim}{\nabla}\, \omega_1^{\delta\epsilon} \cdot \underset{\sim}{\nabla}\, v_1 \, d\underset{\sim}{x} \\ + \int_\Omega \underset{\sim}{a}^\epsilon \underset{\sim}{\nabla}\, \omega_0^{\delta\epsilon} v_1 \, d\underset{\sim}{x} + \int_\Omega a_{33}^\epsilon \omega_1^{\delta\epsilon} v_1 \, d\underset{\sim}{x}. \end{aligned} \tag{8}$$

We define the norm $|||\cdot|||$ by

$$|||(w_0, w_1)|||^2 = \|w_0\|_{H^1(\Omega)}^2 + \delta^2 \|w_1\|_{H^1(\Omega)}^2 + \|w_1\|_{L^2(\Omega)}^2,$$

where $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H^1(\Omega)}$ denotes the usual norms in $L^2(\Omega)$ and $H^1(\Omega)$. The bilinear form $a(\cdot, \cdot)$ is coercive and the solutions $(\omega_0^{\delta\epsilon}, \omega_1^{\delta\epsilon})$ bounded with respect to such norm, as stated in the lemma bellow. Such result, combined with the Lax–Milgram Theorem, guarantees in particular the existence and uniqueness of solutions to (7).

Lemma 3.1. *Let $a(\cdot,\cdot)$ be as in (8), and assume that (2) holds. Then, there exists a constant C independent of δ and ϵ such that*

$$a(w_0, w_1; w_0, w_1) \geq C\|\!|(w_0, w_1)\|\!|^2 \quad \text{for all } w_0,\, w_1 \in H_0(\Omega),$$
$$C\|\!|(\omega_0^{\delta\epsilon}, \omega_1^{\delta\epsilon})\|\!| \leq \|f_0\|_{H^{-1}(\Omega)} + \|f_1\|_{H^{-1}(\Omega)},$$

where $(\omega_0^{\delta\epsilon}, \omega_1^{\delta\epsilon})$ solves (7)

Proof. Let $\underset{-}{\xi} = (\underset{\sim}{\nabla}\, w_0, w_1)$ and $\underset{-}{\zeta} = (\underset{\sim}{\nabla}\, w_1, 0)$. Then

$$a(w_0, w_1; w_0, w_1) = \int_\Omega \underset{-}{\xi} \cdot \underset{=}{a}^\epsilon \underset{-}{\xi} + \frac{\delta^2}{3} \underset{-}{\zeta} \cdot \underset{=}{a}^\epsilon \underset{-}{\zeta}\, d\underset{\sim}{x} \geq C\|\!|(w_0, w_1)\|\!|^2$$

due to (2) and the definition of $\|\!| \cdot \|\!|$. The bound for $\|\!|(\omega_0^{\delta\epsilon}, \omega_1^{\delta\epsilon})\|\!|$ follows from the coercivity of $a(\cdot,\cdot)$ and (7). □ □

Since $\omega_0^{\delta\epsilon}$ is bounded in $H^1(\Omega)$, and $\omega_1^{\delta\epsilon}$ is bounded in $L^2(\Omega)$, there exist $\omega_0^\epsilon \in H^1(\Omega)$ and $\omega_1^\epsilon \in L^2(\Omega)$, and subsequences of $\omega_0^{\delta\epsilon}$ and $\omega_1^{\delta\epsilon}$, such that $\omega_0^{\delta\epsilon} \rightharpoonup \omega_0^\epsilon$ weakly in $H^1(\Omega)$ and $\omega_1^{\delta\epsilon} \rightharpoonup \omega_1^\epsilon$ weakly in $L^2(\Omega)$, as $\delta \to 0$.

Taking $v_0 = 0$ and the limit $\delta \to 0$ in (7), and using that $\delta \underset{\sim}{\nabla}\, \omega_1^{\delta\epsilon}$ is bounded in $L^2(\Omega)$, we obtain

$$\omega_1^\epsilon = -\frac{\underset{\sim}{a}^\epsilon}{a_{33}^\epsilon} \cdot \underset{\sim}{\nabla}\, \omega_0^\epsilon.$$

Thus, (5) is justified. Considering now $v_1 = 0$ and the limit $\delta \to 0$ in (7), we gather that (6) holds in the weak sense. Since (6) has a unique solution, the whole sequence $\omega_0^{\delta\epsilon}$ and $\omega_1^{\delta\epsilon}$ converges as $\delta \to 0$.

Thus, regardless of assumptions on the heterogenuities, in the vanishing thickness limit the hierarchical and exact solutions coincide, and we write that formally as $\lim_{\delta\to 0} \tilde{u}_{3D}^{\delta\epsilon} = \lim_{\delta\to 0} u_{3D}^{\delta\epsilon}$. Hence, making further assumptions with respect to ε (periodicity for instance), and taking the limit with respect to ε, it follows that $\lim_{\varepsilon\to 0} \lim_{\delta\to 0} \tilde{u}_{3D}^{\delta\epsilon} = \lim_{\varepsilon\to 0} u_{3D}^{\delta\epsilon}$.

3.2. **Making $\varepsilon \to 0$ first.** We now consider the asymptotic limit $\varepsilon \to 0$, for a fixed δ, and $\underset{=}{a}^\epsilon$ periodic. The convergence results follow from standard arguments [7], and we thus opt to develop the formal two scale asymptotic expansion [13, 14]. After that we take $\delta \to 0$ and conclude that, again, the exact and model solutions have the same limits.

Assume that $\underset{=}{a}^\epsilon$ is periodic with periodicity ϵ, i.e., there exists a periodic, ε-independent function $\underset{=}{a}$ such that $\underset{=}{a}^\epsilon(\underset{\sim}{x}) = \underset{=}{a}(\varepsilon^{-1}\underset{\sim}{x})$. We assume that $\underset{=}{a}$ has period l_α with respect to the αth coordinate, and define $Y = (0, l_1) \times (0, l_2)$. Let

$$\omega_0^{\delta\epsilon}(\underset{\sim}{x}) \sim w_0^{\delta,0}(\underset{\sim}{x}, \underset{\sim}{y}) + \varepsilon w_0^{\delta,1}(\underset{\sim}{x}, \underset{\sim}{y}) + \varepsilon^2 w_0^{\delta,2}(\underset{\sim}{x}, \underset{\sim}{y}) + \dots,$$
$$\omega_1^{\delta\epsilon}(\underset{\sim}{x}) \sim w_1^{\delta,0}(\underset{\sim}{x}, \underset{\sim}{y}) + \varepsilon w_1^{\delta,1}(\underset{\sim}{x}, \underset{\sim}{y}) + \varepsilon^2 w_1^{\delta,2}(\underset{\sim}{x}, \underset{\sim}{y}) + \dots,$$

where $\underset{\sim}{y} = \epsilon^{-1}\underset{\sim}{x}$ and w_α^j, $\alpha = 1,2$ and $j = 1,2,...$ are periodic with respect to $\underset{\sim}{y}$. Formally substituting the above expansions in the first equation of (4), we gather that

$$
\begin{aligned}
-\epsilon^{-2}\operatorname{div}_{\underset{\sim}{y}}\underset{\approx}{a}\underset{\sim}{\nabla}_{\underset{\sim}{y}} w_0^{\delta,0} - \epsilon^{-1}\big[\operatorname{div}_{\underset{\sim}{y}}\underset{\approx}{a}\underset{\sim}{\nabla}_{\underset{\sim}{x}} w_0^{\delta,0} + \operatorname{div}_{\underset{\sim}{x}}\underset{\approx}{a}\underset{\sim}{\nabla}_{\underset{\sim}{y}} w_0^{\delta,0} + \operatorname{div}_{\underset{\sim}{y}}\underset{\approx}{a}\underset{\sim}{\nabla}_{\underset{\sim}{y}} w_0^{\delta,1} + \operatorname{div}_{\underset{\sim}{y}}\big(\underset{\sim}{a} w_1^{\delta,0}\big)\big] \\
-\operatorname{div}_{\underset{\sim}{x}}\underset{\approx}{a}\underset{\sim}{\nabla}_{\underset{\sim}{x}} w_0^{\delta,0} - \operatorname{div}_{\underset{\sim}{x}}\underset{\approx}{a}\underset{\sim}{\nabla}_{\underset{\sim}{y}} w_0^{\delta,1} - \operatorname{div}_{\underset{\sim}{y}}\underset{\approx}{a}\underset{\sim}{\nabla}_{\underset{\sim}{x}} w_0^{\delta,1} - \operatorname{div}_{\underset{\sim}{y}}\underset{\approx}{a}\nabla_{\underset{\sim}{y}} w_0^{\delta,2} \\
-\operatorname{div}_{\underset{\sim}{x}}\big(\underset{\sim}{a} w_1^{\delta,0}\big) - \operatorname{div}_{\underset{\sim}{y}}\big(\underset{\sim}{a} w_1^{\delta,1}\big) + \cdots = f_0.
\end{aligned}
$$

We conclude that $w_0^{\delta,0}$ is independent of $\underset{\sim}{y}$, and we write $w_0^{\delta,0}(\underset{\sim}{x},\underset{\sim}{y}) =: \omega_0^\delta(\underset{\sim}{x})$. Considering now the second equation of (4), we have

$$
\begin{aligned}
-\frac{\delta^2}{3}\varepsilon^{-2}\operatorname{div}_{\underset{\sim}{y}}\underset{\approx}{a}\underset{\sim}{\nabla}_{\underset{\sim}{y}} w_1^{\delta,0} - \varepsilon^{-1}\frac{\delta^2}{3}\big[\operatorname{div}_{\underset{\sim}{x}}\underset{\approx}{a}\underset{\sim}{\nabla}_{\underset{\sim}{y}} w_1^{\delta,0} + \operatorname{div}_{\underset{\sim}{y}}\underset{\approx}{a}\underset{\sim}{\nabla}_{\underset{\sim}{x}} w_1^{\delta,0} + \operatorname{div}_{\underset{\sim}{y}}\underset{\approx}{a}\underset{\sim}{\nabla}_{\underset{\sim}{y}} w_1^{\delta,1}\big] \\
+\underset{\sim}{a}\cdot\underset{\sim}{\nabla}_{\underset{\sim}{x}}\omega_0^\delta - \frac{\delta^2}{3}\operatorname{div}_{\underset{\sim}{x}}\underset{\approx}{a}\underset{\sim}{\nabla}_{\underset{\sim}{x}} w_1^{\delta,0} - \frac{\delta^2}{3}\operatorname{div}_{\underset{\sim}{x}}\underset{\approx}{a}\underset{\sim}{\nabla}_{\underset{\sim}{y}} w_1^{\delta,1} - \frac{\delta^2}{3}\operatorname{div}_{\underset{\sim}{y}}\underset{\approx}{a}\underset{\sim}{\nabla}_{\underset{\sim}{x}} w_1^{\delta,0} \\
-\frac{\delta^2}{3}\operatorname{div}_{\underset{\sim}{y}}\underset{\approx}{a}\underset{\sim}{\nabla}_{\underset{\sim}{y}} w_1^{\delta,2} + \underset{\sim}{a}\cdot\underset{\sim}{\nabla}_{\underset{\sim}{y}} w_0^{\delta,1} + a_{33}w_1^{\delta,0} + \cdots = \delta f_1.
\end{aligned}
$$

Considering the term with the power ϵ^{-2} we conclude that $w_1^{\delta,0}$ is independent of $\underset{\sim}{y}$, and we write $w_1^0(\underset{\sim}{x},\underset{\sim}{y}) =: \omega_1^\delta(\underset{\sim}{x})$. Grouping the terms with the power ϵ^{-1} in both equations, we have

$$
\operatorname{div}_{\underset{\sim}{y}}\underset{\approx}{a}\underset{\sim}{\nabla}_{\underset{\sim}{y}} w_0^{\delta,1} = -\operatorname{div}_{\underset{\sim}{y}}\underset{\approx}{a}\underset{\sim}{\nabla}_{\underset{\sim}{x}}\omega_0^\delta - \operatorname{div}_{\underset{\sim}{y}}\big(\underset{\sim}{a}\omega_1^\delta\big), \qquad \operatorname{div}_{\underset{\sim}{y}}\underset{\approx}{a}\underset{\sim}{\nabla}_{\underset{\sim}{y}} w_1^{\delta,1} = -\operatorname{div}_{\underset{\sim}{y}}\underset{\approx}{a}\underset{\sim}{\nabla}_{\underset{\sim}{x}}\omega_1^\delta.
$$

To satisfy both equations, we set

$$
w_0^{\delta,1}(\underset{\sim}{x},\underset{\sim}{y}) = \sum_{\beta=1}^2 \overline{\chi}_\beta(\underset{\sim}{y})\frac{\partial\omega_0^\delta}{\partial x_\beta}(\underset{\sim}{x}) + \overline{\chi}_3(\underset{\sim}{y})\omega_1^\delta(\underset{\sim}{x}), \qquad w_1^{\delta,1}(\underset{\sim}{x},\underset{\sim}{y}) = \sum_{\beta=1}^2 \overline{\chi}_\beta(\underset{\sim}{y})\frac{\partial\omega_1^\delta}{\partial x_\beta}(\underset{\sim}{x}),
$$

where we introduce the cell problems for $j = 1,2,3$:

$$
\operatorname{div}\big(\underset{\approx}{a}\nabla\overline{\chi}_j\big) = -\sum_{\alpha=1}^2\frac{\partial a_{\alpha j}}{\partial y_\alpha} \quad \text{in } Y,
$$

plus periodic boundary conditions. Finally, collecting the terms with the power ϵ^0 in the both equations, using periodicity arguments and the definitions for the functions $w_0^{\delta,1}$ and $w_1^{\delta,1}$, we have

$$
\begin{aligned}
-\sum_{\alpha,\beta=1}^2\partial_\alpha\big(\overline{A}_{\alpha\beta}\partial_\beta\omega_0^\delta\big) - \sum_{\alpha=1}^2\partial_\alpha\big(\overline{A}_{\alpha3}\omega_1^\delta\big) = f_0, \\
-\frac{\delta^2}{3}\sum_{\alpha,\beta=1}^2\partial_\alpha\big(\overline{A}_{\alpha\beta}\partial_\beta\overline{w}_1\big) + \sum_{\alpha=1}^2\overline{A}_{3\alpha}\partial_\alpha\omega_0^\delta + \overline{A}_{33}\omega_1^\delta = \delta f_1,
\end{aligned} \tag{9}
$$

where $\partial_\alpha\cdot = \partial\cdot/\partial x_\alpha$ and

$$
\overline{A}_{ij} = \int_Y a_{ij} + \sum_{\beta=1}^2 a_{i\beta}\frac{\partial\overline{\chi}_j}{\partial y_\beta}\,d\underset{\sim}{y} \quad \text{for } i = 1,2,3.
$$

It is very interesting to note that applying hierarchical modeling as dimension reduction technique for the three-dimensional original problem, and then homogenizing the resultant problem is equivalent to homogenize the three-dimensional original problem and then apply the hierarchical modeling technique.

We can now proceed as in Subsection 3.1 and consider the limit $\delta \to 0$, to conclude that $\omega_0^\delta \rightharpoonup \omega_0$ weakly in $H^1(\Omega)$ and $\omega_1^\delta \rightharpoonup \omega_1$ weakly in $L^2(\Omega)$, where

$$w_1 = -\frac{1}{\overline{A}_{33}} \sum_{\alpha=1}^{2} \overline{A}_{3\alpha} \partial_\alpha w_0 \in L^2(\Omega),$$

and $w_0 \in H_0^1(\Omega)$ is the weak solution of

$$\begin{aligned} -2 \operatorname{div}_{\underset{\sim}{x}} \underset{\approx}{B} \underset{\sim}{\nabla}_{\underset{\sim}{x}} w_0 &= f_0 \quad \text{in } \Omega, \\ w_0 &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{10}$$

with

$$B_{\alpha\beta} = \overline{A}_{\alpha\beta} - \frac{\overline{A}_{\alpha 3}\overline{A}_{3\beta}}{\overline{A}_{33}}, \quad \text{for } \alpha, \beta = 1, 2.$$

In conclusion, (10) yields the equation that the limit as $\varepsilon \to 0$ and then $\delta \to 0$ of the hierarchical model solution (which is, in the limit, independent of x_3) must satisfy. It turns out that this statement also holds for the limit of the solution of the original problem [8]. In other words, $\lim_{\delta\to 0} \lim_{\varepsilon\to 0} \tilde{u}_{3D}^{\delta\epsilon} = \lim_{\delta\to 0} \lim_{\varepsilon\to 0} u_{3D}^{\delta\epsilon}$.

4. Conclusion

Dimension reduction techniques face a more difficult task under the presence of heterogeneities. Asymptotic limits, a powerful analysis tool, do not yield good models since the final equations depend on *a priori* assumptions that are too restrictive. This is not the case if hierarchical modeling is employed, and adds yet another reason for preferring hierarchical models over asymptotic ones [5].

The model proposed here is obtained without *any* unreasonable assumptions on the heterogeneities, and the result system is always well-posed. In terms of analysis, if one considers the vanishing thickness limit, the model has the same limit as the exact solution, and again no assumptions on the heterogeneities are needed. One the other hand, homogenizing the hierarchical model is the same as homogenizing the original three-dimensional problem and then reduce dimension. And it turns out that, again, the vanishing thickness limit of the homogenized original and the homogenized model solutions coincide.

[1] Toufic Abboud and Habib Ammari, *Diffraction at a curved grating: TM and TE cases, homogenization*, J. Math. Anal. Appl. **202** (1996), no. 3, 995–1026, DOI 10.1006/jmaa.1996.0357. MR1408364 (98b:78027)

[2] Stephen M. Alessandrini, Douglas N. Arnold, Richard S. Falk, and Alexandre L. Madureira, *Derivation and justification of plate models by variational methods*, Plates and shells (Québec, QC, 1996), CRM Proc. Lecture Notes, vol. 21, Amer. Math. Soc., Providence, RI, 1999, pp. 1–20. MR1696513 (2000j:74055)

[3] Habib Ammari and Chiraz Latiri-Grouz, *Conditions aux limites approchées pour les couches minces périodiques*, M2AN Math. Model. Numer. Anal. **33** (1999), no. 4, 673–693, DOI 10.1051/m2an:1999157 (French, with English and French summaries). MR1726479 (2000k:78015)

[4] Douglas N. Arnold and Alexandre L. Madureira, *Asymptotic estimates of hierarchical modeling*, Math. Models Methods Appl. Sci. **13** (2003), no. 9, 1325–1350, DOI 10.1142/S0218202503002933. MR2005646 (2004j:35074)

[5] Douglas N. Arnold, Alexandre L. Madureira, and Sheng Zhang, *On the range of applicability of the Reissner-Mindlin and Kirchhoff-Love plate bending models*, J. Elasticity **67** (2002), no. 3, 171–185 (2003), DOI 10.1023/A:1024986427134. MR1997951 (2004e:74053)

[6] Ferdinando Auricchio, Carlo Lovadina, and Alexandre L. Madureira, *An asymptotically optimal model for isotropic heterogeneous linearly elastic plates*, M2AN Math. Model. Numer. Anal. **38** (2004), no. 5, 877–897, DOI 10.1051/m2an:2004042. MR2104433 (2005i:74051)

[7] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic analysis for periodic structures*, AMS Chelsea Publishing, Providence, RI, 2011. Corrected reprint of the 1978 original [MR0503330]. MR2839402

[8] D. Caillerie, *Homogénéisation des équations de la diffusion stationnaire dans les domaines cylindriques aplatis*, RAIRO Anal. Numér. **15** (1981), no. 4, 295–319 (French, with English summary). MR642495 (83g:80003)

[9] P. G. Ciarlet and P. Destuynder, *A justification of the two-dimensional linear plate model*, J. Mécanique **18** (1979), no. 2, 315–344 (English, with French summary). MR533827 (80e:73046)

[10] A. C. Carius, *Modelagem hierrquica para a equao de Poisson e para o problema de elasticidade linear em uma placa heterognea*, Laboratrio Nacional de Computao Cientfica, 2012 (Portuguese, with English summary). D.Sc. Thesis.

[11] Philippe G. Ciarlet, *Mathematical elasticity. Vol. II*, Studies in Mathematics and its Applications, vol. 27, North-Holland Publishing Co., Amsterdam, 1997. Theory of plates. MR1477663 (99e:73001)

[12] Doina Cioranescu and Jeannine Saint Jean Paulin, *Homogenization of reticulated structures*, Applied Mathematical Sciences, vol. 136, Springer-Verlag, New York, 1999. MR1676922 (2000d:74064)

[13] G. A. Chechkin, A. L. Piatnitski, and A. S. Shamaev, *Homogenization*, Translations of Mathematical Monographs, vol. 234, American Mathematical Society, Providence, RI, 2007. Methods and applications; Translated from the 2007 Russian original by Tamara Rozhkovskaya. MR2337848 (2008j:35013)

[14] Doina Cioranescu and Patrizia Donato, *An introduction to homogenization*, Oxford Lecture Series in Mathematics and its Applications, vol. 17, The Clarendon Press Oxford University Press, New York, 1999. MR1765047 (2001j:35019)

[15] Monique Dauge and Isabelle Gruais, *Asymptotics of arbitrary order for a thin elastic clamped plate. I. Optimal error estimates*, Asymptotic Anal. **13** (1996), no. 2, 167–197. MR1413859 (98b:73022)

[16] P. Destuynder, *Une théorie asymptotique des plaques minces en élasticité linéaire*, Recherches en Mathématiques Appliquées [Research in Applied Mathematics], vol. 2, Masson, Paris, 1986 (French). MR830660 (87g:73050)

[17] A. M. Il′in, *Matching of asymptotic expansions of solutions of boundary value problems*, Translations of Mathematical Monographs, vol. 102, American Mathematical Society, Providence, RI, 1992. Translated from the Russian by V. Minachin [V. V. Minakhin]. MR1182791 (93g:35016)

[18] Alexandre L. Madureira, *Hierarchical modeling based on mixed principles: asymptotic error estimates*, Math. Models Methods Appl. Sci. **15** (2005), no. 7, 985–1008, DOI 10.1142/S0218202505000662. MR2151796 (2006a:74057)

[19] B. Miara, *Optimal spectral approximation in linearized plate theory*, Appl. Anal. **31** (1989), no. 4, 291–307, DOI 10.1080/00036818908839832. MR1017518 (90g:73078)

[20] M. Vogelius and I. Babuška, *On a dimensional reduction method. I. The optimal selection of basis functions*, Math. Comp. **37** (1981), no. 155, 31–46, DOI 10.2307/2007498. MR616358 (83c:65259a)

Laboratório Nacional de Computação Científica, Av. Getúlio Vargas, 333, 25651-070 Petrópolis - RJ, Brazil,

E-mail address: ana.carius@ifrj.edu.br

Laboratório Nacional de Computação Científica and Fundação getúlio Vargas

E-mail address: alm@lncc.br

E-mail address: alexandre.madureira@fgv.br