DIMENSIONAL REDUCTION FOR PLATES BASED ON MIXED VARIATIONAL PRINCIPLES*

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ABSTRACT

We consider the derivation and rigorous justification of models for thin linearly elastic plates using mixed variational principles.

We consider an isotropic, homogeneous, linearly elastic plate occupying the region $P_t = \Omega \times (-t/2, t/2)$, with Ω a smoothly bounded domain in $\mathbb{R}^2$ and $t \in (0, 1]$. We denote the union of the top and bottom surfaces of the plate by $\partial P_t^{\pm} = \Omega \times \{-t/2, t/2\}$ and the lateral boundary by $\partial P_t^{\mathrm{L}} = \partial\Omega \times (-t/2, t/2)$. We suppose that the plate is loaded by a surface force density $\underline{g} : \partial P_t^{\pm} \to \mathbb{R}^3$ and a volume force density $\underline{f} : P_t \to \mathbb{R}^3$, and is clamped along its lateral boundary. The resulting stress $\underline{\underline{\sigma}}^* : P_t \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ and displacement $\underline{u}^* : P_t \to \mathbb{R}^3$ then satisfy the boundary-value problem

$$\underline{\underline{\underline{A}}}\,\underline{\underline{\sigma}}^* = \underline{\underline{\varepsilon}}(\underline{u}^*), \quad -\underline{\mathrm{div}}\,\underline{\underline{\sigma}}^* = \underline{f} \quad \text{in } P_t, \qquad \underline{\underline{\sigma}}^*\underline{n} = \underline{g} \text{ on } \partial P_t^{\pm}, \qquad \underline{u}^* = 0 \text{ on } \partial P_t^{\mathrm{L}}, \tag{1}$$

where $\underline{\underline{\varepsilon}}(\underline{u}^*)$ denotes the infinitesimal strain tensor and $\underline{\underline{\underline{A}}}$ is the usual isotropic compliance tensor.

We discuss systematic procedures of dimensional reduction of the three-dimensional problem to two-dimensional plate models which proceed from variational formulations of the three-dimensional problem (1). Besides the derivation of models, we also consider their rigorous justification. Namely, we study the convergence to zero of the relative error in energy norm on the three-dimensional plate domain P_t of an approximation of the three-dimensional solution

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determined from the solution of the dimensionally reduced model. For some of the models we show that this error tends to zero, and establish the rate of convergence. A fuller development of the ideas discussed here may be found in [2].

The variational approach to dimensional reduction is systematic and is tied naturally to rigorous convergence theory. These characteristics are shared with another important approach to dimensional reduction of shells, the asymptotic approach developed, for example, in the book [4] of Ciarlet. However the variational approach also differs from this asymptotic analysis in a number of significant ways. The asymptotic approach essentially identifies one particular canonical plate model, the limiting solution of the three-dimensional elastic problem. This is the Kirchhoff–Love model, the superposition of the generalized plane stess model of plate stretching and the biharmonic model of plate bending. By contrast, the variational approach naturally generates a hierarchical family of plate models by using polynomial approximations of increasing degree. In fact, we consider two different mixed variational principles, and each leads to several different hierarchical families of models. It is interesting to note that the simplest plate stretching model arising from a variational approach to dimensional reduction is again generalized plane stress, but the biharmonic plate bending model does *not* arise naturally from this approach. Instead, the simplest bending model arising is (a form of) the Reissner–Mindlin model.

Before proceeding, we summarize some notational conventions. We write first-order tensors (or 3-vectors) with one underbar, second-order tensors (or 3×3 matrices) with two underbars, etc. For tensors in two variables we use undertildes in the same way. Any 3-vector may be expressed in terms of a 2-vector giving its in-plane components and a scalar giving its transverse component, and any 3×3 symmetric matrix may be expressed in terms of a 2×2 symmetric matrix, a 2-vector, and a scalar thus:

$$\underline{v} = \begin{pmatrix} \underset{\sim}{v} \\ v_3 \end{pmatrix}, \quad \underline{\underline{\tau}} = \begin{pmatrix} \underset{\approx}{\tau} & \underset{\sim}{\tau} \\ \underset{\sim}{\tau}^T & \tau_{33} \end{pmatrix}.$$

The starting point for the variational approach to dimensional reduction is the Hellinger–Reissner variational principle. We consider two variant forms of this principle. The first, which we refer to as HR, characterizes $(\underline{\underline{\sigma}}^*, \underline{u}^*)$ as the unique critical point (namely a saddle point) of the HR functional

$$J(\underline{\underline{\tau}}, \underline{v}) = \frac{1}{2}\int_{P_t} \underline{\underline{\underline{\underline{A}}}}\,\underline{\underline{\tau}} : \underline{\underline{\tau}}\, d\underline{x} - \int_{P_t} \underline{\underline{\tau}} : \underline{\underline{\varepsilon}}(\underline{v})\, d\underline{x} + \int_{P_t} \underline{f} \cdot \underline{v}\, d\underline{x} + \int_{\partial P_t^{\pm}} \underline{g} \cdot \underline{v}\, d\underset{\sim}{x}$$

over $\underline{\underline{\Sigma}}^{\bullet} \times \underline{V}^{\bullet} := \underline{\underline{L}}^2(P_t) \times \{\, \underline{v} \in \underline{H}^1(P_t) \,:\, \underline{v} = 0 \text{ on } \partial P_t^{\mathrm{L}} \,\}$. The second form of the Hellinger–Reissner principle, HR$'$ characterizes $(\underline{\underline{\sigma}}^*, \underline{u}^*)$ as the unique critical point (again a saddle point) of the HR$'$ functional

$$J'(\underline{\underline{\tau}}, \underline{v}) = \frac{1}{2}\int_{P_t} \underline{\underline{\underline{\underline{A}}}}\,\underline{\underline{\tau}} : \underline{\underline{\tau}}\, d\underline{x} + \int_{P_t} \underline{\operatorname{div}}\, \underline{\underline{\tau}} \cdot \underline{v}\, d\underline{x} + \int_{P_t} \underline{f} \cdot \underline{v}\, d\underline{x}$$

on $\underline{\underline{\Sigma}}_g^* \times \underline{V}^* := \{\, \underline{\underline{\sigma}} \in \underline{\underline{H}}(\underline{\operatorname{div}}, P_t) \,|\, \underline{\underline{\sigma}}\underline{n} = \underline{g} \text{ on } \partial P_t^{\pm} \,\} \times \underline{L}^2(P)$.

Plate models may be derived by replacing $\underline{\underline{\Sigma}}^{\bullet} \times \underline{V}^{\bullet}$ in HR with subspaces which admit only a specified polynomial dependence on x_3. If the subspaces $\underline{\underline{\Sigma}}$ and $\underline{V}$ are chosen carelessly, there may not exist any such critical point or it may not be unique. We insure a unique solution by insisting that $\underline{\underline{\varepsilon}}(\underline{V}) \subset \underline{\underline{\Sigma}}$. In the simplest example, we seek a saddle-point of the HR functional over $\underline{\underline{\sigma}} \in \underline{\underline{\Sigma}}^\bullet$ with $\underset{\approx}{\sigma}$ linear in x_3, $\underset{\sim}{\sigma}$ constant in x_3, and σ_{33} zero, and $\underline{u} \in \underline{V}^\bullet$ with $\underset{\sim}{u}$ linear and u_3 constant in x_3. This leads to the lowest order model in the family we refer to as HR_1. This family of models and two other families are described in Table 1.

Table 1. The principle plate models based on the HR principle. The degree p is a positive integer.

model	$\deg_3 \underset{\approx}{\sigma}$	$\deg_3 \underset{\sim}{\sigma}$	$\deg_3 \sigma_{33}$	$\deg_3 \underset{\sim}{u}$	$\deg_3 u_3$
$\mathrm{HR}_1(p)$	p	$p-1$	$p-2$	p	$p-1$
$\mathrm{HR}_2(p)$	p	$p-1$	p	p	$p-1$
$\mathrm{HR}_3(p)$	p	$p+1$	p	p	$p+1$

The $\mathrm{HR}_1(1)$ turns out to yield the classical generalized plane stress model for stretching and a form of the Reissner–Mindlin model (with shear correction factor 1) for bending. It is perhaps the simplest way to derive these models.

The model families $\mathrm{HR}_2(p)$ and $\mathrm{HR}_3(p)$ are *minimum energy* or *energy projection models*. Namely, because they satisfy the condition $\underline{\underline{\underline{A}}}^{-1}\,\underline{\underline{\varepsilon}}(\underline{V}) \subset \underline{\underline{\Sigma}}$, the three-dimensional constitutive equation is satisfied exactly and $\underline{u}$ is determined as the minimizer in $\underline{V}$ of the potential energy. In the literature there has been a great deal of attention paid to the minimum energy models (cf., e.g., [3], [7], [8]), and much less to other models arising mixed variational principles. However, the restriction to minimum energy principles eliminates many of the best features of the variational approach.

A striking failure of the minimum energy approach occurs with the simplest minimum energy model, $\mathrm{HR}_2(1)$. It turns out that this is an incorrect plate model, one which is not even consistent with the Kirchhoff–Love reduced problem in the limit of vanishing thickness. More precisely, the $\mathrm{HR}_1(1)$ model equations are of same form as generalized plane stress and Reissner–Mindlin, but these equations contain spurious terms, causing divergence as t tends to 0 (for both stretching and bending). This phenomenon is well-known and has been studied in some generality recently in [7], where it is shown that for a minimum energy method to be consistent in the $t = 0$ limit, the polynomial spaces must be of higher degree. By contrast, the $\mathrm{HR}_1(1)$ method, which uses spaces of *lower* degree than $\mathrm{HR}_2(1)$, is a consistent method.

While the $\mathrm{HR}_2(1)$ model is incorrect, for $p \geq 3$ it can be shown that the $\mathrm{HR}_2(p)$ model is convergent. (For $p = 3$, it can be shown to be identical to a method of Lo, Christensen, and Wu [5].) The $\mathrm{HR}_3(p)$ is also convergent for all $p \geq 1$. However, even in the simplest case ($p = 1$), the models in this family are more complex (involve more dependent variables) than $\mathrm{HR}_1(1)$.

Table 2. The principle plate models based on the HR′ principle. The degree p is a positive integer.

model	$\deg_3 \underset{\approx}{\sigma}$	$\deg_3 \underset{\sim}{\sigma}$	$\deg_3 \sigma_{33}$	$\deg_3 \underset{\sim}{u}$	$\deg_3 u_3$
$\mathrm{HR}'_1(p)$	p	$p-1$	p	p	$p-1$
$\mathrm{HR}'_2(p)$	p	$p+1$	p	p	$p-1$
$\mathrm{HR}'_3(p)$	p	$p+1$	p	p	$p+1$
$\mathrm{HR}'_4(p)$	p	$p+1$	$p+2$	p	$p+1$

Table 2 shows the principle model families for the HR' principle. Here we wish to particularly emphasize the model $\mathrm{HR}'_4(1)$, which is the simplest *complementary energy model*. Namely, $\underline{\underline{\sigma}}$ minimizes the complementary energy $E_c(\underline{\underline{\tau}}) = (1/2)\int_{P_t} \underline{\underline{\underline{A}}}\underline{\underline{\tau}} : \underline{\underline{\tau}}\, d\underline{x}$ over all $\underline{\underline{\tau}} \in \underline{\underline{\Sigma}}_g$ satisfying the equilibrium condition $\underline{\operatorname{div}}\, \underline{\underline{\tau}} = -P_{\underline{V}} \underline{f}$ ($P_{\underline{V}}$ is the L^2-projection onto $\underline{V}$). This model agains gives rise to the classical generalized plane stress stretching equations, but with the load constructed in a more sophisticated way from the three-dimensional loading. The bending model is again Reissner–Mindlin, but with a shear correction factor of 5/6 and, again, more complicated loads. As we discuss below, this method is not only consistent with the Kirchhoff–Love solution in the limit of vanishing thickness, but, more importantly, it is convergent in relative energy norm. This is a strong property not shared by all methods which are consistent in the thin plate limit. In fact, the Kirchhoff–Love solution is itself not convergent in relative energy norm. Morgenstern, in his pioneering work on the energy convergence of the biharmonic plate model [6], showed that three-dimensional displacement and stress fields could be constructed from the biharmonic solution which converge to the full three-dimensional solution in relative energy norm. However the construction of these fields is rather ad-hoc, and not suggested by the biharmonic model itself. By contrast, the approximation delivered by the $\mathrm{HR}'_4(1)$ model is, without any post-processing, convergent.

The key to the error analysis in [6] is the two energies principle or Prager–Synge theorem, and we follow that approach. This approach requires a stress field which is in equilibrium with the imposed volume and surface loads. Generally such a field is not trivial to construct (especially if volume loading is present—this case wasn't treated in [6]). However the $\mathrm{HR}'_4(1)$ method, being a complementary energy method, automatically generates such a stress field. Combining the two energies principle with careful *a priori* estimates of the plate model solutions, we are able to obtain precise bounds on the energy error $\|\underline{\underline{\varepsilon}}(\underline{u}^* - \underline{u})\|_{0,P_t} + \|\underline{\underline{\sigma}}^* - \underline{\underline{\sigma}}\|_{0,P_t}$ in terms of the thickness t and various norms of the loading functions $\underline{f}$ and $\underline{g}$. For example, if the surface load is purely in-plane and is even in x_3 and the volume load vanishes, we find that $|\underline{\underline{\varepsilon}}(\underline{u}^* - \underline{u})\|_{0,P_t} + \|\underline{\underline{\sigma}}^* - \underline{\underline{\sigma}}\|_{0,P_t} \le$ const., while $\|\underline{\underline{\varepsilon}}(\underline{u})\|_{0,P_t}$ and $\|\underline{\underline{\sigma}}\|_{0,P_t}$ behave as $O(t^{-1/2})$. Thus

$$\frac{\|\underline{\underline{\varepsilon}}(\underline{u}^* - \underline{u})\|_{0,P_t}}{\|\underline{\underline{\varepsilon}}(\underline{u})\|_{0,P_t}} + \frac{\|\underline{\underline{\sigma}}^* - \underline{\underline{\sigma}}\|_{0,P_t}}{\|\underline{\underline{\sigma}}\|_{0,P_t}} \le Ct^{1/2},$$

so the $\mathrm{HR}'_1(4)$ plate model converges with order $t^{1/2}$ measured in relative energy norm in this stretching situation. The same result holds for many other loading cases.

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