HIERARCHICAL MODELING OF PIEZOELECTRIC PLATES

ALEXANDRE L. MADUREIRA

Abstract. We use variational techniques to derive a class of two-dimensional models for three-dimensional linearly elastic piezoelectric plates. The models result from a mixed formulation for the original problem within spaces of functions with polynomial dependence in the transverse direction. We show that the resulting system of equations is well-posed, and then discuss the asymptotic consistency of the simplest of such models.

1. Introduction

Dimension reduction is a powerful tool to model physical phenomena that occur in slender domains, and as more complex problems are considered, it is useful to have a mathematically sound technique to do so. Variational arguments yield just that, and our goal in this paper is to derive a simple model for piezoelectric plates. Our work was particularly motivated by the investigation developed by Sène [21], and Raoult and Sène [19].

In this paper we use variational principles to develop two-dimensional models for static piezoelectric plates. For the importance and applications of such problem, the reader can check [13, 22].

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The first assumption is that a piezoelectric material is occupying a plate domain given by $P = \Omega \times (-\varepsilon, \varepsilon)$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary $\partial\Omega$. The union of the plate's top and bottom surfaces are given by $\partial P_\pm = \Omega \times \{-\varepsilon, \varepsilon\}$, and $\partial P_{\mathrm{L}} = \partial\Omega \times (-\varepsilon, \varepsilon)$ denotes the lateral surface of the plate. We denote a typical point in P by $\underline{x} = (\underset{\sim}{x}, x_3)$, where $\underset{\sim}{x} \in \Omega$ and $x_3 \in (-\varepsilon, \varepsilon)$.

The problem is to find the displacement $\underline{u}^\epsilon$, the electrical potential ϕ^ϵ, the stress tensor $\underline{\underline{\sigma}}^\epsilon$, and the electrical displacement $\underline{D}^\epsilon$ of the plate subject to prescribed internal force density $\underline{f} : P \to \mathbb{R}^3$, surface force density $\underline{g} : \partial P_\pm \to \mathbb{R}^3$, and electric potential $\phi_{bc} : \partial P_\pm \to \mathbb{R}$. The constitutive relations are

$$\underline{\underline{\sigma}}^\epsilon = \underline{\underline{\underline{C}}}\,\underline{\underline{e}}(\underline{u}^\epsilon) - \underline{\nabla}\,\phi^\epsilon \underline{\underline{\underline{Q}}}, \qquad \underline{D}^\epsilon = \underline{\underline{\underline{Q}}}\,\underline{\underline{e}}(\underline{u}^\epsilon) + \underline{\underline{d}}\,\underline{\nabla}\,\phi^\epsilon, \tag{1}$$

or componentwise,

$$\sigma_{ij}^\epsilon = \sum_{k,l=1}^{3} C_{ijkl} e_{kl}(\underline{u}^\epsilon) - \sum_{k=1}^{3} \partial_k \phi^\epsilon Q_{kij}, \qquad D_i^\epsilon = \sum_{k,l=1}^{3} Q_{ikl} e_{kl}(\underline{u}^\epsilon) + \sum_{k=1}^{3} d_{ik} \partial_k \phi^\epsilon,$$

for $i, j = 1, 2, 3$. The equilibrium equations are

$$-\underline{\operatorname{div}}\,\underline{\underline{\sigma}}^\epsilon = \underline{f}, \quad \operatorname{div} \underline{D}^\epsilon = 0 \qquad \text{in } P,$$

with the boundary conditions

$$\underline{u}^\epsilon = 0, \quad \underline{D}^\epsilon \cdot \underline{n} = 0 \qquad \text{on } \partial P_{\mathrm{L}},$$

$$\underline{\underline{\sigma}}^\epsilon \underline{n} = \underline{g}, \quad \phi^\epsilon = \phi_{bc} \qquad \text{on } \partial P_\pm.$$

The rigidity tensor and the infinitesimal strain tensor are given by

$$\underline{\underline{\underline{C}}}\,\underline{\underline{e}}(\underline{u}^\epsilon) = 2\mu\,\underline{\underline{e}}(\underline{u}^\epsilon) + \lambda \operatorname{div} \underline{u}^\epsilon\ \underline{\underline{\delta}}, \qquad \underline{\underline{e}}(\underline{u}^\epsilon) = \frac{1}{2}\big(\underline{\underline{\nabla}} + \underline{\underline{\nabla}}^t\big)\underline{u}^\epsilon,$$

where μ, λ are the Lamé coefficients, and $\underline{\underline{\delta}}$ the identity matrix. For simplicity, we follow [16] and assume that the dielectric tensor is of the form

$$\underline{\underline{d}} = \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d_{33} \end{pmatrix},$$

where d and d_{33} are positive constants. We also assume that the piezoelectric tensor is such that the only nonzero constants are [16]

$$Q_{333}, \quad Q_{113} = Q_{223}, \quad \text{and } Q_{311} = Q_{322},$$

and that the symmetry relations

$$Q_{ijk} = Q_{ikj} \quad \text{for } i, k, j = 1, 2, 3,$$

hold.

As stated above, our goal is to develop, in a consistent mathematical framework, two-dimensional models for the problem just described. There is an extensive literature dealing with the simpler problem of linearly elastic plates. Regarding the modeling of piezoelectric plates, there are derivations based on geometric and mechanical *a priori* assumptions, see for instance [4], or [22] and references therein.

On the mathematical side, some authors generalized the asymptotic arguments that Ciarlet and collaborators used for the linearly elastic plate problem [12]. In particular, Sène [21] showed that as the plate thickness goes to zero, the solution converges in a proper sense to the solution to a biharmonic (17) and a membrane equation (16). See also Maugin and

Attou [16], Weller and Licht [23], and Canon and Lenczner [11] for further developments using such approach.

The way we proceed is different since it is not "asymptotic" in principle, and we find our models using mixed variational formulations. The approach is based on firm mathematical grounds, and the equations form a sequence of *hierarchical models* that become more accurate as the order of the model grow. See [1], and also [3, 6, 7, 14] and references therein, for linearly elastic plates. See [9, 22] for a review of the engineering literature that resorts to variational arguments.

Before proceeding, we need to introduce some notation. The 3×3 symmetric tensors are denoted in Greek letters with double underbars, as in $\underline{\underline{\sigma}}$, $\underline{\underline{\tau}}$. The symbol $\underline{\underline{\delta}}$ denote the identity tensor. For 2×2 symmetric tensors, we use Greek letters with double under-tildes. Similarly, we write vectors in italic letters. If they belong to $\mathbb{R}^3$, they have an under bar and if they belong to $\mathbb{R}^2$, they have an under-tilde. We can then decompose each tensor and vector as in

$$\underline{\underline{\sigma}} = \begin{pmatrix} \underset{\approx}{\sigma} & \underset{\sim}{\sigma} \\ \times & \sigma_{33} \end{pmatrix}, \quad \underline{u} = \begin{pmatrix} \underset{\sim}{u} \\ u_3 \end{pmatrix}.$$

We use four under bars (four under tildes) for fourth order tensors acting on 3×3 (2×2) symmetric tensors. Similar notation holds for third order tensors, and the operators divergence and gradient obey similar notation rules.

Accordingly, if $O \subset \mathbb{R}^d$, $d = 1, 2, 3$, is an open set, then $\underline{\underline{L}}^2(O)$ is the set of 3×3 symmetric matrices which components are square integrable functions in O, and $\underline{L}^2(O)$ and $L^2(O)$ are the set of vector and scalar square integrable functions defined in O. Similar definitions hold for $\underline{\underline{H}}^s(O)$, $\underline{H}^s(O)$ and $H^s(O)$, the Sobolev space of order s, for a real number s. We denote the norms of those spaces by $\|\cdot\|_{L^2(O)}$ and $\|\cdot\|_{H^s(O)}$, and the semi-norms by $|\cdot|_{H^s(O)}$.

The symbol ∂_i denotes the derivative with respect to the variable x_i, where $i=1,2,3$. We denote by c an arbitrary positive constant that might depend on Ω, $\underline{f}$, $\underline{g}$, ϕ_{bc}, and on the material parameters, but does not depend on ε, $\underline{u}$, ϕ, etc.

We now briefly describe the contents of the present paper. In Section 2 we rewrite the piezoelectric problem in a variational form, and define a two-dimensional model. After that, in Section 3, we discuss the asymptotic consistency issue. In 4 we discuss some aspects of the present investigation, and finally, in the Appendix, we perform the computations that led to our model.

2. Variational formulations and Hierarchical Modeling

Our first step is to rewrite the piezoelectric problem in a variational form. Let

$$\underline{V}(P)=\{\underline{v}\in\underline{H}^1(P):\ \underline{v}=0\text{ on }\partial P_{\mathrm{L}}\},\qquad \Psi_{\phi_{bc}}(P)=\{\psi\in H^1(P):\ \psi=\phi_{bc}\text{ on }\partial P_\pm\},$$

and we endow these spaces with the $H^1(P)$ norms. We search for $(\underline{u}^\epsilon,\phi^\epsilon)\in\underline{V}(P)\times\Psi_{\phi_{bc}}(P)$ such that

$$a\big((\underline{u}^\epsilon,\phi^\epsilon),(\underline{v},\psi)\big)=l(\underline{v},\psi)\quad\text{for all }(\underline{v},\psi)\in\underline{V}(P)\times\Psi_0(P),$$

where

$$a\big((\underline{u}^\epsilon,\phi^\epsilon),(\underline{v},\psi)\big)=\int_P\big[\underline{\underline{\underline{\underline{C}}}}\,\underline{\underline{e}}(\underline{u}^\epsilon)-\underline{\nabla}\,\phi^\epsilon\underline{\underline{\underline{Q}}}\big]:\underline{\underline{e}}(\underline{v})\,d\underline{x}+\int_P\big[\underline{\underline{\underline{Q}}}\,\underline{\underline{e}}(\underline{u}^\epsilon)+\underline{\underline{d}}\,\underline{\nabla}\,\phi^\epsilon\big]\cdot\underline{\nabla}\,\psi\,d\underline{x},$$

$$l(\underline{v},\psi)=\int_P\underline{f}\cdot\underline{v}\,d\underline{x}+\int_{\partial P_\pm}\underline{g}\cdot\underline{v}\,d\underline{x}.$$

Existence and uniqueness of solution follows immediately from Lax–Milgram Theorem since

$$a\big((\underline{u}^\epsilon,\phi^\epsilon),(\underline{u}^\epsilon,\phi^\epsilon)\big)=\int_P(\underline{\underline{\underline{C}}}\,\underline{\underline{e}}(\underline{u}^\epsilon)-\underline{\nabla}\,\phi^\epsilon\underline{\underline{\underline{Q}}})\,:\,\underline{\underline{e}}(\underline{u}^\epsilon)\,d\underline{x}+\int_P(\underline{\underline{\underline{Q}}}\,\underline{\underline{e}}(\underline{u}^\epsilon)+\underline{\underline{d}}\,\underline{\nabla}\,\phi^\epsilon)\cdot\underline{\nabla}\,\phi^\epsilon\,d\underline{x}$$
$$=\int_P\underline{\underline{\underline{C}}}\,\underline{\underline{e}}(\underline{u}^\epsilon)\,:\,\underline{\underline{e}}(\underline{u}^\epsilon)\,d\underline{x}+\int_P\underline{\underline{d}}\,\underline{\nabla}\,\phi^\epsilon\cdot\underline{\nabla}\,\phi^\epsilon\,d\underline{x}\geq c\big(\|\underline{u}^\epsilon\|^2_{H^1(P)}+\|\phi^\epsilon\|^2_{H^1(P)}\big).$$

Above, we used that

$$(\underline{\underline{\underline{Q}}}\underline{\underline{\tau}})\cdot\underline{v}=(\underline{v}\underline{\underline{\underline{Q}}}):\underline{\underline{\tau}},\tag{2}$$

for all $\underline{\underline{\tau}}\in\mathbb{R}^{3\times3}_{\text{sym}}$, $\underline{v}\in\mathbb{R}^3$.

We now develop a mixed formulation for the same problem. Note that $\underline{\underline{\sigma}}^\epsilon\in\underline{\underline{L}}^2(P)$, $\underline{D}^\epsilon\in\underline{L}^2(P)$, $\underline{u}^\epsilon\in\underline{V}(P)$, $\phi^\epsilon\in\Psi_{\phi_{bc}}(P)$ satisfy

$$\int_P\underline{\underline{\underline{\underline{A}}}}\underline{\underline{\sigma}}^\epsilon\,:\,\underline{\underline{\tau}}\,d\underline{x}-\int_P\underline{\underline{e}}(\underline{u}^\epsilon)\,:\,\underline{\underline{\tau}}\,d\underline{x}-\int_P\underline{\nabla}\,\phi^\epsilon\underline{\underline{\underline{Q}}}\,:\,\underline{\underline{\underline{\underline{A}}}}\underline{\underline{\tau}}\,d\underline{x}=0\quad\text{for all }\underline{\underline{\tau}}\in\underline{\underline{L}}^2(P),\tag{3.i}$$

$$\int_P\underline{D}^\epsilon\cdot\underline{H}\,d\underline{x}-\int_P\underline{\underline{\underline{Q}}}\,\underline{\underline{e}}(\underline{u}^\epsilon)\cdot\underline{H}\,d\underline{x}+\int_P\underline{\underline{d}}\,\underline{\nabla}\,\phi^\epsilon\cdot\underline{H}\,d\underline{x}=0\quad\text{for all }\underline{H}\in\underline{L}^2(P),\tag{3.ii}$$

$$\int_P\underline{\underline{\sigma}}^\epsilon\,:\,\underline{\underline{e}}(\underline{v})\,d\underline{x}=\int_P\underline{f}\cdot\underline{v}\,d\underline{x}+\int_{\partial P_\pm}\underline{g}\cdot\underline{v}\,d\underline{x}\quad\text{for all }\underline{v}\in\underline{V}(P),\tag{3.iii}$$

$$\int_P\underline{D}^\epsilon\cdot\underline{\nabla}\,\psi\,d\underline{x}=0\quad\text{for all }\psi\in\Psi_0(P),\tag{3.iv}$$

where $\underline{\underline{\underline{\underline{A}}}}=\underline{\underline{\underline{\underline{C}}}}^{-1}$.

If we set

$$b\big((\underline{\underline{\sigma}}^\epsilon,\underline{D}^\epsilon),(\underline{\underline{\tau}},\underline{H})\big)=\int_P \underline{\underline{\underline{\underline{A}}}}\,\underline{\underline{\sigma}}^\epsilon : \underline{\underline{\tau}}\,d\underline{x}+\int_P \underline{D}^\epsilon\cdot\underline{H}\,d\underline{x},$$

$$b_1\big((\underline{\underline{\tau}},\underline{H}),(\underline{u}^\epsilon,\phi^\epsilon)\big)=-\int_P \underline{\underline{e}}(\underline{u}^\epsilon) : \underline{\underline{\tau}}\,d\underline{x}-\int_P \underline{\nabla}\,\phi^\epsilon \underline{\underline{\underline{Q}}} : \underline{\underline{\underline{\underline{A}}}}\,\underline{\underline{\tau}}\,d\underline{x}$$
$$-\int_P \underline{\underline{\underline{Q}}}\,\underline{\underline{e}}(\underline{u}^\epsilon)\cdot\underline{H}\,d\underline{x}+\int_P \underline{\underline{d}}\,\underline{\nabla}\,\phi^\epsilon\cdot\underline{H}\,d\underline{x},$$

$$b_2\big((\underline{\underline{\sigma}}^\epsilon,\underline{D}^\epsilon),(\underline{v},\psi)\big)=\int_P \underline{\underline{\sigma}}^\epsilon : \underline{\underline{e}}(\underline{v})\,d\underline{x}+\int_P \underline{D}^\epsilon\cdot\underline{\nabla}\,\psi\,d\underline{x},$$

we have that

$$\begin{aligned}&b\big((\underline{\underline{\sigma}}^\epsilon,\underline{D}^\epsilon),(\underline{\underline{\tau}},\underline{H})\big)+b_1\big((\underline{\underline{\tau}},\underline{H}),(\underline{u}^\epsilon,\phi^\epsilon)\big)=0,\\ &b_2\big((\underline{\underline{\sigma}}^\epsilon,\underline{D}^\epsilon),(\underline{v},\psi)\big)=l(\underline{v},\psi),\end{aligned}\tag{4}$$

for all $(\underline{\underline{\tau}},\underline{H})\in\underline{\underline{L}}^2(P)\times\underline{L}^2(P)$ and $(\underline{v},\psi)\in\underline{V}(P)\times\Psi_0(P)$.

To show existence and uniqueness of solution of the above mixed formulation, it is enough to follow [17, 8, 10], and show that $b(\cdot,\cdot)$ is coercive (it is!), and that for all $(\underline{v},\psi)\in\underline{V}(P)\times\Psi_0(P)$,

$$\sup_{(\underline{\underline{\tau}},\underline{H})\in\underline{\underline{L}}^2(P)\times\underline{L}^2(P)}\frac{b_\alpha\big((\underline{\underline{\tau}},\underline{H}),(\underline{v},\psi)\big)}{\|(\underline{\underline{\tau}},\underline{H})\|_{L^2(P)\times L^2(P)}}\ge c\|(\underline{v},\psi)\|_{\underline{V}(P)\times\Psi_0(P)}\quad\text{for }\alpha=1,2.$$

The above inf-sup condition is trivial for $\alpha=2$ since $\underline{\underline{e}}(\underline{V}(P))\subset\underline{\underline{L}}^2(P)$ and $\underline{\nabla}(\Psi_0(P))\subset\underline{L}^2(P)$. For $\alpha=1$, it is sufficient to notice that

$$\begin{aligned}b_1\big((-\underline{\underline{\underline{\underline{C}}}}\,\underline{\underline{e}}(\underline{v}),\underline{\nabla}\,\psi),(\underline{v},\psi)\big)&=\int_P \underline{\underline{\underline{\underline{C}}}}\,\underline{\underline{e}}(\underline{v}) : \underline{\underline{e}}(\underline{v})\,d\underline{x}+\int_P \underline{\nabla}\,\psi\underline{\underline{\underline{Q}}} : \underline{\underline{e}}(\underline{v})\,d\underline{x}\\ &-\int_P \underline{\underline{\underline{Q}}}\,\underline{\underline{e}}(\underline{v})\cdot\underline{\nabla}\,\psi\,d\underline{x}+\int_P \underline{\underline{d}}\,\underline{\nabla}\,\psi\cdot\underline{\nabla}\,\psi\,d\underline{x}\ge c\big(\|\underline{v}\|^2_{H^1(P)}+\|\psi\|^2_{H^1(P)}\big),\end{aligned}\tag{5}$$

where we again apply (2).

Solving the mixed problem (4) within subspaces of functions that are polynomials in the transverse direction we derive piezoelectric plate models. For instance, let

$$
\begin{aligned}
&\underset{-}{V}(P,p) = \{\underset{-}{v} \in \underset{-}{V}(P) : \deg_3 \underset{\sim}{v} \le p,\ \deg_3 v_3 \le p+1\},\\
&\Psi_{\phi_{bc}}(P,p) = \{\psi \in \Psi_{\phi_{bc}}(P) : \deg_3 \psi \le p+1\},\\
(6)\qquad &\underset{=}{L}^2(P,p) = \{\underset{=}{\tau} \in \underset{=}{L}^2(P) : \deg_3 \underset{\approx}{\tau} \le p,\ \deg_3 \underset{\sim}{\tau} \le p+1,\ \deg_3 \tau_{33} \le p\},\\
&\underset{-}{L}^2(P,p) = \{\underset{-}{H} \in \underset{-}{L}^2(P) : \deg_3 \underset{\sim}{H} \le p+1,\ \deg_3 H_3 \le p\}.
\end{aligned}
$$

For $\underset{\sim}{v} \in \underset{\sim}{L}^2(P)$ we write $\deg_3 \underset{\sim}{v} \le p$ meaning that the components of v are polynomials of degree at most p with coefficients in Ω. The interpretation for $p < 0$ is that $v = 0$. Similar interpretation holds for the other tensors. The representation below indicates the degrees of $\underset{-}{v}$, ψ, $\underset{=}{\tau}$, $\underset{-}{H}$ in the spaces (6):

$$
\deg \underset{-}{v} = \begin{pmatrix} p \\ p+1 \end{pmatrix}, \qquad \deg \psi = \begin{pmatrix} p+1 \end{pmatrix},
$$

$$
\deg \underset{=}{\tau} = \begin{pmatrix} p & p+1 \\ p+1 & p \end{pmatrix}, \qquad \deg \underset{-}{H} = \begin{pmatrix} p+1 \\ p \end{pmatrix}.
$$

We now search for $\underset{=}{\sigma} \in \underset{=}{L}^2(P,p)$, $\underset{-}{D} \in \underset{-}{L}^2(P,p)$, $\underset{-}{u} \in \underset{-}{V}(P,p)$, $\phi \in \Psi_{\phi_{bc}}(P,p)$ such that

$$
\begin{aligned}
&b\big((\underset{=}{\sigma}, \underset{-}{D}), (\underset{=}{\tau}, \underset{-}{H})\big) + b_1\big((\underset{=}{\tau}, \underset{-}{H}), (\underset{-}{u}, \phi)\big) = 0,\\
(7)\qquad &b_2\big((\underset{=}{\sigma}, \underset{-}{D}), (\underset{-}{v}, \psi)\big) = l(\underset{-}{v}, \psi),
\end{aligned}
$$

for all $(\underset{=}{\tau}, \underset{-}{H}) \in \underset{=}{L}^2(P,p) \times \underset{-}{L}^2(P,p)$ and $(\underset{-}{v}, \psi) \in \underset{-}{V}(P,p) \times \Psi_0(P,p)$.

The degrees in (6) are one possibility, the simplest we could find. Other combinations of polynomial degrees yield different models, but not all combinations yield well-posed problems. Moreover, even if the final equations are well-posed, the model might not be "asymptotically consistent", in a sense that we make clear further ahead.

As in the original formulation (4), it follows for the spaces in (6) that $\underset{=}{e}(\underline{V}(P,p)) \subset \underset{=}{L}^2(P,p)$, and $\underline{\nabla}(\Psi_0(P),p) \subset \underline{L}^2(P,p)$. Also, (5) holds for $(\underline{v},\psi) \in V(P,p) \times \Psi_0(P,p)$. Thus, the inf-sup conditions hold and the model problem (7) is well-posed for all p.

Also note that since

$$\underset{\equiv}{C}\,\underset{=}{e}\big(\underline{V}(P,p)\big) - \underline{\nabla}\,\Psi_{\phi_{bc}}(P,p)\underset{\equiv}{Q} \subset \underset{=}{L}^2(P,p), \qquad \underset{\equiv}{Q}\,\underset{=}{e}\big(\underline{V}(P,p)\big) + \underset{=}{d}\,\underline{\nabla}\,\Psi_{\phi_{bc}}(P,p) \subset \underline{L}^2(P,p),$$

the constitutive equations (1) are enforced exactly.

Before presenting the simplest of such models we define

$$\underline{g}^0 = \frac{1}{2}[\,\underline{g}(\underset{\sim}{x},\varepsilon) + \underline{g}(\underset{\sim}{x},-\varepsilon)], \quad \underline{g}^1 = \frac{1}{2}[\,\underline{g}(\underset{\sim}{x},\varepsilon) - \underline{g}(\underset{\sim}{x},-\varepsilon)],$$

$$\underline{f}^0(\underset{\sim}{x}) = \int_{-\varepsilon}^{\varepsilon} \underline{f}(\underline{x})\,dz, \quad \underline{f}^1(\underset{\sim}{x}) = \varepsilon^{-1}\int_{-\varepsilon}^{\varepsilon} \underline{f}(\underline{x})x_3\,dx_3, \quad f_3^2(\underset{\sim}{x}) = \varepsilon^{-1}\int_{-\varepsilon}^{\varepsilon} f_3(\underline{x})L_2(x_3)\,dx_3,$$

where $L_2(z) = (3z^2 - \varepsilon^2)/2$. Similar definitions hold for ϕ_{bc}^0 and ϕ_{bc}^1. Let $\underset{\approx}{A}$ be the two-dimensional version of the compliance tensor with the inverse

$$\underset{\approx}{A}^{-1}\underset{\approx}{\tau} = 2\mu\left[\underset{\approx}{\tau} + \frac{\lambda}{2\mu+\lambda}\,\mathrm{tr}(\underset{\approx}{\tau})\,\underset{\approx}{\delta}\right]. \tag{8}$$

Here, $\mathrm{tr}(\cdot)$ indicates the trace operator.

As in the linearly elastic plate modeling, the solution decouples in bending and stretching components, so we consider each part separately. We show next the resulting equations for

$p=1$, but postpone the details to the Appendix. Assume the approximate displacement $\underline{u}$, and electrical potential ϕ are given by

(9)

$$\underline{u}(\underline{x}) = \begin{pmatrix} \underset{\sim}{\eta}(\underset{\sim}{x}) \\ \\ \rho(\underset{\sim}{x})x_3 \end{pmatrix} + \begin{pmatrix} -\underset{\sim}{\theta}(\underset{\sim}{x})x_3 \\ \\ \omega(\underset{\sim}{x}) + \omega_2(\underset{\sim}{x})L_2(x_3) \end{pmatrix}, \quad \phi(\underline{x}) = \phi_{bc}^0(\underset{\sim}{x}) + \varepsilon^{-1}x_3\phi_{bc}^1(\underset{\sim}{x}) + (\varepsilon^2 - L_2)\phi_2(\underset{\sim}{x}),$$

where $\underset{\sim}{\eta}$, ρ, $\underset{\sim}{\theta}$, ω, ω_2, ϕ_2 are unknown. Also, the approximate stress tensor $\underline{\underline{\sigma}}$, and electrical displacement $\underline{D}$ are as

(10)

$$\underline{\underline{\sigma}}(\underline{x}) = \begin{pmatrix} \underset{\approx}{\sigma}^0(\underset{\sim}{x}) & \underset{\sim}{\sigma}^1(\underset{\sim}{x})x_3 \\ \\ \underset{\sim}{\sigma}^1(\underset{\sim}{x})^t x_3 & \sigma_{33}^0(\underset{\sim}{x}) \end{pmatrix} + \begin{pmatrix} \underset{\approx}{\sigma}^1(\underset{\sim}{x})x_3 & \underset{\sim}{\sigma}^0(\underset{\sim}{x}) + \underset{\sim}{\sigma}^2(\underset{\sim}{x})L_2(x_3) \\ \\ [\underset{\sim}{\sigma}^0(\underset{\sim}{x}) + \underset{\sim}{\sigma}^2(\underset{\sim}{x})L_2(x_3)]^t & \sigma_{33}^1(\underset{\sim}{x})x_3 \end{pmatrix},$$

$$\underline{D}(\underline{x}) = \begin{pmatrix} \underset{\sim}{D}^1(\underset{\sim}{x}) \\ \\ D_3^0(\underset{\sim}{x}) \end{pmatrix} + \begin{pmatrix} \underset{\sim}{D}^0(\underset{\sim}{x}) + \underset{\sim}{D}^2(\underset{\sim}{x})L_2(x_3) \\ \\ D_3^1(\underset{\sim}{x})x_3 \end{pmatrix},$$

where $\underset{\approx}{\sigma}^0$, $\underset{\sim}{\sigma}^1$, σ_{33}^0, $\underset{\approx}{\sigma}^1$, $\underset{\sim}{\sigma}^0$, $\underset{\sim}{\sigma}^2$, σ_{33}^1, $\underset{\sim}{D}^1$, D_3^0, $\underset{\sim}{D}^0$, $\underset{\sim}{D}^2$, D_3^1 need to be determined.

For the stretching part, we find that $\underset{\sim}{\eta}$ and ρ satisfy the equations

(11)

$$-\underset{\sim}{\operatorname{div}} \underset{\approx}{A}^{-1}\,\underset{\approx}{e}(\underset{\sim}{\eta}) - \frac{\lambda^2}{2\mu+\lambda}\,\underset{\sim}{\nabla}\operatorname{div}\underset{\sim}{\eta} - \lambda\,\underset{\sim}{\nabla}\,\rho = \underset{\sim}{l} \quad \text{in } \Omega,$$

$$-\frac{\varepsilon^2}{3}\mu\,\Delta\,\rho + \lambda\operatorname{div}\underset{\sim}{\eta} + (2\mu+\lambda)\rho = l_3 \quad \text{in } \Omega,$$

$$\underset{\sim}{\eta} = 0, \quad \rho = 0 \quad \text{on } \partial\Omega,$$

where

$$\underset{\sim}{l} = \varepsilon^{-1}\Big(\frac{1}{2}\underset{\sim}{f}^0 + \underset{\sim}{g}^0 + Q_{311}\,\underset{\sim}{\nabla}\,\phi_{bc}^1\Big), \qquad l_3 = \frac{1}{2}f_3^1 + g_3^1 - \varepsilon^{-1}Q_{333}\phi_{bc}^1 + \frac{\varepsilon}{3}Q_{113}\,\Delta\,\phi_{bc}^1.$$

After finding $\underset{\sim}{\eta}$ and ρ, the stress tensor and the electrical displacement are computable from

(12.i) $$\underset{\approx}{\sigma}^0 = \underset{\approxeq}{A}^{-1}\,\underset{\approx}{e}(\underset{\sim}{\eta}) + \frac{\lambda^2}{2\mu+\lambda}\operatorname{div}\underset{\sim}{\eta}\underset{\approx}{\delta} + \lambda\rho\underset{\approx}{\delta} + \varepsilon^{-1}Q_{311}\phi^1_{bc}\underset{\approx}{\delta},$$

(12.ii) $$\underset{\sim}{\sigma}^1 = \mu\,\underset{\sim}{\nabla}\,\rho + Q_{113}\varepsilon^{-1}\,\underset{\sim}{\nabla}\,\phi^1_{bc},$$

(12.iii) $$\sigma^0_{33} = \lambda\operatorname{div}\underset{\sim}{\eta} + (2\mu+\lambda)\rho + \varepsilon^{-1}Q_{333}\phi^1_{bc},$$

(12.iv) $$\underset{\sim}{D}^1 = Q_{113}\,\underset{\sim}{\nabla}\,\rho - \varepsilon^{-1}d\,\underset{\sim}{\nabla}\,\phi^1_{bc}, \qquad D^0_3 = Q_{311}\operatorname{div}\underset{\sim}{\eta} + Q_{333}\rho - \varepsilon^{-1}d_{33}\phi^1_{bc}.$$

For the bending part, $\underset{\sim}{\theta}$, ω, ω_2, ϕ_2 solve

(13.i)
$$\underset{\sim}{\operatorname{div}}\,\underset{\approxeq}{A}^{-1}\,\underset{\approx}{e}(\underset{\sim}{\theta}) + 3\varepsilon^{-2}\mu(-\underset{\sim}{\theta} + \underset{\sim}{\nabla}\,\omega) - \lambda\,\underset{\sim}{\nabla}\left(-\frac{\lambda}{2\mu+\lambda}\operatorname{div}\underset{\sim}{\theta} + 3\omega_2\right) + 3(Q_{113}+Q_{311})\,\underset{\sim}{\nabla}\,\phi_2 = \underset{\sim}{F},$$

(13.ii) $$2\varepsilon\mu\operatorname{div}(-\underset{\sim}{\theta} + \underset{\sim}{\nabla}\,\omega) + 2\varepsilon^3 Q_{113}\,\Delta\,\phi_2 = F_3,$$

(13.iii)
$$-\frac{2\varepsilon^5}{5}\mu\,\Delta\,\omega_2 + 2\varepsilon^3(2\mu+\lambda)\left(-\frac{\lambda}{2\mu+\lambda}\operatorname{div}\underset{\sim}{\theta} + 3\omega_2\right) + \frac{2\varepsilon^5}{5}Q_{113}\,\Delta\,\phi_2 - 6\varepsilon^3 Q_{333}\phi_2 = F_4,$$

(13.iv)
$$Q_{113}\operatorname{div}(-\underset{\sim}{\theta} + \underset{\sim}{\nabla}\,\omega) - Q_{311}\operatorname{div}\underset{\sim}{\theta} - \frac{\varepsilon^2}{5}Q_{113}\,\Delta\,\omega_2 + 3Q_{333}\omega_2 - \frac{6\varepsilon^2}{5}d\,\Delta\,\phi_2 + 3d_{33}\phi_2 = F_5,$$

where

$$\underset{\sim}{F} = \varepsilon^{-2}\frac{3}{2}(\underset{\sim}{f}^1 + 2\underset{\sim}{g}^1) - 3\varepsilon^{-2}Q_{113}\,\underset{\sim}{\nabla}\,\phi^0_{bc}, \qquad F_3 = -\varepsilon f^0_3 - 2g^0_3 - 2\varepsilon Q_{113}\,\Delta\,\phi^0_{bc},$$

$$F_4 = \varepsilon f^2_3 + 2\varepsilon^2 g^0_3, \qquad F_5 = d\,\Delta\,\phi^0_{bc}.$$

with the boundary conditions

$$\underset{\sim}{\theta} = 0, \quad \omega = \omega_2 = 0, \quad Q_{113}\frac{\partial}{\partial n}[\omega - \frac{\varepsilon^2}{5}\omega_2] - \frac{6}{5}d\varepsilon^2\frac{\partial \phi_2}{\partial n} = d\frac{\partial \phi_{bc}^0}{\partial n} \quad \text{on } \partial\Omega.$$

Given $\underset{\sim}{\theta}$, ω, ω_2 and ϕ_2, the stress tensor and the electrical displacement are easily calculated as below:

$$\underset{\approx}{\sigma}^1 = -\underset{\approx}{A}^{-1}\,\underset{\approx}{e}(\underset{\sim}{\theta}) - \frac{\lambda^2}{2\mu+\lambda}\operatorname{div}\underset{\sim}{\theta}\underset{\approx}{\delta} + 3\lambda\omega_2\underset{\approx}{\delta} - 3Q_{311}\phi_2\underset{\approx}{\delta}, \tag{14.i}$$

$$\underset{\sim}{\sigma}^0 = \mu(-\underset{\sim}{\theta} + \underset{\sim}{\nabla}\,\omega) + Q_{113}(\underset{\sim}{\nabla}\,\phi_{bc}^0 + \varepsilon^2\,\underset{\sim}{\nabla}\,\phi_2), \tag{14.ii}$$

$$\underset{\sim}{\sigma}^2 = \mu\,\underset{\sim}{\nabla}\,\omega_2 - Q_{113}\,\underset{\sim}{\nabla}\,\phi_2, \tag{14.iii}$$

$$\sigma_{33}^1 = -\lambda\operatorname{div}\theta - 3Q_{333}\phi_2 + 3(2\mu+\lambda)\omega_2, \tag{14.iv}$$

$$\underset{\sim}{D}^0 = Q_{113}(-\underset{\sim}{\theta} + \underset{\sim}{\nabla}\,\omega) - d\,\underset{\sim}{\nabla}(\phi_{bc}^0 + \varepsilon^2\phi_2), \tag{14.v}$$

$$\underset{\sim}{D}^2 = Q_{113}\,\underset{\sim}{\nabla}\,\omega_2 + d\,\underset{\sim}{\nabla}\,\phi_2, \tag{14.vi}$$

$$D_3 = -Q_{311}\operatorname{div}\underset{\sim}{\theta} + 3Q_{333}\omega_2 + 3d_{33}\phi_2. \tag{14.vii}$$

3. Asymptotic consistency

Considering the sequence of plate problem parameterized by the thickness ε, it is possible to show that the three-dimensional solution converges in a proper sense to a solution of two-dimensional problems. This was done by Sène [21] for the piezoelectric problem, as we point out in the Introduction. We present here the limit equations.

The asymptotic limits of $\underline{u}^\epsilon$ and ϕ^ϵ are $\underline{u}_{KL}$ and ϕ_{KL}, where

(15)
$$\underline{u}_{KL}(\underline{x}) = \begin{pmatrix} \varepsilon\underset{\sim}{\zeta}(\underset{\sim}{x}) - x_3 \underset{\sim}{\nabla}\, \zeta_3(\underset{\sim}{x}) \\ \zeta_3(\underset{\sim}{x}) \end{pmatrix},$$

$$\phi_{KL}(\underline{x}) = \phi^0_{bc} + \left(Q_{311} - \frac{\lambda}{\lambda+2\mu} Q_{333}\right) \frac{\varepsilon^2 - x_3^2}{p_{33}}\, \Delta\, \zeta_3 + \varepsilon^{-1} x_3 \phi^1_{bc}, \quad p_{33} = \frac{(Q_{333})^2}{\lambda+2\mu} + d_{33}.$$

The function $\underset{\sim}{\zeta}$ solves

$$-\varepsilon \underset{\sim}{\operatorname{div}}\, \underset{\approx}{A}^{-1}\, \underset{\approx}{e}(\underset{\sim}{\zeta}) = \underset{\sim}{l} - \varepsilon^{-1} \frac{\lambda}{\lambda+2\mu} Q_{333}\, \underset{\sim}{\nabla}\, \phi^1_{bc} \quad \text{in } \Omega, \qquad \underset{\sim}{\zeta} = 0 \quad \text{on } \partial\Omega, \tag{16}$$

and ζ_3 solves the biharmonic equation

$$\varepsilon^3 B\, \Delta^2\, \zeta_3 = \varepsilon \operatorname{div} \underset{\sim}{f}^1 + f^0_3 + \varepsilon \operatorname{div} \underset{\sim}{g}^1 + g^0_3 \quad \text{in } \Omega, \qquad \zeta_3 = \frac{\partial \zeta_3}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{17}$$

where

$$B = \frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)} + \frac{2}{3p_{33}} \left(Q_{311} - \frac{\lambda}{\lambda+2\mu} Q_{333}\right)^2. \tag{18}$$

An important issue in dimensional reduction modeling is the *asymptotic consistency*, i.e., the relative modeling error, say in the $L^2(P)$ norm, should go to zero with ε. That means that the solution of the model should have the same asymptotic behavior as the original three-dimensional solution. Not all assumptions on the subspaces of $\underline{V}(P)$ etc, lead to consistent models. For an instance of this phenomenom see [18].

To investigate the consistency, we make the following scaling assumptions on the loads:

(19)
$$\underline{f}(\underline{x}) = \big(\varepsilon \check{\underset{\sim}{f}}(\underset{\sim}{x}, \varepsilon^{-1}x_3), \varepsilon^2 \check{f}_3(\underset{\sim}{x}, \varepsilon^{-1}x_3)\big), \qquad \underline{g}(\underline{x}) = \big(\varepsilon^2 \check{\underset{\sim}{g}}(\underset{\sim}{x}, \varepsilon^{-1}x_3), \varepsilon^3 \check{g}_3(\underset{\sim}{x}, \varepsilon^{-1}x_3)\big),$$
$$\phi_{bc}(\underline{x}) = \varepsilon^2 \check{\phi}_{bc}(\underset{\sim}{x}, \varepsilon^{-1}x_3),$$

where $\check{\underline{f}} : \Omega \times (-1,1) \to \mathbb{R}^3$, $\check{\underline{g}} : \Omega \times \{-1,1\} \to \mathbb{R}^3$, and $\check{\phi}_{bc} : \Omega \times \{-1,1\} \to \mathbb{R}$ are all ε-independent functions.

Theorem 1. *Assume that the plate is under a nontrivial pure stretching regime, that is,* $\underline{u}_{KL} = \varepsilon(\underset{\sim}{\zeta}, 0) \neq 0$. *Then, under the scaling assumptions* (19)*, the relative error estimate*

$$\frac{\|\underline{u}_{KL} - \underline{u}_S\|_{L^2(P)}}{\|\underline{u}_{KL}\|_{L^2(P)}} \le c\varepsilon, \tag{20}$$

holds, where $\underline{u}_S = (\underset{\sim}{\eta}, x_3\rho)$.

Proof. From (19) we gather that

$$\|\underset{\sim}{l}\|_{H^{-1}(\Omega)} + \|l_3\|_{L^2(\Omega)} \le c\varepsilon, \tag{21}$$

and that $|\underset{\sim}{\zeta}|_{H^1(\Omega)}$ is bounded above and below by a constant.

Next, multiplying the first equation in (11) by $(2\mu+\lambda)\underset{\sim}{\eta}$, the second by $(2\mu+\lambda)\rho$, integrating by parts and adding the resulting equations, we gather that

$$(2\mu+\lambda)\int_\Omega \underset{\approx}{A}^{-1}\,\underset{\approx}{e}(\underset{\sim}{\eta}) : \underset{\approx}{e}(\underset{\sim}{\eta})\,d\underset{\sim}{x} + \frac{\varepsilon^2}{3}\mu(2\mu+\lambda)\int_\Omega |\underset{\sim}{\nabla}\,\rho|^2\,d\underset{\sim}{x} + \int_\Omega [\lambda\operatorname{div}\underset{\sim}{\eta} + (2\mu+\lambda)\rho]^2\,d\underset{\sim}{x}$$
$$= (2\mu+\lambda)\int_\Omega \underset{\sim}{l}\cdot\underset{\sim}{\eta} + l_3\rho\,d\underset{\sim}{x} = \int_\Omega (2\mu+\lambda)\underset{\sim}{l}\cdot\underset{\sim}{\eta}\,d\underset{\sim}{x} + \int_\Omega l_3[\lambda\operatorname{div}\underset{\sim}{\eta} + (2\mu+\lambda)\rho]\,d\underset{\sim}{x} - \int_\Omega l_3\lambda\operatorname{div}\underset{\sim}{\eta}\,d\underset{\sim}{x}.$$

Then the stability estimate

$$\|\underset{\sim}{\eta}\|_{H^1(\Omega)} + \varepsilon\|\rho\|_{H^1(\Omega)} + \|\lambda\operatorname{div}\underset{\sim}{\eta} + (2\mu+\lambda)\rho\|_{L^2(\Omega)} \le c(\|\underset{\sim}{l}\|_{H^{-1}(\Omega)} + \|l_3\|_{L^2(\Omega)})$$

holds. Thus, from the triangle inequality,

$$\|(2\mu+\lambda)\rho\|_{L^2(\Omega)} \le \|\lambda\operatorname{div}\underset{\sim}{\eta} + (2\mu+\lambda)\rho\|_{L^2(\Omega)} + \|\lambda\operatorname{div}\underset{\sim}{\eta}\|_{L^2(\Omega)} \le c(\|\underset{\sim}{l}\|_{H^{-1}(\Omega)} + \|l_3\|_{L^2(\Omega)}).$$

It follows from (21) that

$$\|\underset{\sim}{\eta}\|_{H^1(\Omega)} \le c\varepsilon, \quad \|\rho\|_{H^1(\Omega)} \le c, \quad \|\rho\|_{L^2(\Omega)} \le c\varepsilon. \tag{22}$$

Let $\underset{\sim}{E} = \underset{\sim}{\eta} - \varepsilon\underset{\sim}{\zeta}$. Thus, it follows from (11), (16) that

$$-\underset{\sim}{\operatorname{div}}\, \underset{\approxeq}{A}^{-1}\, \underset{\approx}{e}(\underset{\sim}{E}) = \frac{\lambda}{2\mu+\lambda}\, \underset{\sim}{\nabla}\, r \quad \text{in } \Omega,$$

$$\underset{\sim}{E} = 0 \quad \text{on } \partial\Omega.$$

where $r = \varepsilon^2(\mu/3)\,\Delta\,\rho + l_3 + \varepsilon^{-1}Q_{333}\phi^1_{bc}$. If $\underset{\sim}{V} = \{\underset{\sim}{v} \in \underset{\sim}{H}^2(\Omega) \cap \underset{\sim}{H}^1_0(\Omega) : \|\underset{\sim}{v}\|_2 = 1\}$, then

$$\begin{aligned}\|\underset{\sim}{E}\|_{L^2(\Omega)} &= \sup_{\underset{\sim}{\eta}\in\underset{\sim}{V}} \int_\Omega \underset{\sim}{E} \cdot \underset{\sim}{\operatorname{div}}\, \underset{\approxeq}{A}\, \underset{\approx}{e}(\underset{\sim}{\eta})\, d\underset{\sim}{x} = \sup_{\underset{\sim}{\eta}\in\underset{\sim}{V}} \int_\Omega \underset{\approxeq}{A}\, \underset{\approx}{e}(\underset{\sim}{\eta}) : \underset{\approx}{e}(\underset{\sim}{E})\, d\underset{\sim}{x} \\ &= \frac{-\lambda}{2\mu+\lambda} \sup_{\underset{\sim}{\eta}\in\underset{\sim}{V}} \int_\Omega r \operatorname{div} \underset{\sim}{\eta}\, d\underset{\sim}{x} \le c\varepsilon^2 \sup_{\underset{\sim}{\eta}\in\underset{\sim}{V}} \int_\Omega \Delta\,\rho \operatorname{div} \underset{\sim}{\eta}\, d\underset{\sim}{x} + c\|l_3 + \varepsilon^{-1}Q_{333}\phi^1_{bc}\|_{L^2(\Omega)}.\end{aligned} \tag{23}$$

Now, for $\underset{\sim}{\eta} \in \underset{\sim}{V}$,

$$\int_\Omega \Delta\,\rho \operatorname{div} \underset{\sim}{\eta}\, d\underset{\sim}{x} = -\int_\Omega \underset{\sim}{\nabla}\,\rho \cdot \underset{\sim}{\nabla} \operatorname{div} \underset{\sim}{\eta}\, d\underset{\sim}{x} + \Big\langle \frac{\partial\rho}{\partial n}, \operatorname{div} \underset{\sim}{\eta} \Big\rangle \le c\|\rho\|_{H^1(\Omega)}.$$

where $\langle\cdot,\cdot\rangle$ indicates the duality between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. Applying the above inequality in (23), and also using (19) and (22) we gather that

$$\|\underset{\sim}{E}\|_{L^2(\Omega)} \le c\varepsilon^2. \tag{24}$$

The final result follows then from (22), (24), and from the fact that $\underset{\sim}{\zeta}$ is ε-independent under the scaling assumptions (19). $\square$

Remark. The estimate (20) does not measure the modeling error with respect to the exact three-dimensional solution, only to its asymptotic limit $\underline{u}_{KL}$. What (20) guarantees is that the solution to the model proposed here is consistent, i.e., converges to the correct limit.

We next present formal arguments that lead to the strong conjecture that the bending model (13) is also asymptotically consistent. The idea is to assume

$$\theta \sim \theta^0 + \mathrm{HOT}, \qquad \omega \sim \omega^0 + \mathrm{HOT}, \qquad \omega_2 \sim \omega_2^0 + \mathrm{HOT}, \qquad \phi_2 \sim \phi_2^0 + \mathrm{HOT},$$

where the "HOT" (Higher Order Terms) are, as the name suggests, terms which L^2 norms behave at least as $O(\varepsilon)$. Based on the scalings (19), we gather that

$$\underset{\sim}{F} = O(1), \qquad F_3 = O(\varepsilon^3), \qquad F_4 = O(\varepsilon^5), \qquad F_5 = O(\varepsilon^2).$$

From (13.i), we gather that $\theta^0 = \underset{\sim}{\nabla}\,\omega^0$. From (13.iii), (13.iv), we obtain

$$\omega_2^0 = \frac{\lambda}{3(2\mu+\lambda)}\operatorname{div}\theta^0 + \frac{Q_{333}}{2\mu+\lambda}\phi_2^0,$$

$$-Q_{311}\operatorname{div}\theta^0 + 3Q_{333}\omega_2^0 + 3d_{33}\phi_2^0 = 0,$$

and then

$$\phi_2^0 = \frac{1}{3p_{33}}\left(Q_{311} - Q_{333}\frac{\lambda}{2\mu+\lambda}\right)\operatorname{div}\theta^0,$$

where p_{33} is defined in (15).

Next, from (13.i), (13.ii),

$$\operatorname{div}\underset{\sim}{\operatorname{div}}\underset{\approx}{A}^{-1}\underset{\approx}{e}(\underset{\sim}{\theta}) + 3Q_{113}\,\Delta\,\phi_2 - \lambda\,\Delta\left(-\frac{\lambda}{2\mu+\lambda}\operatorname{div}\underset{\sim}{\theta} + 3\omega_2\right) + 3(Q_{113}+Q_{311})\,\Delta\,\phi_2$$
$$= \operatorname{div}\underset{\sim}{F} - \frac{3}{2}\varepsilon^{-3}F_3.$$

Thus,

$$\frac{2}{3}\operatorname{div}\underset{\sim}{\operatorname{div}}\underset{\approx}{A}^{-1}\underset{\approx}{e}(\theta^0) + 2(-Q_{113}+c_1)\,\Delta\,\phi_2^0 = \frac{2}{3}\operatorname{div}\underset{\sim}{F} - \varepsilon^{-3}F_3,$$

where

$$c_1 = -\frac{\lambda Q_{333}}{2\mu+\lambda} + Q_{113} + Q_{311}.$$

Finally,

$$\operatorname{div} \underset{\sim}{\operatorname{div}} \underset{\approx}{A}^{-1} \underset{\approx}{e}(\underset{\sim}{\nabla}\,\omega^0) + \frac{1}{p_{33}}(-Q_{113}+c_1)(Q_{311} - Q_{333}\frac{\lambda}{2\mu+\lambda})\,\Delta^2\,\omega^0 = \operatorname{div} \underset{\sim}{F} - \frac{3}{2}\varepsilon^{-3}F_3,$$

and using the identities

$$\operatorname{div} \underset{\sim}{\operatorname{div}} \underset{\approx}{A}^{-1} \underset{\approx}{e}(\underset{\sim}{\nabla}\,\omega^0) = \frac{4\mu(\mu+\lambda)}{2\mu+\lambda}\,\Delta^2\,\omega^0, \qquad -Q_{113}+c_1 = -\frac{\lambda Q_{333}}{2\mu+\lambda} + Q_{311},$$

we gather that

$$\left[\frac{4\mu(\mu+\lambda)}{2\mu+\lambda} + \frac{1}{p_{33}}\left(Q_{311} - Q_{333}\frac{\lambda}{2\mu+\lambda}\right)^2\right]\Delta^2\,\omega = \operatorname{div} \underset{\sim}{F} - \frac{3}{2}\varepsilon^{-3}F_3. \tag{25}$$

Thus (18), (25) indicate that our bending model derived through hierarchical modeling is indeed consistent.

The model (13) is not as simple as one would hope for, due to that many unknowns involved. It is desirable to obtain a further reduced model, i.e., a system of equations only in terms of ω and $\underset{\sim}{\theta}$, as in Reissner–Mindlin models, and only afterwards compute the other quantities of interest.

It is somewhat straightforward [2] to show that

$$\begin{gathered} \underset{\sim}{\operatorname{div}} \underset{\approx}{A}^{-1} \underset{\approx}{e}(\underset{\sim}{\theta}) + 3\varepsilon^{-2}\mu(-\underset{\sim}{\theta} + \underset{\sim}{\nabla}\,\omega) + c_2\,\underset{\sim}{\nabla}\operatorname{div}\theta = \underset{\sim}{F}, \\ 2\varepsilon\mu\operatorname{div}(-\underset{\sim}{\theta} + \underset{\sim}{\nabla}\,\omega) = F_3, \end{gathered} \tag{26}$$

where

$$c_2 = \frac{1}{p_{33}}\left(Q_{311} - Q_{333}\frac{\lambda}{2\mu+\lambda}\right)^2,$$

yields a consistent model. Based on the formal considerations of the previous sections, one could compute

$$\omega_2 \approx \frac{1}{3(2\mu+\lambda)}\left[\lambda + \frac{Q_{333}}{p_{33}}\left(Q_{311} - Q_{333}\frac{\lambda}{2\mu+\lambda}\right)\right]\operatorname{div}\theta,$$
$$\phi_2 \approx \frac{1}{3p_{33}}\left(Q_{311} - Q_{333}\frac{\lambda}{2\mu+\lambda}\right)\operatorname{div}\theta.$$

Of course such procedure is *ad hoc*, and thus not fully satisfactory. Nonetheless, it yields a consistent model that is as simple as the usual Reissner–Mindlin models, what is good news.

4. Discussion

The holy grail of dimensional reduction is to obtain the simplest possible model that is "good enough" for most application, and computationally feasible. Our criteria for "good enough" is asymptotic consistency, and here the references [21, 19] play a leading role. It is thus wise to ask if it is possible to derive a simpler model that is asymptotically consistent through variational arguments.

A positive answer to the above question would be good news. Indeed, Alessandrini et al. [1] obtains a simpler linearly elastic plate model, denoted $\mathrm{HR}_1(1)$. For sure, such work considers no piezoelectricity, but the stretching equations involves only a two-dimensional vector unknown, instead of the coupled system (11). And for the bending part, the $\mathrm{HR}_1(1)$ model of [1] requires solving for three scalar unknowns instead of the four unknowns required in (13) (disregarding the electrical potential contribution). Unfortunaly, for the present piezoelectric problem, such simpler assumptions on the load does lead to consistent models.

Using the notation of [1], our model is closer to the more complicated $\mathrm{HR}_3(1)$, which is the simplest consistent minimum energy model. In the case of bending of linearly elastic plates, this is the $(1,1,2)$ model of Babuska and Li [7, 20]. See also [15] for a complete description of such models.

To be fair with the present derivation, it led to the intriguing system (26), which we recall, is consistent. Is there a variational way to derive such system without *ad hoc* considerations? How good is this model, i.e., what is the convergence rate of its solutions to the exact, three-dimensional solutions?

An alternative to derive models is to use variants of Hellinger–Reisner principle, as in [1, 5], but it is not so clear how to do so for piezoelectric materials.

Finally, we understand that our choices for rigidity, dielectric and piezoelectric tensors are not as general as would be desirable to model “real life” materials. But we hope that even in this simpler setting, our modeling efforts shed some light and help in the quest of developing provably good plate models.

5. Appendix

In this Appendix we provide the main steps to derive (11), (12), (13), and (14) from (7), for $p = 1$.

From (8) we gather that

$$\underset{\approx}{A}\underset{\approx}{\tau} = \frac{1}{2\mu}\left[\underset{\approx}{\tau} - \frac{\lambda}{2\mu+3\lambda}\operatorname{tr}(\underset{\approx}{\tau})\,\underset{\approx}{\delta}\right], \qquad \underset{=}{A}\underset{=}{\tau} = \begin{pmatrix} \underset{\approx}{A}\underset{\approx}{\tau} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\tau_{33}\underset{\approx}{\delta} & \frac{1}{2\mu}\underset{\sim}{\tau} \\ \frac{1}{2\mu}\underset{\sim}{\tau}^t & \frac{\mu+\lambda}{\mu(2\mu+3\lambda)}\tau_{33} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\operatorname{tr}(\underset{\approx}{\tau}) \end{pmatrix}.$$

5.1. **A stretching model.** Assume that (9), (10) holds, and let

$$\underline{v} = \begin{pmatrix} \underset{\sim}{v} \\ v_3 x_3 \end{pmatrix}, \qquad \underset{=}{\tau} = \begin{pmatrix} \underset{\approx}{\tau} & \underset{\sim}{\tau}^1 x_3 \\ (\underset{\sim}{\tau}^1)^t x_3 & \tau_{33} \end{pmatrix}, \quad \underline{H} = \begin{pmatrix} \underset{\sim}{H} x_3 \\ H_3 \end{pmatrix},$$

where $\underline{v} \in \underline{V}(P,1)$, $\underline{\underline{\tau}} \in \underline{\underline{L}}^2(P,1)$ and $\underline{H} \in \underline{L}^2(P,1)$. Using the first constitutive equation (3.i) we integrate with respect to x_3 and find

$$\underset{\approx\approx}{A}\underset{\approx}{\sigma}^0 - \frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{33}^0\underset{\approx}{\delta} - \underset{\approx}{e}(\underset{\sim}{\eta}) - \varepsilon^{-1}Q_{311}\phi_{bc}^1\underset{\approx\approx}{A}\underset{\approx}{\delta} + \varepsilon^{-1}Q_{333}\phi_{bc}^1\frac{\lambda}{2\mu(2\mu+3\lambda)}\underset{\approx}{\delta} = 0,$$

$$\frac{2\varepsilon^3}{3}\frac{1}{\mu}\underset{\sim}{\sigma}^1 - \frac{2\varepsilon^3}{3}\underset{\sim}{\nabla}\rho - \frac{2Q_{113}}{3\mu}\varepsilon^2\underset{\sim}{\nabla}\phi_{bc}^1 = 0,$$

$$-\varepsilon\frac{\lambda}{\mu(2\mu+3\lambda)}\operatorname{tr}(\underset{\approx}{\sigma}^0) + 2\varepsilon\frac{\mu+\lambda}{\mu(2\mu+3\lambda)}\sigma_{33}^0 - 2\varepsilon\rho + \frac{2\lambda Q_{311}}{\mu(2\mu+3\lambda)}\phi_{bc}^1 - 2Q_{333}\phi_{bc}^1\frac{\mu+\lambda}{\mu(2\mu+3\lambda)} = 0.$$

Thus, (12.i), (12.ii), (12.iii) follows. Similarly, (3.ii) yields (12.iv).

Analogously to the constitutive equation, integrating the equilibrium equation in x_3, we find the equilibrium condition

$$\int_\Omega 2\varepsilon\underset{\approx}{\sigma}^0 : \underset{\approx}{e}(\underset{\sim}{v}) + \frac{2\varepsilon^3}{3}\underset{\sim}{\sigma}^1\cdot\underset{\sim}{\nabla}v_3 + 2\varepsilon\sigma_{33}^0 v_3\,d\underset{\sim}{x} = \int_\Omega(\underset{\sim}{f}^0 + 2\underset{\sim}{g}^0)\cdot\underset{\sim}{v} + (\varepsilon f_3^1 + 2\varepsilon g_3^1)v_3^1\,d\underset{\sim}{x},$$

$$\text{for all } \underset{\sim}{v} \in \underset{\sim}{H}_0^1(\Omega) \text{ and all } v_3 \in H_0^1(\Omega).$$

Hence, from (12.i), (12.ii), (12.iii), we obtain (11).

5.2. **A Bending Model.** We assume again (9), (10), and that

$$\underline{v} = \begin{pmatrix} x_3\underset{\sim}{v} \\ v_3^0 + v_3^2 L_2(x_3)\end{pmatrix}, \qquad \underline{\underline{\tau}} = \begin{pmatrix} \underset{\approx}{\tau}x_3 & \underset{\sim}{\tau}^0 + \underset{\sim}{\tau}^2 L_2(x_3) \\ [\underset{\sim}{\tau}^0 + \underset{\sim}{\tau}^2 L_2(x_3)]^t & \tau_{33}x_3\end{pmatrix},$$

$$\underline{H} = \begin{pmatrix} \underset{\sim}{H}^0 + \underset{\sim}{H}^2 L_2(x_3) \\ H_3 x_3\end{pmatrix}, \qquad \psi = (\varepsilon^2 - L_2)\psi^2,$$

where $\underline{v} \in \underline{V}(P,1)$, $\underline{\underline{\tau}} \in \underline{\underline{L}}^2(P,1)$, $\underline{H} \in \underline{L}^2(P,1)$, and $\psi \in \Psi_0(P,1)$.

Integrating (3.i) in the transverse direction we gather

$$\frac{2\varepsilon^3}{3}\underset{\approx}{A}\underset{\sim}{\sigma}^1 - \frac{\varepsilon^3\lambda}{3\mu(2\mu+3\lambda)}\sigma^1_{33}\underset{\approx}{\delta} + \frac{2\varepsilon^3}{3}\underset{\approx}{e}(\theta) + 2\varepsilon^3 Q_{311}\phi_2\underset{\approx}{A}\underset{\sim}{\delta} - \frac{\varepsilon^3\lambda Q_{333}}{\mu(2\mu+3\lambda)}\phi_2\underset{\approx}{\delta} = 0,$$

$$\frac{2\varepsilon}{\mu}\underset{\sim}{\sigma}^0 - 2\varepsilon(-\theta + \underset{\sim}{\nabla}\,\omega) - \frac{2\varepsilon Q_{113}}{\mu}(\underset{\sim}{\nabla}\,\phi^0_{bc} + \varepsilon^2\underset{\sim}{\nabla}\,\phi_2) = 0,$$

$$\frac{2\varepsilon^5}{5\mu}\underset{\sim}{\sigma}^2 - \frac{2\varepsilon^5}{5}\underset{\sim}{\nabla}\,\omega_2 + \frac{2\varepsilon^5}{5\mu}Q_{113}\underset{\sim}{\nabla}\,\phi_2 = 0,$$

$$-\frac{\varepsilon^3\lambda}{3\mu(2\mu+3\lambda)}\operatorname{tr}(\underset{\approx}{\sigma}^1) + \frac{2\varepsilon^3(\mu+\lambda)}{3\mu(2\mu+3\lambda)}\sigma^1_{33} - 2\varepsilon^3\omega_2 - \frac{2\varepsilon^3\lambda Q_{311}}{\mu(2\mu+3\lambda)}\phi_2 + \frac{2\varepsilon^3(\mu+\lambda)Q_{333}}{\mu(2\mu+3\lambda)}\phi_2 = 0.$$

and then (14.i–14.iv) holds. In the same fashion, (14.v–14.vii) follow from (3.ii).

To find now the equilibrium conditions we integrate (3.iii) and then

$$\int_\Omega \frac{2\varepsilon^3}{3}\underset{\approx}{\sigma}^1 : \underset{\approx}{e}(\underset{\sim}{v}) + 2\varepsilon\underset{\sim}{\sigma}^0\cdot(\underset{\sim}{v} + \underset{\sim}{\nabla}\,v^0_3) + \frac{2\varepsilon^5}{5}\underset{\sim}{\sigma}^2\cdot\underset{\sim}{\nabla}\,v^2_3 + 2\varepsilon^3\sigma^1_{33}v^2_3\,d\underset{\sim}{x}$$
$$= \int_\Omega \varepsilon\underset{\sim}{f}^1\cdot\underset{\sim}{v} + \varepsilon f^0_3 v^0_3 + \varepsilon f^2_3 v^2_3 + 2\varepsilon\underset{\sim}{g}^1\cdot\underset{\sim}{v} + 2g^0_3 v^0_3 + 2\varepsilon^2 g^0_3 v^2_3\,d\underset{\sim}{x}.$$

Equations (13.i–13.iii) follow then from (14.i–14.iv).

Finally from (3.iv),

$$\int_\Omega 2\varepsilon^3\underset{\sim}{D}^0\cdot\underset{\sim}{\nabla}\,\psi^2 - \frac{2\varepsilon^5}{5}\underset{\sim}{D}^2\cdot\underset{\sim}{\nabla}\,\psi^2 - 2\varepsilon^3 D_3\psi^2\,d\underset{\sim}{x} = 0,$$

i.e.,

$$-\operatorname{div}\underset{\sim}{D}^0 + \frac{\varepsilon^2}{5}\operatorname{div}\underset{\sim}{D}^2 - D_3 = 0 \quad \text{in } \Omega, \qquad (\underset{\sim}{D}^0 - \frac{\varepsilon^2}{5}\underset{\sim}{D}^2)\cdot\underset{\sim}{n} = 0 \quad \text{on } \partial\Omega.$$

and then (13.i) results from (14.v–14.vii).

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Departamento de Matemática Aplicada, Laboratório Nacional de Computação Científica, Av. Getúlio Vargas 333, 25651-070 Petrópolis - RJ, Brazil

E-mail address: alm@lncc.br