A MULTISCALE FINITE ELEMENT METHOD
FOR PARTIAL DIFFERENTIAL EQUATIONS
POSED IN DOMAINS WITH ROUGH BOUNDARIES

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ABSTRACT. We propose and analyze a finite element scheme of multiscale type to deal with elliptic partial differential equations posed in domains with rough boundaries. There is no need to assume that the boundary is periodic in any sense, so the method is quite general. On the other hand, if the boundary is periodic we prove convergence of the scheme.

1. Introduction

Several applications require solving PDEs in domains with rough boundaries. Finding exact solutions is in general out of question, and numerically approximating the problem can be hard since discretizing the boundaries requires a sophisticated and quite refined, i.e. expensive, mesh. That is because good quality meshes for such domains often over-refine unnecessarily the interior of the domain.

Assuming that the wrinkles of the boundary are, in a proper sense, periodic, several authors proposed effective boundary conditions, also known as wall laws, on a mollified boundary that somehow upscaled the small scale geometry [1–6, 8–10, 12, 14, 25, 26]. This was possible for several operators and geometries, and involved solving cell problems, as is typical in homogenization procedures. Also, the article [13] recently presented the case of random, spatially homogeneous wrinkles.

Despite the remarkable advances just mentioned, to the best of our knowledge, the case of nonperiodic wrinkles was left untouched, with the exception noted above.

On the other hand, some numerical methods for PDEs with oscillatory coefficients were recently suggested and analyzed [11, 16–24, 30]. A desirable feature of these numerical schemes is that there is no need to derive a homogeneous equation, and only then discretize. On the contrary, the methods tackle the problems in their original forms.

The goal of the present paper is to propose a Multiscale Finite Element Method (MsFEM) to numerically solve PDEs in domains with oscillatory boundaries.

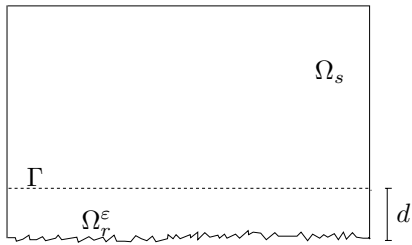Consider the sequence of problems parameterized by $\varepsilon < 1$: find u^ε such that

$$(1.1) \qquad \begin{aligned} -\Delta u^\varepsilon &= f \quad \text{in } \Omega^\varepsilon, \\ u^\varepsilon &= 0 \quad \text{on } \partial\Omega^\varepsilon, \end{aligned}$$

FIGURE 1. The domain Ω^ε.

for $f \in L^2(\Omega^\varepsilon)$. For simplicity, we assume that the domain Ω^ε is a rectangle with a rough bottom, as depicted in Figure 1. For $d > \varepsilon$, let

(1.2) $\Omega^\varepsilon = \{\mathbf{x} = (x, y) \in \mathbb{R}^2 \ : \ 0 < x < 1, \ -d + \psi_r^\varepsilon(x) < y < 1\},$

where the Lipschitz function ψ_r^ε is such that $\psi_r^\varepsilon(0) = \psi_r^\varepsilon(1) = 0$, and $\|\psi_r^\varepsilon\|_{L^\infty(0,1)} < \varepsilon$. The graph of ψ_r^ε defines Γ_r^ε, the lower boundary of Ω^ε:

$$\Gamma_r^\varepsilon = \{(x, y) \in \partial\Omega^\varepsilon \ : \ y = -d + \psi_r^\varepsilon(x)\}.$$

For the sake of the forthcoming analysis, it is convenient to consider Ω^ε as the union of an ε-independent domain and a "rough rectangle"; see (2.1) below. The parameter d indicates the "thickness" (disregarding the wrinkles) of the rugged part.

Next, we outline the contents of this paper. In Section 2, we introduce a numerical scheme to solve (1.1), and in Section 3, we do an error analysis for the periodic case.

We now briefly introduce and explain some basic notation that we use throughout this paper. As usual, if D is an open set, then $L^2(D)$ is the set of square integrable functions in D, and $H^s(D)$ is the corresponding Sobolev space of order s, for a real number s. We denote the norms of those spaces by $\|\cdot\|_{L^2(D)}$ and $\|\cdot\|_{H^s(D)}$. Without loss of generality, we have chosen to work in two dimensions. Nonetheless, all that follows can be generalized to the three-dimensional case. Bold fonts indicate two-dimensional vectors, and we denote by c a generic constant (not necessarily the same in all occurrences) which is independent of ε, but may depend on Sobolev norms of f.

2. Multiscale finite element method

We propose in this section a multiscale numerical method to approximate the solutions of (1.1). Variants of this method were successfully employed in different contexts, and we extend their use to the present problem.

In the MsFEM [19–24], the basis functions are local (elementwise) solutions of the original problem, and should capture the local features of the problem to upscale this information into the global formulation.

In our setting, we propose a Galerkin method where the basis functions in the trial and test spaces have local support, as in the traditional finite element method. Nevertheless, the present method is unusual since it does not use polygonal shape functions. Actually, the elements are not even triangles or quadrilaterals, but have a "rough geometry" whenever they intersect the rough boundary. The basic idea behind the method is that the influence of the geometry should be captured by the
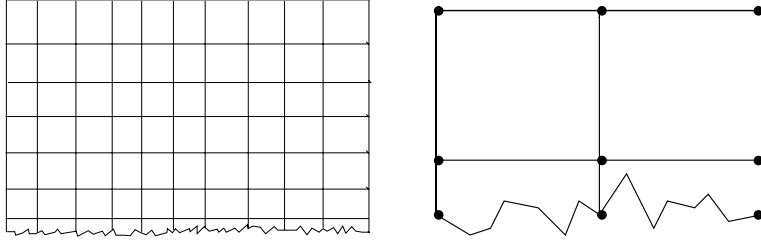
FIGURE 2. A mesh for Ω^ε, and a patch of elements of $\mathcal{T}_h$ intercepting Γ_r^ε.

basis functions without affecting the dimension of the finite element space. So our multiscale method is of Galerkin type, where the basis functions are local solutions of the operator under consideration (Laplacian), possibly in a rough element.

Before discretizing Ω^ε in finite elements, we define

$$(2.1) \qquad \Omega_s = (0,1) \times (0,1), \qquad \Omega_r^\varepsilon = \Omega^\varepsilon \backslash \overline{\Omega_s}, \qquad \Gamma = (0,1) \times \{0\}.$$

Note that

$$(2.2) \qquad \Omega_r^\varepsilon = \{\mathbf{x} = (x,y) \in \Omega^\varepsilon : 0 < x < 1, \, -d + \psi_r^\varepsilon(x) < y < 0\}.$$

Let $N \in \mathbb{N}$ and $h = 1/(N+1) > \varepsilon$. For $i = 0, \ldots, N$, let $K_i^\varepsilon = \{(x,y) \in \Omega_r^\varepsilon : ih < x < (i+1)h\}$. Note that such elements define a tiling of Ω_r^ε. Next, we introduce a Cartesian mesh for Ω_s using squares of side h. Such a procedure induces a partition $\mathcal{T}_h$ of Ω^ε in finite elements, not all of them being squares. Indeed, if $K \in \mathcal{T}_h \cap \Omega_s$, then K is a square of size h. Otherwise, K has a rough bottom, and straight lateral and top edges; see Figure 2. The mesh $\mathcal{T}_h$ has as nodal points the set $\mathcal{N} = \{(ih, jh) \in \Omega^\varepsilon : i = 1, \ldots, N, \, j = 0, \ldots, N\}$.

For each node $\mathbf{x}_i \in \mathcal{N} \cap \Omega_s$, we associate a piecewise bilinear polynomial $\lambda_i \in H_0^1(\Omega^\varepsilon)$ such that $\lambda_i(\mathbf{x}_j) = \delta_{ij}$ for all $\mathbf{x}_j \in \mathcal{N}$.

For each node $\mathbf{x}_i \in \mathcal{N} \cap \Gamma$, we define $\lambda_i \in H_0^1(\Omega^\varepsilon)$ such that

$$(2.3)$$
$$-\Delta \lambda_i = 0 \quad \text{in} \ \bigcup_{j=0}^{N} K_j^\varepsilon,$$

$$\lambda_i(\mathbf{x}_j) = \delta_{ij} \quad \text{for all } \mathbf{x}_j \in \mathcal{N}, \qquad \lambda_i \text{ is linear on } \bigcup_{j=0}^{N} \partial K_j^\varepsilon \cap \Omega^\varepsilon.$$

We extend λ_i to Ω_s by imposing that λ_i is piecewise bilinear in Ω_s.

Using the functions just defined, we set

$$V_h^\varepsilon = \text{span}\{\lambda_i\} \subset H_0^1(\Omega^\varepsilon).$$

The MsFEM solution $u_h^\varepsilon \in V_h^\varepsilon$ is simply the Galerkin approximation of u^ε in V_h^ε, i.e,

$$(2.4) \qquad \int_{\Omega^\varepsilon} \nabla u_h^\varepsilon(\mathbf{x}) \cdot \nabla v_h(\mathbf{x}) \, d\mathbf{x} = \int_{\Omega^\varepsilon} f(\mathbf{x}) v_h(\mathbf{x}) \, d\mathbf{x} \qquad \text{for all } v_h \in V_h^\varepsilon.$$

Remark 2.1. Note that the computation of each basis function is ε-dependent, thus expensive. Although the cost can be mitigated by parallelization, this is clearly a drawback. Nonetheless, as noted by Marcus Sarkis [31], at least for two-dimensional

problems, it is possible to obtain a quite narrow banded stiffness matrix by numbering the nodes properly.

Remark 2.2. The present method is particularly attractive if the problem (1.1) is to be solved repeatedly for different source terms. Indeed, the basis functions need to be computed only once for a given geometry, and the size of the stiffness matrix is ε-independent.

3. Numerical analysis

In this section we perform the error analysis for the numerical schemes proposed. Such analysis is based on asymptotic methods, and we restrict ourselves to the case of periodic wrinkles, when the solution has a well-known structure. We do not present the arguments leading to the development of the terms in the expansion, but we rather simply define the terms. Madureira and Valentin [25, 26] consider a complete asymptotics for a curved rough boundary.

By *periodic wrinkles*, we mean that $\psi_r^\varepsilon(x) = \varepsilon\psi_r(x/\varepsilon)$, where ψ_r is periodic with period 1, and $\|\psi_r\|_{L^\infty(\mathbb{R})} < 1$. In what follows, we assume for simplicity that f vanishes in Ω_r^ε. We also assume that there exist constants $\gamma \in (0, 1]$, and c_0 positive such that

$$(3.1) \qquad\qquad c_0\varepsilon^{1-\gamma} \le d,$$

where we recall that the parameter d helps to define the rough domain Ω_r^ε. See Figure 1, and equations (1.2), (2.2).

Remark 3.1. The restriction (3.1) indicates that d cannot be "too small". Indeed, the rough boundary originates ripples in u^ε that decay exponentially as y/ε grows, and it is important in the numerical analysis that the oscillatory part of the solution in Ω_s is at most polynomially small with respect to ε. So, the distance d helps to define the region Ω_s which is free from the high-gradient part of the solution.

3.1. **Asymptotic expansion for the exact solution.** To make clear how the solution of equation (1.1) depends on the small parameter ε, we expand u^ε in a formal power series with respect to ε.

We develop the expansion using the decomposition (2.1) of Ω in Ω_r^ε and Ω_s, and the expansions are coupled by boundary conditions on Γ. In Ω_s the asymptotic expansion of u^ε is composed by smooth terms, and in Ω_r^ε the expansion is made up of a highly oscillatory part which decays exponentially to zero away from Γ_r^ε, plus a linear function in y. The first few terms of the expansion are as follows:

$$(3.2) \quad u^\varepsilon(\mathbf{x}) \sim \begin{cases} (d + y - \varepsilon W - \varepsilon z)\dfrac{\partial u^0}{\partial y}(x, 0) + (d + y)\dfrac{\partial u^1}{\partial y}(x, 0) + \dots & \text{in } \Omega_r^\varepsilon, \\ u^0(\mathbf{x}) + u^1(\mathbf{x}) + \dots & \text{in } \Omega_s. \end{cases}$$

The first term of the asymptotics solves

$$(3.3) \qquad \begin{aligned} -\Delta u^0 &= f \quad \text{in } \Omega_s, \\ u^0 = 0 \quad \text{on } \partial\Omega_s, &\qquad u^0 = 0 \quad \text{on } \Omega_r^\varepsilon. \end{aligned}$$

Remark 3.2. In (3.2) and the remainder of this paper, the term $\partial u^0/\partial y(x, 0)$ denotes the restriction of $\partial u^0/\partial y|_{\Omega_s}$ on Γ. The same convention holds for $\partial u^1/\partial y(x, 0)$, etc.
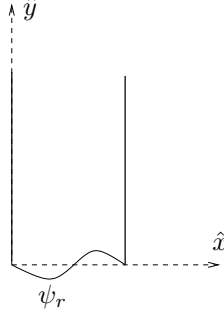
FIGURE 3. The cell domain.

To continue the description of the expansion, it is necessary to introduce a *cell problem*. This is no different from other singularly perturbed problems, perhaps elliptic PDEs with highly oscillatory coefficients being the most notorious. Such cell problems are an essential part in up-scaling procedures, bringing information related to the small scale geometry into the large scale behavior of the solution.

In the present case, the cell problem is defined in the semi-infinite strip Ω_r, which "contains" the geometry of the wrinkles,

$$\Omega_r = \{(\hat{x}, \hat{y}) \in \mathbb{R}^2 \,:\, \hat{x} \in (0, 1),\, \hat{y} > \psi_r(\hat{x})\},$$

i.e., Ω_r occupies the region delimited by straight lateral boundaries at $\hat{x} = 0$ and $\hat{x} = 1$, and by the lower boundary $\Gamma_r = \{(\hat{x}, \psi_r(\hat{x})) \in \mathbb{R}^2 \,:\, \hat{x} \in (0,1)\}$; see Figure 3.

We define $C^\infty_{\mathrm{per}}(\Omega_r)$ by restricting to Ω_r the functions in $C^\infty(\mathbb{R}^2)$ which are one-periodic with respect to $\hat{\theta}$. Let $H^1_{\mathrm{per}}(\Omega_r)$ be the closure of $C^\infty_{\mathrm{per}}(\Omega_r)$ with respect to the $H^1(\Omega_r)$ norm. We also introduce the space of exponentially decaying functions

$$S(\Omega_r) = \{w \in H^1_{\mathrm{per}}(\Omega_r) \,:\, w(\hat{x}, \hat{y})e^{\alpha\hat{y}} \in H^1(\Omega_r) \text{ for some } \alpha > 0\}.$$

It is possible to show [7, 26] that there exists unique $w \in S(\Omega_r)$, and a unique constant z such that

(3.4) $$\Delta\, w = 0 \quad \text{in } \Omega_r, \qquad w = \hat{y} - z \quad \text{on } \Gamma_r.$$

Moreover, $0 \leq z \leq \|\psi_r\|_{L^\infty(0,1)}$, and $\|w\|_{H^1(\Omega_r)} \leq c$, where c depends on the geometry only. Both z and w are related to the boundary layers that naturally occur in the original problem. Let $W(x, y) = w(\varepsilon^{-1}x, \varepsilon^{-1}(d + y))$ in Ω_r^ε.

Finally, let

$$-\Delta\, u^1 = 0 \quad \text{in } \Omega_s,$$

(3.5) $$u^1 = (d - \varepsilon z)\frac{\partial u^0}{\partial y} \quad \text{on } \Gamma, \qquad u^1 = 0 \quad \text{on } \partial\Omega_s\backslash\Gamma, \qquad u^1 = 0 \quad \text{on } \Omega_r^\varepsilon.$$

Remark 3.3. The above terms are slightly different from what usually appears in the literature. The difference is due to our definition of d, since usually $d = d_0\varepsilon$, where d_0 is a predefined constant. See for instance [25, 26].

Albeit (3.2) is formal, it is possible to show [26] that if

$$e(\mathbf{x}) = u^\varepsilon(\mathbf{x}) - u^0(\mathbf{x}) - u^1(\mathbf{x}) + (y - \varepsilon W)\frac{\partial u^0}{\partial y}(x, 0)\chi_r^\varepsilon,$$

where χ_r^ε is the characteristic function of Ω_r^ε, and under the assumption (3.1), there exists an ε-independent constant c such that

$$(3.6) \qquad \|e\|_{H^1(\Omega^\varepsilon)} \le cd^{3/2}, \qquad \|e\|_{H^1(\Omega_s)} \le cd^2.$$

To obtain the above estimates, it is useful to have the following bounds for each individual term. Such bounds follow from classical regularity estimates, and changes of coordinates.

Lemma 3.4. *Let u^0, u^1, and W be defined as above. Then there exists an ε-independent constant c such that*

$$\|u^0\|_{H^1(\Omega_s)} + d^{-1}\|u^1\|_{H^1(\Omega_s)} + \varepsilon^{-1/2}\|W\|_{L^2(\Omega_r^\varepsilon)} + \varepsilon^{1/2}\|\nabla W\|_{L^2(\Omega_r^\varepsilon)} \le c.$$

Proof. The estimate for u^0 follows immediately from (3.3), and (3.5) yields the bound for u^1. The estimates related to W come from the change of coordinates $\hat{x} = \varepsilon^{-1}x$, $\hat{y} = \varepsilon^{-1}(d + y)$, since, e.g.,

$$\int_{\Omega_r^\varepsilon} |W(\mathbf{x})|^2 \, d\mathbf{x} = \varepsilon \int_{\Omega_r} |w(\hat{\mathbf{x}})|^2 \, d\hat{\mathbf{x}}, \qquad \int_{\Omega_r^\varepsilon} |\nabla W(\mathbf{x})|^2 \, d\mathbf{x} = \varepsilon^{-1} \int_{\Omega_r} |\nabla w(\hat{\mathbf{x}})|^2 \, d\hat{\mathbf{x}}.$$

$\square$

3.2. Analysis of the multiscale finite element method.

The goal of this subsection is to prove a convergence result for the MsFEM. We first note that Poincaré's inequality holds uniformly with respect to ε [26], i.e., there exists an ε-independent constant such that

$$\|v\|_{L^2(\Omega^\varepsilon)} \le |v|_{H^1(\Omega^\varepsilon)},$$

for all $v \in H_0^1(\Omega^\varepsilon)$. With that, we conclude that Céa's Lemma also holds uniformly in ε, as we state below.

Lemma 3.5 (Céa's Lemma). *Let $u^\varepsilon \in H_0^1(\Omega^\varepsilon)$ be the weak solution of (1.1), and $u_h^\varepsilon \in V_h^\varepsilon$ the solution of (2.4). Then there exists an ε-independent constant c such that*

$$\|u^\varepsilon - u_h^\varepsilon\|_{H^1(\Omega^\varepsilon)} \le c \inf_{v_h^\varepsilon \in V_h^\varepsilon} |u^\varepsilon - v_h^\varepsilon|_{H^1(\Omega^\varepsilon)}.$$

Hence, the task now is to find a good approximation for u^ε in V_h^ε. As in [23], we use the asymptotics of both u^ε and the basis functions as a tool. We present here the asymptotic expansion in K_i^ε of the basis function λ_i defined in (2.3), corresponding to a nodal point $(ih, 0) \in \Gamma \cap \mathcal{N}$. The asymptotics of λ_i in K_{i-1}^ε is similar. In K_i^ε we have that

$$(3.7) \qquad \lambda_i(\mathbf{x}) = \frac{1}{h(d - \varepsilon z)}\Big[\big(d + y - \varepsilon W - \varepsilon z\big)(x_{i+1} - x) + \varepsilon\theta_i(\mathbf{x}) + \varepsilon r_i(\mathbf{x})\Big],$$

where $x_{i+1} = (i+1)h$, and θ_i solves

$$-\Delta \theta_i = 0 \quad \text{in } K_i^\varepsilon,$$

$$\theta_i(x, y) = (x_{i+1} - x)\left[W(x, y) - \frac{zy}{d}\right] \quad \text{on } \partial K_i^\varepsilon \backslash \Gamma_r^\varepsilon, \qquad \theta_i = 0 \quad \text{on } \Gamma_r^\varepsilon.$$

Also,

$$-\Delta r_i = 2\frac{\partial W}{\partial x} \quad \text{in } K_i^\varepsilon, \qquad r_i = 0 \quad \text{on } \partial K_i^\varepsilon.$$

Note that the expansion (3.2) in Ω_r^ε and (3.7) are quite similar since in both cases the lower order terms involve the expression $d + y - \varepsilon W - \varepsilon z$ times a function of x only, that is, $\partial u^0/\partial y$ in (3.2), and $(x_{i+1} - x)/[h(d - \varepsilon z)]$ in (3.7).

The following lemma presents an upper bound for the H^1 norms of r_i and θ_i.

Lemma 3.6. *Let r_i and θ_i be defined as above. Then there exists a constant c independent of ε and h such that*

$$\|r_i\|_{H^1(K_i^\varepsilon)} \le c\varepsilon^{1/2}h^{1/2}, \qquad \|\theta_i\|_{H^1(K_i^\varepsilon)} \le ch.$$

Proof. It follows from standard estimates that $\|r_i\|_{H^1(K_i^\varepsilon)} \le \|W\|_{L^2(K_i^\varepsilon)}$. Then, arguing as in the proof of Lemma 3.4 we obtain the first estimate. Next, again from standard estimates, $\|\nabla\theta_i\|_{L^2(K_i^\varepsilon)} \le c\|\theta_i\|_{H^{1/2}(\partial K_i^\varepsilon\setminus\Gamma_r^\varepsilon)}$. Note that

$$\|\theta_i\|_{H^{1/2}(\partial K_i^\varepsilon\setminus\Gamma_r^\varepsilon)} \le h\|W\|_{H^{1/2}(\partial K_i^\varepsilon\setminus\Gamma_r^\varepsilon)} + hz.$$

From the exponential decay properties of W, and trace and interpolation inequalities, we gather that $\|W\|_{H^{1/2}(\partial K_i^\varepsilon\setminus\Gamma_r^\varepsilon)} \le c$. Thus, the result follows. $\quad\square$

Theorem 3.7. *Let u^ε be the solution of (1.1), and $u_h^\varepsilon \in V_h^\varepsilon$ be the solution of (2.4). Assume further (3.1), and that the trace of $\partial u^0/\partial y|_{\Omega_s}$ on Γ belongs to $H^2(\Gamma)$, where u^0 solves (3.3). Then there exists a constant c independent of ε and h such that*

$$\|u^\varepsilon - u_h^\varepsilon\|_{H^1(\Omega^\varepsilon)} \le c(h + d^{3/2} + \varepsilon h^{-1/2} + \varepsilon^{3/2}h^{-1}).$$

Proof. We shall base our proof on Lemma 3.5 and the asymptotics for u^ε and the functions in V_h^ε. Let $\Upsilon(\mathbf{x}) = (y - \varepsilon W)\partial u^0/\partial y(x,h)\chi_r^\varepsilon$. Using the triangle inequality and (3.6), it follows that

$$(3.8) \quad |u^\varepsilon - v_h|_{H^1(\Omega^\varepsilon)} \le |u^\varepsilon - u^0 - u^1 + \Upsilon|_{H^1(\Omega^\varepsilon)} + |u^0 + u^1 + \Upsilon - v_h|_{H^1(\Omega^\varepsilon)}$$

$$\le cd^{3/2} + |u^0 + u^1 + \Upsilon - v_h|_{H^1(\Omega^\varepsilon)}$$

for all $v_h \in V_h^\varepsilon$. We choose v_h to be $u_I(\mathbf{x}) = \sum_{\mathbf{x}_i \in \mathcal{N}}[u_0(\mathbf{x}_i) + u_1(\mathbf{x}_i)]\lambda_i(\mathbf{x})$. Then, in Ω_s, u_I is piecewise bilinear, continuous, and interpolates $u^0 + u^1$. Thus

$$(3.9) \qquad\qquad |u^0 + u^1 - u_I|_{H^1(\Omega_s)} \le ch.$$

In Ω_r^ε,

$$u_I(\mathbf{x}) = \left(d + y - \varepsilon W - \varepsilon z\right)I_h\left(\frac{\partial u^0}{\partial y}\right)(x) + R_I,$$

$$R_I(\mathbf{x}) = \varepsilon h^{-1}\sum_{i=1}^{N}\frac{\partial u^0}{\partial y}(x_i, 0)[\theta_i(\mathbf{x}) + r_i(\mathbf{x})],$$

where $I_h(\partial u^0/\partial y)(\cdot)$ is the piecewise linear interpolant of $\partial u^0/\partial y(\cdot, 0)$ in $(0,1)$. Let $e_I(x,y) = \partial u^0/\partial y(x,h) - I_h(\partial u^0/\partial y)(x)$ be the interpolation error. We need the estimates [15, 29]

$$\|e_I\|_{L^2(0,1)} + h\|\frac{\partial e_I}{\partial x}\|_{L^2(0,1)} + h\|e_I\|_{L^\infty(0,1)} + h^2\|\frac{\partial e_I}{\partial x}\|_{L^\infty(0,1)} \le ch^2\|\frac{\partial u^0}{\partial y}\|_{H^2(0,1)}.$$

We compute
(3.10)

$$|\Upsilon - u_I|_{H^1(\Omega_r^\varepsilon)} \le |ye_I|_{H^1(\Omega_r^\varepsilon)} + \varepsilon|we_I|_{H^1(\Omega_r^\varepsilon)} + |d - \varepsilon z||I_h(\frac{\partial u^0}{\partial y})|_{H^1(\Omega_r^\varepsilon)} + |R_I|_{H^1(\Omega_r^\varepsilon)}.$$

Estimating each term we have

$$|ye_I|^2_{H^1(\Omega_r^\varepsilon)} \le d\int_{\Omega_r^\varepsilon}|\frac{\partial e_I}{\partial x}|^2\,d\mathbf{x} + \int_{\Omega_r^\varepsilon}|e_I|^2\,d\mathbf{x} \le cd^2h^2 + cdh^4,$$

$$|we_I|^2_{H^1(\Omega_r^\varepsilon)} \le \|e_I\|^2_{L^\infty(0,1)}|w|^2_{H^1(\Omega_r^\varepsilon)} + \|\frac{\partial e_I}{\partial x}\|^2_{L^\infty(0,1)}|w|^2_{L^2(\Omega_r^\varepsilon)} \le c\varepsilon^{-1}h^2 + c\varepsilon,$$

$$|d - \varepsilon z|^2|I_h(\frac{\partial u^0}{\partial y})|^2_{H^1(\Omega_r^\varepsilon)} \le cd^2|I_h(\frac{\partial u^0}{\partial y})|^2_{H^1(\Omega_r^\varepsilon)} \le cd^2\int_{\Omega_r^\varepsilon}|\frac{\partial I_h(\frac{\partial u^0}{\partial y})}{\partial x}|^2\,d\mathbf{x} \le cd^3.$$

Using Lemma 3.6 we finally estimate

$$|R_I|^2_{H^1(K_i^\varepsilon)} \le c\varepsilon^2 h^{-2}(\|\theta_i\|^2_{H^1(K_i^\varepsilon)} + \|r_i\|^2_{H^1(K_i^\varepsilon)}) \le c(\varepsilon^2 + \varepsilon^3 h^{-1}).$$

Adding up over the elements we obtain $|R_I|_{H^1(K_i^\varepsilon)} \le c(\varepsilon h^{-1/2} + \varepsilon^{3/2}h^{-1})$. Finally, from (3.10) we gather that

$$(3.11) \quad |\Upsilon - u_I|_{H^1(\Omega_r^\varepsilon)} \le c(dh + d^{1/2}h^2 + \varepsilon^{1/2}h + \varepsilon^{3/2} + d^{3/2} + \varepsilon h^{-1/2} + \varepsilon^{3/2}h^{-1}).$$

The theorem follows from (3.1), (3.8), (3.9), and (3.11). $\qquad\square$

Remark 3.8. If $d \le c_1 h^{2/3}$, and $\varepsilon \le c_2 h$ for some constants c_1, c_2, then

$$(3.12) \qquad\qquad\qquad \|u^\varepsilon - u_h^\varepsilon\|_{H^1(\Omega^\varepsilon)} \le ch + c\varepsilon h^{-1/2}.$$

In practice it is wise to choose d as small as possible, reducing the computational costs associated with computing the basis functions λ_i.

In Theorem 3.7 and estimate (3.12), the $\varepsilon h^{-1/2}$ error is related to the choice of (linear) boundary conditions for the λ_i which originates the troublesome term θ_i in (3.7). Of course, the exact solution is far from being linear close to the rough boundary, and such a mismatch indicates that indeed θ_i is spurious.

Such a type of *resonance* error is not an exclusivity of the present method. For instance, some multiscale methods (e.g. MsFEM, and Residual Free Bubbles) proposed to tackle PDEs with rapidly varying coefficients also suffer from the same malady, and various strategies were devised to overcome it [22], [23], [24], [30].

One promising strategy to avoid such boundary effects in our case is oversampling [19, 21, 22, 24]. The goal is to avoid the restriction that λ_i is linear over the edges. Consider the element K_i^ε, and the enlarged ficticious element

$$\hat{K}_i^\varepsilon = \{(x, y) \in \Omega_r^\varepsilon : ih - l < x < (i+1)h + l\},$$

for some $l > 0$. Let the auxiliary multiscale functions ψ_i, ψ_{i+1} be the solutions of (2.3) in $\hat{K}_i^\varepsilon$. We define then λ_i as

$$\lambda_i = c_i^1 \psi_i + c_i^2 \psi_{i+1},$$

and the constants c_i^1, c_i^2 are uniquely determined from the restriction $\lambda_i(\mathbf{x}_j) = \delta_{ij}$ for $j = i, i+1$. The numerical solution is defined by (2.4), where again $V_h^\varepsilon = \text{span}\{\lambda_i\}$.

To conclude the definition of such an oversampling method, it is necessary to define l. Such a constant needs to be large enough so that the effect in K_i^ε of the boundary condition on $\partial\hat{K}_i^\varepsilon \cap \Omega^\varepsilon$ is negligible. On the other hand, increasing l augments computational costs. A good compromise would be $l = O(\varepsilon)$.

Note that the method becomes nonconforming as λ_i might have jumps over the edges, and thus $V_h^\varepsilon \not\subset H^1(\Omega^\varepsilon)$ in general.

4. Conclusion

In this work, we propose a multiscale finite element scheme to deal with PDEs posed in domains with rough boundary. Previous techniques approach this problem with homogenization techniques. As far as we now, the exceptions are [27, 28], but their methods seem more complicated than ours.

The present method is quite general, and its definition does not assume any special feature of the wrinkles. Since the scheme is based on solutions of local problems, parallelization is trivial. Of course, the method is still expensive since local problems depend on ε, but it is much cheaper than using pure piecewise linear FEM in the whole domain. However, if the wrinkles are periodic, it is still cheaper to use traditional wall-laws.

Concerning estimates, an error analysis is available for the periodic case only, where models using wall-laws have the $H^1(\Omega^\varepsilon)$ norm error as $h + \varepsilon^{1/2}$, and the present method converges as $h + \varepsilon h^{-1/2}$. The $h^{-1/2}$ term is related to a resonance error, also present in other multiscale methods applied to PDEs with highly oscillatory coefficients [19, 21–24, 30]. Oversampling techniques might ameliorate such a problem.

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