

AN ASYMPTOTICALLY OPTIMAL MODEL FOR ISOTROPIC HETEROGENEOUS LINEARLY ELASTIC PLATES

FERDINANDO AURICCHIO, CARLO LOVADINA, AND ALEXANDRE L. MADUREIRA

ABSTRACT. In this paper, we derive and analyze a Reissner–Mindlin–like model for isotropic heterogeneous linearly elastic plates. The modeling procedure is based on a Hellinger–Reissner principle, which we modify to derive consistent models. Due to the material heterogeneity, the classical polynomial profiles for the plate shear stress are replaced by more sophisticated choices, that are asymptotically correct. In the homogeneous case we recover a Reissner–Mindlin model with $5/6$ as shear correction factor. Asymptotic expansions are used to estimate the modeling error.

We remark that our derivation is *not* based on asymptotic arguments only. Thus, the model obtained is more sophisticated (and accurate) than simply taking the asymptotic limit of the three dimensional problem. Moreover, we do not assume periodicity of the heterogeneities.

1. INTRODUCTION

Laminate slender structures are often adopted in practical applications for the excellent ratio between mechanical performances and weight. However, in terms of modeling they present several challenges.

In fact, although occupying a three dimensional domain in space, slender bodies are characterized by having a one- or two-dimensional “aspect”. Accordingly, *dimension reduction models* are posed in domains with at least one dimension less than the original problem, but the model solution should approximate as close as possible the original three-dimensional domain solution.

The modeling complexity is particularly significant for the case of slender structures made of heterogeneous materials, also in the case of two dimensional planar plate-like bodies. There have been numerous modeling attempts trying to incorporate the influence that the heterogeneity has on the solution in such problems (see for instance [4], [11], and references therein).

In this paper, we derive (and analyze) a version of the Reissner-Mindlin equations for the case of a plate made by isotropic heterogeneous materials. To reach this goal, we use the approach of Reference [1] for the development of a model. Such a technique relies on a *variational approach* and it is based essentially on two ingredients: the choice of a variational principle *and* the choice of proper subspaces.

Due to the material heterogeneity, in the following we need to develop an ad-hoc variational principle, which is different from the ones adopted in Reference [1], effective only for the case of homogeneous materials. Moreover, we also have to modify the subspaces in which we look for the solution; in fact, the classical polynomial profiles for the shear stress valid for the

This work has been partly supported by I.A.N. C.N.R. of Pavia, Italy.

case of homogeneous materials are now replaced by more sophisticated choices. In particular, such new profiles are derived through considerations based on asymptotic expansions.

As a result of our analysis, we obtain equations (26)–(29) defining our candidate for approximating the three dimensional heterogeneous plate problem solution.

We would like to emphasize that a plate model based on variational principles differs substantially from a model which result from considering asymptotic limits. Indeed, the former is defined by a system of singularly perturbed equations, and the corresponding solution presents a complex behavior with respect to the thickness.

On the other hand, models originated by asymptotic considerations are of a different nature. The best known example is the biharmonic plate model (cf. [6], [7], [9] for homogeneous isotropic plates, and [5], [13] for the heterogeneous case). The corresponding equations do *not* depend singularly on the plate thickness, and usually have limited applicability, as shown in [2], [6].

The outline of the paper is as follows. In the next section, we describe the traditional variational approaches and why it is not possible to use them. In Section 3 we detail the variational principle upon which we base our derivation and present a description of the proposed model. Finally, in Section 4, we prove the consistency of our solution and some other convergence results. We postpone to various appendices most of the details regarding derivation of the model, asymptotic expansions, and convergence results.

Before proceeding, we introduce and explain some notation. We use one underbar for first order tensors in three variables, two underbars for second order tensors in three variables, etc. Similar notation holds with undertildes for tensors in two variables. We can then decompose 3-vectors and 3×3 matrices as follows:

$$\underline{u} = \begin{pmatrix} \underline{u} \\ u_3 \end{pmatrix}, \quad \underline{\underline{\sigma}} = \begin{pmatrix} \underline{\underline{\sigma}} & \underline{\underline{\sigma}} \\ \underline{\underline{\sigma}}^T & \sigma_{33} \end{pmatrix}.$$

Moreover, throughout the paper we make use of the following operators that assigns for a given tensor its moments:

$$(1) \quad \begin{cases} I^k(h) = \varepsilon^{-k-1} \int_{-\varepsilon}^{\varepsilon} h(x_3) x_3^k dx_3, & \tilde{I}^k(\tilde{h}) = \varepsilon^{-k-1} \int_{-\varepsilon}^{\varepsilon} \tilde{h}(x_3) x_3^k dx_3, \\ \underline{\underline{I}}^k(\underline{\underline{h}}) = \varepsilon^{-k-1} \int_{-\varepsilon}^{\varepsilon} \underline{\underline{h}}(x_3) x_3^k dx_3, & \underline{\underline{\underline{I}}}^k(\underline{\underline{\underline{h}}}) = \varepsilon^{-k-1} \int_{-\varepsilon}^{\varepsilon} \underline{\underline{\underline{h}}}(x_3) x_3^k dx_3. \end{cases}$$

2. PLATE MODEL CONSTRUCTION BY VARIATIONAL APPROACH

We consider a linearly elastic body occupying the three-dimensional domain $P^\varepsilon = \Omega \times (-\varepsilon, \varepsilon)$, where Ω is a bounded two-dimensional domain. Clamped on the lateral boundary $\partial P_L^\varepsilon = \partial\Omega \times (-\varepsilon, \varepsilon)$, the body is under the action of a surface force density $\underline{g}^\varepsilon$ on its top and bottom $\partial P_\pm^\varepsilon = \Omega \times \{-\varepsilon, \varepsilon\}$, and a volume force density $\underline{f}^\varepsilon$. The equations of linear elasticity state that the displacement $\underline{u}^\varepsilon : P^\varepsilon \rightarrow \mathbb{R}^3$, and the stress $\underline{\underline{\sigma}}^\varepsilon : P^\varepsilon \rightarrow \mathbb{R}_{\text{sym}}^{3 \times 3}$ satisfy

$$(2) \quad \begin{cases} \underline{\underline{A}} \underline{\underline{\sigma}}^\varepsilon = \underline{\underline{e}}(\underline{u}^\varepsilon), & -\underline{\underline{\text{div}}} \underline{\underline{\sigma}}^\varepsilon = \underline{f}^\varepsilon \quad \text{in } P^\varepsilon, \\ \underline{u}^\varepsilon = 0 \text{ on } \partial P_L^\varepsilon, & \underline{\underline{\sigma}}^\varepsilon \underline{n} = \underline{g}^\varepsilon \text{ on } \partial P_\pm^\varepsilon, \end{cases}$$

where $\underline{e}(\underline{u}^\varepsilon) = (\underline{\nabla} \underline{u}^\varepsilon + \underline{\nabla}^T \underline{u}^\varepsilon)/2$ is the symmetric part of the gradient of $\underline{u}^\varepsilon$, and \underline{A} is the compliance tensor. Limiting the discussion to an isotropic material, the compliance tensor \underline{A} is defined as

$$(3) \quad \underline{A}\underline{\tau} = \frac{1+\nu}{E}\underline{\tau} - \frac{\nu}{E}\text{tr}(\underline{\tau})\underline{\delta},$$

for every second-order tensor $\underline{\tau}$, being $\underline{\delta}$ the identity tensor. The Young's modulus E and the Poisson's ratio ν are material parameters, which might depend on the transverse variable x_3 , being however independent of the planar variable \underline{x} .

A way to systematically derive two dimensional plate models is the *variational approach* proposed in [1]. This procedure consists in choosing a variational principle, which corresponds to a weak formulation of (2). The plate model is then derived restricting the test and trial spaces to suitable subspaces.

In [1] the following two different variational principles were considered to treat the case of *isotropic and homogeneous* bodies.

- The couple $(\underline{u}^\varepsilon, \underline{\sigma}^\varepsilon)$, solution of Problem (2), is the unique critical point of the functional

$$(4) \quad J(\underline{v}, \underline{\tau}) = \frac{1}{2} \int_{P^\varepsilon} \underline{A}\underline{\tau} : \underline{\tau} \, d\underline{x} + \int_{P^\varepsilon} \underline{v} \cdot \underline{\text{div}} \underline{\tau} \, d\underline{x} + \int_{P^\varepsilon} \underline{f} \cdot \underline{v} \, d\underline{x}$$

on $\underline{L}^2(P^\varepsilon) \times \underline{\Sigma}_{\underline{g}^\varepsilon}$, where $\underline{\Sigma}_{\underline{g}^\varepsilon} = \{ \underline{\sigma} : \underline{\sigma} \in \underline{H}(\underline{\text{div}}, P^\varepsilon), \underline{\sigma} \underline{n} = \underline{g}^\varepsilon \text{ on } \partial P^\varepsilon_\pm \}$.

Following the variational approach of Reference [1], plate models are derived restricting (4) to particular subspaces of $\underline{L}^2(P^\varepsilon) \times \underline{\Sigma}_{\underline{g}^\varepsilon}$, characterized by having specific polynomial dependences in the transverse direction.

It is interesting to notice that not all choices of subspaces lead to a well-posed PDE system, as well as some specific choices lead to models that, although well-posed, are divergent in a sense that we make clear later on.

It is also interesting to recall that, starting from functional (4) and adopting two different subspaces, it is possible to obtain two versions of the Reissner-Mindlin model, both with the shear correction factor 5/6. By "version", we mean that the left hand side of the equations are the same as the standard Reissner-Mindlin ones, up to the shear correction factor, while the right hand side may differ.

- The couple $(\underline{u}^\varepsilon, \underline{\sigma}^\varepsilon)$, solution of Problem (2), is the unique critical point of the functional

$$(5) \quad J^*(\underline{v}, \underline{\tau}) = \frac{1}{2} \int_{P^\varepsilon} \underline{A}\underline{\tau} : \underline{\tau} \, d\underline{x} - \int_{P^\varepsilon} \underline{e}(\underline{v}) : \underline{\tau} \, d\underline{x} + \int_{P^\varepsilon} \underline{f} \cdot \underline{v} \, d\underline{x} + \int_{\partial P^\varepsilon_\pm} \underline{g} \cdot \underline{v} \, d\underline{x}$$

on $\{ \underline{v} \in \underline{H}^1(P^\varepsilon) : \underline{v} = 0 \text{ on } \partial P^\varepsilon_L \} \times \underline{L}^2(P^\varepsilon)$.

Again, modeling is possible by searching for critical points in subspaces with certain polynomial dependence.

It is interesting to notice that the minimum energy models are a particular instance of this approach, and that the simplest model derived using this approach is not a minimum energy model.

Finally, we recall that starting from the functional (5) and adopting specific subspaces, the resulting system is a variant of Reissner–Mindlin model, with shear correction factor 1. The other models based on (5) are either more complicate than Reissner–Mindlin or divergent.

2.1. Definition of model consistency. As discussed in the Introduction, every plate model is supposed to approximate the 3D solution for the limiting “thin” plate case. Accordingly, we say that a plate model is *consistent*, or *convergent*, if for every “reasonable” choice of loads it holds

$$(6) \quad \lim_{\varepsilon \rightarrow 0} \frac{\|\underline{u}^\varepsilon - \underline{u}^{M,\varepsilon}\|_{H^1(P^\varepsilon)}}{\|\underline{u}^\varepsilon\|_{H^1(P^\varepsilon)}} = 0,$$

where $\underline{u}^\varepsilon$ are the 3D displacements and $\underline{u}^{M,\varepsilon}$ are the model displacements, respectively.

More precisely, let us suppose that there exist a set of ε -independent loads \underline{f} and \underline{g} defined as $\underline{f} : \Omega \times (-1, 1) \rightarrow \mathbb{R}^3$, $\underline{g} : \Omega \times \{-1, 1\} \rightarrow \mathbb{R}^3$ and such that

$$(7) \quad \begin{cases} \underline{f}(\underline{x}, \hat{x}_3) = \varepsilon^{-1} \underline{f}^\varepsilon(\underline{x}, \varepsilon \hat{x}_3), & f_3(\underline{x}, \hat{x}_3) = \varepsilon^{-2} f_3^\varepsilon(\underline{x}, \varepsilon \hat{x}_3), \\ \underline{g}(\underline{x}, \hat{x}_3) = \varepsilon^{-2} \underline{g}^\varepsilon(\underline{x}, \varepsilon \hat{x}_3), & g_3(\underline{x}, \hat{x}_3) = \varepsilon^{-3} g_3^\varepsilon(\underline{x}, \varepsilon \hat{x}_3), \end{cases}$$

for $\hat{x}_3 \in (-1, 1)$. Also, let us assume that the functions

$$\hat{\nu}(\hat{x}_3) = \nu(\varepsilon \hat{x}_3), \quad \hat{E}(\hat{x}_3) = E(\varepsilon \hat{x}_3) \quad \text{for } \hat{x}_3 \in (-1, 1),$$

are independent of ε and, since there is no risk of confusion, we still write ν and E , even in the domain $(-1, 1)$.

Under these hypotheses, as detailed in Appendix B, $\underline{u}^\varepsilon$ converges asymptotically to

$$\underline{u}_L(\underline{x}) = \begin{pmatrix} \varepsilon \zeta^1(\underline{x}) - x_3 \nabla \zeta_3^0(\underline{x}) \\ \zeta_3^0(\underline{x}) \end{pmatrix},$$

where ζ^1 and ζ_3^0 solve the following system of equations

$$(8) \quad \begin{cases} \varepsilon \operatorname{div} \left[\underset{\approx}{I}^0(\underset{\approx}{A}^{-1}) \underset{\approx}{e}(\zeta^1) \right] - \varepsilon \operatorname{div} \left[\underset{\approx}{I}^1(\underset{\approx}{A}^{-1}) \underset{\approx}{e}(\nabla \zeta_3^0) \right] = -\underset{\approx}{I}^0(\underline{f}^\varepsilon) - 2\varepsilon^{-1} \underline{g}^{\varepsilon,\varepsilon} & \text{in } \Omega, \\ \varepsilon \operatorname{div} \operatorname{div} \left[\underset{\approx}{I}^1(\underset{\approx}{A}^{-1}) \underset{\approx}{e}(\zeta^1) \right] - \varepsilon \operatorname{div} \operatorname{div} \left[\underset{\approx}{I}^2(\underset{\approx}{A}^{-1}) \underset{\approx}{e}(\nabla \zeta_3^0) \right] = -\varepsilon^{-1} \underset{\approx}{I}^0(\underline{f}_3^\varepsilon) \\ \quad - \underset{\approx}{I}^1(\operatorname{div} \underline{f}^\varepsilon) - 2\varepsilon^{-1} \operatorname{div} \underline{g}^{\varepsilon,\circ} - 2\varepsilon^{-2} \underline{g}_3^{\varepsilon,\varepsilon} & \text{in } \Omega, \\ \zeta^1 = 0, \quad \zeta_3^0 = \frac{\partial \zeta_3^0}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

with $\underset{\approx}{A}$ defined as

$$(9) \quad \underset{\approx}{A} \underset{\approx}{\tau} = \frac{(1 + \nu)}{E} \underset{\approx}{\tau} - \frac{\nu}{E} \operatorname{tr}(\underset{\approx}{\tau}) \underset{\approx}{\delta},$$

and with the even and odd parts of \underline{g} defined as

$$(10) \quad \underline{g}^{\varepsilon,\varepsilon}(\underline{x}) = \frac{1}{2} [\underline{g}^\varepsilon(\underline{x}, \varepsilon) + \underline{g}^\varepsilon(\underline{x}, -\varepsilon)], \quad \underline{g}^{\varepsilon,\circ}(\underline{x}) = \frac{1}{2} [\underline{g}^\varepsilon(\underline{x}, \varepsilon) - \underline{g}^\varepsilon(\underline{x}, -\varepsilon)].$$

We say that the plate model is *consistent*, or *convergent*, if condition (6) is satisfied for all loads such that at least one of the functions ζ^1 and ζ_3^0 are nonzero.

Remark. Note that Reference [12] uses a slightly different definition of consistency. \square

Remark. Note that (8) couples the membrane and bending problems. Nevertheless, if ν and E are even functions (symmetric plates), then it is easy to see that (8) decouples into the membrane equation for ζ^1 , and the biharmonic equation for ζ_3^0 . \square

2.2. Problems with the Hellinger–Reissner Principle. Taking into account the previous considerations, it would be natural to choose functional (4) as a starting point to derive a plate model also for the case of heterogeneous materials. Unfortunately, the direct application of functional (4) is not possible since, due to the heterogeneity, it leads to divergent models, as discussed below.

Let us consider the case where $g_3 = 0$ and ν and E are even functions of x_3 . Since the three-dimensional problem (2) decouples in bending and membrane equations, for the sake of simplicity we consider only the membrane contribution.

The simplest model based on (4) is obtained by assuming that trial/test in-plane displacements and in-plane stresses are constant along the thickness (cf. [1] for the details)

$$(11) \quad \underline{u}^M(\underline{x}) = \begin{pmatrix} \underline{\eta}(\underline{x}) \\ 0 \end{pmatrix}, \quad \underline{\sigma}^M(\underline{x}) = \underline{\underline{\sigma}}(\underline{x}), \quad \sigma_{33}^M(\underline{x}) = 0, \quad \underline{\underline{\sigma}}^M(\underline{x}) = \varepsilon^{-1} x_3 \underline{\underline{g}}^{\varepsilon, \varepsilon}(\underline{x}),$$

for some $\underline{\eta}$ and $\underline{\underline{\sigma}}$. Searching now for critical points, we obtain the following membrane equation

$$(12) \quad \operatorname{div} \left[4I_{\underline{\underline{\approx}}}^0(A)^{-1} \underline{\underline{e}}(\underline{\eta}) \right] = -I_{\underline{\underline{\approx}}}^0(\underline{f}^\varepsilon) - 2\varepsilon^{-1} \underline{\underline{g}}^{\varepsilon, \varepsilon} \quad \text{in } \Omega.$$

On the other hand, for this specific (uncoupled) problem the 3D in-plane displacement field $\underline{u}^\varepsilon$ converges asymptotically to $\varepsilon \zeta^1$, solution of (cf. (8))

$$(13) \quad \operatorname{div} \left[I_{\underline{\underline{\approx}}}^0(A^{-1}) \underline{\underline{e}}(\varepsilon \zeta^1) \right] = -I_{\underline{\underline{\approx}}}^0(\underline{f}^\varepsilon) - 2\varepsilon^{-1} \underline{\underline{g}}^{\varepsilon, \varepsilon} \quad \text{in } \Omega,$$

Hence, the obtained membrane model is divergent since for a general heterogeneous material

$$(14) \quad 4I_{\underline{\underline{\approx}}}^0(A)^{-1} \neq I_{\underline{\underline{\approx}}}^0(A^{-1}).$$

Remark. Note that for homogeneous plates $4I_{\underline{\underline{\approx}}}^0(A)^{-1} = I_{\underline{\underline{\approx}}}^0(A^{-1})$; accordingly, the above membrane model is consistent, since $\underline{u}^M = \underline{\eta} = \varepsilon \zeta^1$.

To summarize, the extension of the techniques detailed in [1] to the case of heterogeneous plates leads to the following two difficulties.

- The variational principle (4) is no longer suitable.
- Polynomial profiles in the transverse direction are no more satisfactory.

In particular, the second point is due to the fact that now the profiles for the stress tensor components have a complicated shape, depending on the material heterogeneities. Accordingly, assuming polynomial profiles for all the unknowns may lead to a poor representation of the stress tensor in the transverse direction.

In what follows we overcome these difficulties by

- Choosing a different variational principle.

- Choosing a subspace for the stress tensor which takes into account the heterogeneity of the plate.

3. A NEW MODEL

3.1. A new functional. As shown in Eq. (14), the processes of inverting and homogenizing do not commute in the case of heterogeneous tensors, yielding an incorrect constitutive relation for the plate planar components. To overcome this difficulty, we try to impose *directly in the model* the following planar constitutive equation

$$(15) \quad \underline{\underline{\sigma}}^\varepsilon = \underline{\underline{A}}^{-1} \underline{\underline{e}}(\underline{\underline{u}}^\varepsilon) + \frac{\nu}{E} \sigma_{33}^\varepsilon \underline{\underline{A}}^{-1} \underline{\underline{\delta}}.$$

To reach this goal, we first introduce the space $\underline{\underline{S}}_{g^\varepsilon}$, defined by

$$(16) \quad \underline{\underline{S}}_{g^\varepsilon} = \{ \underline{\underline{s}} : \underline{\underline{B}}\underline{\underline{s}} \in \underline{\underline{\Sigma}}_{g^\varepsilon} \}, \quad \text{where } \underline{\underline{B}}\underline{\underline{s}} = \begin{pmatrix} \underline{\underline{A}}^{-1}\underline{\underline{s}} & \underline{\underline{s}} \\ \underline{\underline{s}}^T & s_{33} \end{pmatrix}.$$

We then consider the *new* functional

$$(17) \quad J^\bullet(\underline{\underline{v}}, \underline{\underline{t}}) = \int_{P^\varepsilon} \underline{\underline{A}}\underline{\underline{B}}\underline{\underline{t}} : \underline{\underline{B}}\underline{\underline{t}} \, d\underline{\underline{x}} + \int_{P^\varepsilon} \underline{\underline{v}} \cdot \underline{\underline{\operatorname{div}}} \underline{\underline{B}}\underline{\underline{t}} \, d\underline{\underline{x}} + \int_{P^\varepsilon} \underline{\underline{v}} \cdot \underline{\underline{f}} \, d\underline{\underline{x}}$$

defined on $\underline{\underline{L}}^2(P^\varepsilon) \times \underline{\underline{S}}_{g^\varepsilon}$. It is easily seen that $(\underline{\underline{u}}^\varepsilon, \underline{\underline{\sigma}}^\varepsilon) \in \underline{\underline{L}}^2(P^\varepsilon) \times \underline{\underline{\Sigma}}_{g^\varepsilon}$ is the unique critical point of functional (4), if and only if the couple $(\underline{\underline{u}}^\varepsilon, \underline{\underline{s}}^\varepsilon) = (\underline{\underline{u}}^\varepsilon, \underline{\underline{B}}^{-1}\underline{\underline{\sigma}}^\varepsilon) \in \underline{\underline{L}}^2(P^\varepsilon) \times \underline{\underline{S}}_{g^\varepsilon}$ is the unique critical point of functional (17). Since $\underline{\underline{s}}^\varepsilon = \underline{\underline{B}}^{-1}\underline{\underline{\sigma}}^\varepsilon$, it follows that it holds (cf. (16))

$$(18) \quad \underline{\underline{s}}^\varepsilon = \underline{\underline{e}}(\underline{\underline{u}}^\varepsilon) + \frac{\nu}{E} \sigma_{33}^\varepsilon \underline{\underline{\delta}} = \underline{\underline{A}}\underline{\underline{\sigma}}^\varepsilon, \quad \underline{\underline{s}}^\varepsilon = \underline{\underline{\sigma}}^\varepsilon, \quad s_{33}^\varepsilon = \sigma_{33}^\varepsilon.$$

Moreover, we have that $(\underline{\underline{u}}^\varepsilon, \underline{\underline{s}}^\varepsilon) \in \underline{\underline{L}}^2(P^\varepsilon) \times \underline{\underline{S}}_{g^\varepsilon}$ satisfies the weak equations

$$(19) \quad \begin{cases} \int_{P^\varepsilon} \underline{\underline{A}}\underline{\underline{B}}\underline{\underline{s}}^\varepsilon : \underline{\underline{B}}\underline{\underline{t}} \, d\underline{\underline{x}} + \int_{P^\varepsilon} \underline{\underline{u}}^\varepsilon \cdot \underline{\underline{\operatorname{div}}} \underline{\underline{B}}\underline{\underline{t}} \, d\underline{\underline{x}} = 0 & \text{for all } \underline{\underline{t}} \in \underline{\underline{S}}_0, \\ - \int_{P^\varepsilon} \underline{\underline{\operatorname{div}}} \underline{\underline{B}}\underline{\underline{s}}^\varepsilon \cdot \underline{\underline{v}} \, d\underline{\underline{x}} = \int_{P^\varepsilon} \underline{\underline{f}}^\varepsilon \cdot \underline{\underline{v}} \, d\underline{\underline{x}} & \text{for all } \underline{\underline{v}} \in \underline{\underline{L}}^2(P^\varepsilon), \end{cases}$$

where $\underline{\underline{S}}_0 = \{ \underline{\underline{s}} : \underline{\underline{B}}\underline{\underline{s}} \in \underline{\underline{\Sigma}}_0 \}$. The model we are going to derive is based on the weak formulation (19) and, similarly to [1], it will be obtained by looking for critical points of J^\bullet within suitable subspaces $\underline{\underline{\hat{L}}}^2(P^\varepsilon) \times \underline{\underline{\hat{S}}}_{g^\varepsilon} \subset \underline{\underline{L}}^2(P^\varepsilon) \times \underline{\underline{S}}_{g^\varepsilon}$.

Remark. We notice that Auricchio and Sacco in [4] already used the constitutive equation (15) to develop their model. \square

3.2. Subspace choice. Our choices of profiles for displacements and stresses are based on the asymptotic analysis of the three-dimensional solution. To obtain the simplest possible convergent model, we decided to exclude the profile of the asymptotic limit of σ_{33}^ε . As we

describe in Appendix B,

$$(20) \quad \begin{cases} \underline{u}^\varepsilon(\underline{x}) & \sim \begin{pmatrix} \varepsilon \zeta^1(\underline{x}) - x_3 \nabla \zeta_3^0(\underline{x}) \\ \zeta_3^0(\underline{x}) \end{pmatrix} + \dots, \\ \underline{\sigma}^\varepsilon(\underline{x}) & \sim \underset{\approx}{A}^{-1} \underset{\approx}{e}(\varepsilon \zeta^1(\underline{x}) - x_3 \nabla \zeta_3^0(\underline{x})) + \dots, \\ \underline{\sigma}^\varepsilon(\underline{x}) & \sim \varepsilon^2 \nabla \operatorname{div} \zeta^1(\underline{x}) p^m(x_3) + \varepsilon^2 \nabla \Delta \zeta_3^0(\underline{x}) p^b(x_3) + \varepsilon^2 \underline{p}^l(\underline{x}) + \dots, \end{cases}$$

where ζ^1 and ζ_3^0 solve (8), and

$$(21) \quad \begin{cases} p^m(x_3) = \frac{I^0\left(\frac{E\nu}{1-\nu^2}\right)}{I^0\left(\frac{E}{1+\nu}\right)} \varepsilon^{-1} \int_{-\varepsilon}^{x_3} \frac{E}{1+\nu} d\xi - \varepsilon^{-1} \int_{-\varepsilon}^{x_3} \frac{E\nu}{1-\nu^2} d\xi, \\ p^b(x_3) = \frac{I^1\left(\frac{E}{1-\nu^2}\right)}{I^0\left(\frac{E}{1+\nu}\right)} \varepsilon^{-1} \int_{-\varepsilon}^{x_3} \frac{E}{1+\nu} d\xi - \varepsilon^{-2} \int_{-\varepsilon}^{x_3} \frac{E}{1-\nu^2} \xi d\xi, \\ \underline{p}^l(\underline{x}) = \frac{\varepsilon^{-3}}{I^0\left(\frac{E}{1+\nu}\right)} \left(\int_{-\varepsilon}^{\varepsilon} \underline{f}^\varepsilon d\xi + 2\underline{g}^{\varepsilon,\varepsilon} \right) \int_{-\varepsilon}^{x_3} \frac{E}{1+\nu} d\xi - \varepsilon^{-2} \int_{-\varepsilon}^{x_3} \underline{f}^\varepsilon d\xi - \varepsilon^{-2} \underline{g}^\varepsilon(\underline{x}, -\varepsilon). \end{cases}$$

Remark. Note that $p^m(\pm\varepsilon) = p^b(\pm\varepsilon) = 0$. Moreover, $\varepsilon^2 \underline{p}^l(\underline{x}, \varepsilon) = \underline{g}^\varepsilon(\underline{x}, \varepsilon)$ and $\varepsilon^2 \underline{p}^l(\underline{x}, -\varepsilon) = -\underline{g}^\varepsilon(\underline{x}, -\varepsilon)$. Moreover, \underline{p}^l depends only on the load. \square

Remark. We remark that the leading terms of the variables involved in (21) have the following shapes *in the transverse direction*:

$$\begin{cases} \text{a linear polynomial for } \underline{u}^\varepsilon; \\ \text{a constant for } u_3^\varepsilon; \\ \text{a linear polynomial for } \underline{s}^\varepsilon = \underset{\approx}{A} \sigma^\varepsilon; \\ \text{a linear combination of the functions (in general: not polynomials)} \\ \quad p^m, p^b \text{ and } \underline{p}^l \text{ for } \underline{s}^\varepsilon. \quad \square \end{cases}$$

We are now ready to introduce the subspaces $\hat{\underline{L}}^2(P^\varepsilon) \subset \underline{L}^2(P^\varepsilon)$ and $\hat{\underline{S}}_{\underline{g}^\varepsilon} \subset \underline{S}_{\underline{g}^\varepsilon}$, respectively for the displacement and the stress-related unknowns, by setting

$$(22) \quad \hat{\underline{L}}^2(P^\varepsilon) = \left\{ \underline{u}^{RM} = \begin{pmatrix} \underline{\eta}(\underline{x}) - x_3 \underline{\phi}(\underline{x}) \\ \omega(\underline{x}) \end{pmatrix} \right\},$$

$$(23) \quad \hat{\underline{S}}_{\underline{g}^\varepsilon} = \left\{ \underline{s}^{RM} = \begin{pmatrix} \underline{s}^m(\underline{x}) + \varepsilon^{-1} x_3 \underline{s}^b(\underline{x}) & \underline{s}^{RM} \\ \underline{s}^{RMT} & \underline{g}_3^{\varepsilon,0}(\underline{x}) + \varepsilon^{-1} x_3 \underline{g}_3^{\varepsilon,\varepsilon}(\underline{x}) \end{pmatrix} \right\},$$

where $\underline{\eta}$, $\underline{\phi}$, ω , \underline{s}^m and \underline{s}^b are functions defined on Ω .

Moreover, for the shear stress \underline{s}^{RM} there are two possibilities, since p^m and p^b can be linearly dependent. This occurs for instance, if ν is constant, and consequently $p^m = 0$. Hence, we assume that

$$(24) \quad \underline{s}^{RM}(\underline{x}) = \varepsilon^2 \underline{p}^l(\underline{x}) + \begin{cases} \underline{s}^m(\underline{x}) p^m(x_3) + \underline{s}^b(\underline{x}) p^b(x_3) & \text{if } p^m, p^b \text{ are lin. indep.,} \\ \underline{s}^b(\underline{x}) p^b(x_3) & \text{otherwise.} \end{cases}$$

3.3. Model derivation. Restricting the test and trial spaces as discussed in subsection 3.2, problem (19) reduces to find $(\underline{u}^{RM}, \underline{s}^{RM}) \in \hat{\underline{L}}^2(P^\varepsilon) \times \hat{\underline{S}}_{g^\varepsilon}$ solution of

$$(25) \quad \begin{cases} \int_{P^\varepsilon} \underline{ABs}^{RM} : \underline{Bt} \, d\underline{x} + \int_{P^\varepsilon} \underline{u}^{RM} \cdot \underline{\operatorname{div}} \underline{Bt} \, d\underline{x} = 0 & \text{for all } \underline{t} \in \hat{\underline{S}}_0, \\ - \int_{P^\varepsilon} \underline{\operatorname{div}} \underline{Bs}^{RM} \cdot \underline{v} \, d\underline{x} = \int_{P^\varepsilon} \underline{f}^\varepsilon \cdot \underline{v} \, d\underline{x} & \text{for all } \underline{v} \in \hat{\underline{L}}^2(P^\varepsilon). \end{cases}$$

Due to the structure of $\hat{\underline{L}}^2(P^\varepsilon)$ and $\hat{\underline{S}}_{g^\varepsilon}$, the above system (25) can be seen as a system whose unknowns are the following functions, all defined on the mid-plane Ω :

$$\begin{cases} \text{the unknowns } \underline{\eta}, \underline{\phi} \text{ and } \omega \text{ related to the kinematic fields;} \\ \text{the unknowns } \underline{s}^m \text{ and } \underline{s}^b \text{ related to in-plane stress;} \\ \text{the unknowns } \underline{s}^b \text{ and } \underline{s}^m \text{ related to shear stress.} \end{cases}$$

In particular, our approximation to $\underline{u}^\varepsilon$ is given by

$$(26) \quad \underline{u}^{RM} = \begin{pmatrix} \underline{\eta}(x) - x_3 \underline{\phi}(x) \\ \omega(x) \end{pmatrix}.$$

From the weak formulation (25) it is possible to show that $\underline{\eta}$, $\underline{\phi}$ and ω are uniquely determined by the following system of partial differential equations on Ω :

$$(27) \quad \boxed{\begin{cases} \varepsilon \underline{\operatorname{div}} \left[\underline{I}^0(\underline{A}^{-1}) \underline{e}(\underline{\eta}) \right] - \varepsilon^2 \underline{\operatorname{div}} \left[\underline{I}^1(\underline{A}^{-1}) \underline{e}(\underline{\phi}) \right] = \varepsilon^2 \underline{l}_0 + \varepsilon^4 \underline{l}_1 & \text{in } \Omega, \\ \varepsilon^2 \underline{\operatorname{div}} \left[\underline{I}^1(\underline{A}^{-1}) \underline{e}(\underline{\eta}) \right] - \varepsilon^3 \underline{\operatorname{div}} \left[\underline{I}^2(\underline{A}^{-1}) \underline{e}(\underline{\phi}) \right] + \varepsilon C^S (\underline{\phi} - \underline{\nabla} \omega) \\ \qquad \qquad \qquad = \varepsilon^3 \underline{I}^0(\underline{p}^l) - \varepsilon^3 \underline{l}_5 + \varepsilon^3 \underline{l}_2 + \varepsilon^5 \underline{l}_3 & \text{in } \Omega, \\ \varepsilon C^S \underline{\operatorname{div}}(\underline{\phi} - \underline{\nabla} \omega) = \varepsilon^3 \underline{\operatorname{div}} \underline{I}^0(\underline{p}^l) - \varepsilon^3 \underline{\operatorname{div}} \underline{l}_5 + \varepsilon^3 \underline{l}_4 & \text{in } \Omega, \\ \omega = 0, \quad \underline{\phi} = \underline{\eta} = 0 & \text{on } \partial\Omega, \end{cases}}$$

where

$$(28) \quad C^S = \begin{pmatrix} I^0(p^m) & I^0(p^b) \end{pmatrix} \underline{M} \begin{pmatrix} I^0(p^m) \\ I^0(p^b) \end{pmatrix}, \quad \underline{M} = \begin{cases} \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix}^{-1} & \text{if } d_{11}d_{22} - (d_{12})^2 \neq 0, \\ \frac{1}{d_{22}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \text{otherwise,} \end{cases}$$

$$d_{11} = I^0 \left(\frac{2(1+\nu)}{E} (p^m)^2 \right), \quad d_{12} = I^0 \left(\frac{2(1+\nu)}{E} p^m p^b \right), \quad d_{22} = I^0 \left(\frac{2(1+\nu)}{E} (p^b)^2 \right).$$

For the bending equation, we have

$$(34) \quad \left\{ \begin{array}{l} -\frac{2\varepsilon^3}{3} \operatorname{div} \underset{\cong}{A}^{-1} \underset{\cong}{e}(\phi) + \varepsilon \frac{5}{6} \frac{E}{1+\nu} (\phi - \nabla \omega) = -\varepsilon^3 \underset{\cong}{l}_5 - \varepsilon^2 \frac{2\nu}{3(1-\nu)} \nabla \underset{\cong}{g}_3^{\varepsilon, \varepsilon} \quad \text{in } \Omega, \\ \varepsilon \frac{5}{6} \frac{E}{1+\nu} \operatorname{div}(\phi - \nabla \omega) = F + 2g_3^{\varepsilon, \varepsilon} + \frac{\varepsilon}{3} \operatorname{div} g^{\varepsilon, 0} \quad \text{in } \Omega, \\ \omega = 0, \quad \phi = 0 \quad \text{on } \partial\Omega. \end{array} \right.$$

The loads are given by

$$(35) \quad \left\{ \begin{array}{l} \underset{\cong}{l}_5 = \frac{5}{4} \varepsilon^{-1} \underset{\cong}{I}^1(\underline{f}^\varepsilon) - \frac{5}{12} \varepsilon^{-1} \underset{\cong}{I}^3(\underline{f}^\varepsilon) + \frac{5}{3} \varepsilon^{-2} g^{\varepsilon, 0}, \\ F = -\frac{1}{4} \varepsilon^2 \operatorname{div} \underset{\cong}{I}^1(\underline{f}^\varepsilon) + \frac{5}{12} \varepsilon^2 \operatorname{div} \underset{\cong}{I}^3(\underline{f}^\varepsilon) + \varepsilon I^0(f_3^\varepsilon). \end{array} \right.$$

The problem defined by (34)–(35) differs from the bending problem found in [1]. Indeed, the right hand side of the (34) is new, and it incorporates higher order moments of \underline{f} . Notice also that we recover a model of Reissner–Mindlin type, with shear correction factor 5/6.

4. MODEL CONSISTENCY AND SOME CONVERGENCE RESULTS

4.1. Model consistency. We now study the consistency of our model, by performing an asymptotic expansion with respect to ε for both the 3D displacement solution and the model displacement solution. The key point for proving that (cf. (6))

$$(36) \quad \lim_{\varepsilon \rightarrow 0} \frac{\|\underline{u}^\varepsilon - \underline{u}^{RM}\|_{H^1(P^\varepsilon)}}{\|\underline{u}^\varepsilon\|_{H^1(P^\varepsilon)}} = 0$$

is to recognize that the leading terms of both the expansions coincide. More precisely, as shown in Appendix B, the 3D displacement solution $\underline{u}^\varepsilon(\underline{x})$ admits the following expansion

$$(37) \quad \underline{u}^\varepsilon(\underline{x}) = \underline{u}_L(\underline{x}) + \text{higher order terms}$$

where

$$(38) \quad \underline{u}_L(\underline{x}) = \begin{pmatrix} \varepsilon \zeta^1(\underline{x}) - x_3 \nabla \zeta_3^0(\underline{x}) \\ \zeta_3^0(\underline{x}) \end{pmatrix}.$$

On the other hand (cf. Appendix C), the model displacement solution $\underline{u}^{RM}(\underline{x})$ can be written as

$$(39) \quad \underline{u}^{RM}(\underline{x}) = \underline{u}_L^{RM}(\underline{x}) + \text{higher order terms}$$

where

$$(40) \quad \underline{u}_L^{RM}(\underline{x}) = \begin{pmatrix} \varepsilon \eta^1(\underline{x}) - x_3 \phi^0(\underline{x}) \\ \omega^0(\underline{x}) \end{pmatrix}.$$

We have the following result.

Lemma 1. For the leading terms \underline{u}_L and \underline{u}_L^{RM} defined by (38) and (40), it holds

$$\underline{u}_L = \underline{u}_L^{RM}.$$

Proof. From (60) of Appendix B, we have that the terms ζ^1 and ζ_3^0 involved in $\underline{u}_L(\underline{x})$ (cf. (38)) are uniquely determined by

$$(41) \quad \begin{cases} \operatorname{div} \left[I^0(A^{-1}) \underline{e}(\zeta^1) \right] - \operatorname{div} \left[I^1(A^{-1}) \underline{e}(\nabla \zeta_3^0) \right] = -\varepsilon^{-1} I^0(\underline{f}^\varepsilon) - 2\varepsilon^{-2} \underline{g}^{\varepsilon, \varepsilon} & \text{in } \Omega, \\ \operatorname{div} \operatorname{div} \left[I^1(A^{-1}) \underline{e}(\zeta^1) \right] - \operatorname{div} \operatorname{div} \left[I^2(A^{-1}) \underline{e}(\nabla \zeta_3^0) \right] = -\varepsilon^{-2} I^0(\underline{f}_3^\varepsilon) \\ \quad - \varepsilon^{-1} I^1(\operatorname{div} \underline{f}^\varepsilon) - 2\varepsilon^{-2} \operatorname{div} \underline{g}^{\varepsilon, \varepsilon} - 2\varepsilon^{-3} \underline{g}_3^{\varepsilon, \varepsilon} & \text{in } \Omega, \\ \zeta^1 = 0, \quad \zeta_3^0 = \frac{\partial \zeta_3^0}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

On the other hand, from (77) of Appendix C, we have that $\eta^1, \phi^0, \omega^0 \in H_0^1(\Omega)$ solve

$$(42) \quad \begin{cases} \operatorname{div} \left[I^0(A^{-1}) \underline{e}(\eta^1) \right] - \operatorname{div} \left[I^1(A^{-1}) \underline{e}(\phi^0) \right] = \underline{l}_0, \\ \operatorname{div} \operatorname{div} \left[I^1(A^{-1}) \underline{e}(\eta^1) \right] - \operatorname{div} \operatorname{div} \left[I^2(A^{-1}) \underline{e}(\nabla \omega^0) \right] = \operatorname{div} \underline{l}_2 - \underline{l}_4 \\ \phi^0 - \nabla \omega^0 = 0. \end{cases}$$

Hence, $\phi^0 = \nabla \omega^0$ and $\partial \omega^0 / \partial n = 0$, so that η^1 and ω^0 are determined by

$$(43) \quad \begin{cases} \operatorname{div} \left[I^0(A^{-1}) \underline{e}(\eta^1) \right] - \operatorname{div} \left[I^1(A^{-1}) \underline{e}(\nabla \omega^0) \right] = \underline{l}_0, \\ \operatorname{div} \operatorname{div} \left[I^1(A^{-1}) \underline{e}(\eta^1) \right] - \operatorname{div} \operatorname{div} \left[I^2(A^{-1}) \underline{e}(\nabla \omega^0) \right] = \operatorname{div} \underline{l}_2 - \underline{l}_4 \\ \eta^1 = 0, \quad \omega^0 = \frac{\partial \omega^0}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Recalling (29), we see that system (41) is the same as (43). It follows that $\zeta^1 = \eta^1$ and $\zeta_3^0 = \omega^0$, so that $\underline{u}_L = \underline{u}_L^{RM}$. \square

In the result below, we prove the consistency of our model. To be able to obtain relative convergence estimates, we shall assume that either $\zeta_3^0 \neq 0$ or $\zeta^1 \neq 0$. We shall actually assume this hypothesis throughout the paper.

Theorem 2. *Let $\underline{u}^\varepsilon$ and \underline{u}^{RM} be defined by (2), and (26)–(29). Then there exists a constant $C = C(\Omega, \underline{f}, \underline{g})$ independent of ε , such that*

$$(44) \quad \frac{\|\underline{u}^\varepsilon - \underline{u}^{RM}\|_{H^1(P^\varepsilon)}}{\|\underline{u}^\varepsilon\|_{H^1(P^\varepsilon)}} \leq C\varepsilon.$$

Proof. In this proof, we consider only the case when the third component of \underline{u}_L , i.e. ζ_3^0 , is not identically zero. The case when $\zeta_3^0 = 0$ but $\zeta^1 \neq 0$ can be handled with the same technique. Since $\underline{u}_L = \underline{u}_L^{RM}$, from the triangle inequality it holds

$$(45) \quad \frac{\|\underline{u}^\varepsilon - \underline{u}^{RM}\|_{H^1(P^\varepsilon)}}{\|\underline{u}^\varepsilon\|_{H^1(P^\varepsilon)}} \leq \frac{\|\underline{u}^\varepsilon - \underline{u}_L\|_{H^1(P^\varepsilon)} + \|\underline{u}_L^{RM} - \underline{u}^{RM}\|_{H^1(P^\varepsilon)}}{\|\underline{u}^\varepsilon\|_{H^1(P^\varepsilon)}}.$$

From Theorem 7 of Appendix B it follows

$$(46) \quad \|\underline{u}^\varepsilon - \underline{u}_L\|_{H^1(P^\varepsilon)} \leq c\varepsilon^{3/2}.$$

On the other hand, since

$$(47) \quad \underline{u}_L^{RM}(\underline{x}) - \underline{u}^{RM} = \begin{pmatrix} (\varepsilon \eta^1(\underline{x}) - \eta(\underline{x})) - x_3(\phi^0(\underline{x}) - \phi(\underline{x})) \\ \omega^0(\underline{x}) - \omega(\underline{x}), \end{pmatrix}$$

from Theorem 8 of Appendix C an integration along x_3 leads to

$$(48) \quad \|\underline{u}_L^{RM} - \underline{u}^{RM}\|_{H^1(P^\varepsilon)} \leq c\varepsilon^{3/2}.$$

Collecting (46) and (48) we obtain

$$(49) \quad \|\underline{u}^\varepsilon - \underline{u}_L\|_{H^1(P^\varepsilon)} + \|\underline{u}_L^{RM} - \underline{u}^{RM}\|_{H^1(P^\varepsilon)} \leq c\varepsilon^{3/2}.$$

Furthermore, a lower bound for $\underline{u}^\varepsilon$ easily follows since

$$(50) \quad \|\underline{u}^\varepsilon\|_{H^1(P^\varepsilon)} \geq \|u_3^\varepsilon\|_{H^1(P^\varepsilon)} \geq \|\zeta_3^0\|_{H^1(P^\varepsilon)} - \|u_3^\varepsilon - \zeta_3^0\|_{H^1(P^\varepsilon)} \geq c\varepsilon^{1/2},$$

for ε sufficiently small. Hence, from (45), (49) and (50) we have

$$(51) \quad \frac{\|\underline{u}^\varepsilon - \underline{u}^{RM}\|_{H^1(P^\varepsilon)}}{\|\underline{u}^\varepsilon\|_{H^1(P^\varepsilon)}} \leq C\varepsilon.$$

□

Remark. From the proof of Theorem 2, it is easily seen that we have indeed

$$\frac{\|\underline{u}^\varepsilon - \underline{u}^{RM}\|_{H^1(P^\varepsilon)}}{\|\underline{u}^\varepsilon\|_{H^1(P^\varepsilon)}} \leq C\varepsilon, \quad \frac{\|u_3^\varepsilon - u_3^{RM}\|_{H^1(P^\varepsilon)}}{\|u_3^\varepsilon\|_{H^1(P^\varepsilon)}} \leq C\varepsilon.$$

□

4.2. Other Convergence Results. In this subsection we collect some results concerning the modeling error for the stress field. Since the proofs are rather involved, we postpone them to appendix D, for the sake of readability. We begin by recalling that $\underline{\sigma}^\varepsilon$ is the 3D stress solution (cf. (2)), while $\underline{\sigma}^{RM}$ is the model stress solution recovered by (32).

The next Theorem gives an error estimate for the *planar* stress components.

Theorem 3. *There exists a constant $C = C(\Omega, \underline{f}, \underline{g})$ independent of ε , such that*

$$\frac{\|\underline{\sigma}^\varepsilon - \underline{\sigma}^{RM}\|_{L^2(P^\varepsilon)}}{\|\underline{\sigma}^\varepsilon\|_{L^2(P^\varepsilon)}} \leq C\varepsilon^{1/2}.$$

□

Regarding the convergence of the *shear* stress, we could obtain weaker results. In particular, due to boundary layer effects, only *interior* estimates have been developed. Nevertheless, the following theorem shows $O(\varepsilon)$ convergence for the *averaged* shear stress.

Theorem 4. *Let Ω_0 be a domain such that $\bar{\Omega}_0 \subset \Omega$. Then there exists a constant $C = C(\Omega, \Omega_0, \underline{f}, \underline{g})$ independent of ε , such that*

$$\frac{\|I^0(\underline{\sigma}^\varepsilon - \underline{\sigma}^{RM})\|_{L^2(\Omega_0)}}{\|I^0(\underline{\sigma}^\varepsilon)\|_{L^2(\Omega_0)}} \leq C\varepsilon.$$

□

Theorem 4 can be used to obtain interior convergence estimates for the shear stress (not only for its *average*) in two particular situations.

The first one concerns the case of symmetric plates under bending, as displayed by the following corollary.

Corollary 5. Assume that ν and E are even functions. Assume also that the loads induce a pure bending state, i.e.

- f^ε is an odd function, and f_3^ε is an even function, with respect to x_3 ;
- $\underline{g}^\varepsilon = (g^{\varepsilon,o}, g_3^{\varepsilon,e})$.

Let Ω_0 be a domain such that $\bar{\Omega}_0 \subset \Omega$, and set $P_0^\varepsilon = \Omega_0 \times (-\varepsilon, \varepsilon)$. Then there exists a constant $C = C(\Omega, \Omega_0, \underline{f}, \underline{g})$ independent of ε , such that

$$\frac{\|\underline{\mathcal{Q}}^\varepsilon - \underline{\mathcal{Q}}^{RM}\|_{L^2(P_0^\varepsilon)}}{\|\underline{\mathcal{Q}}^\varepsilon\|_{L^2(P_0^\varepsilon)}} \leq C\varepsilon.$$

□

The second case is when the polynomials p^m and p^b are linearly dependent functions. For instance, this occurs whenever ν is constant. Indeed, we have the following result.

Corollary 6. Assume that p^m and p^b are linearly dependent functions. Let Ω_0 be a domain such that $\bar{\Omega}_0 \subset \Omega$, and set $P_0^\varepsilon = \Omega_0 \times (-\varepsilon, \varepsilon)$. Then there exists a constant $C = C(\Omega, \Omega_0, \underline{f}, \underline{g})$ independent of ε , such that

$$\frac{\|\underline{\mathcal{Q}}^\varepsilon - \underline{\mathcal{Q}}^{RM}\|_{L^2(P_0^\varepsilon)}}{\|\underline{\mathcal{Q}}^\varepsilon\|_{L^2(P_0^\varepsilon)}} \leq C\varepsilon.$$

□

REFERENCES

- [1] S.M. Alessandrini, D.N. Arnold, R.S. Falk and A.L. Madureira, Derivation and Justification of Plate Models by Variational Methods, *Centre de Recherches Mathematiques, CRM Proceedings and Lecture Notes*, 1999.
- [2] D.N. Arnold, A.L. Madureira and S. Zhang, On the range of applicability of the Reissner–Mindlin and Kirchhoff–Love plate bending models, *Journal of Elasticity*, to appear.
- [3] D.N. Arnold and R.S. Falk, Asymptotic analysis of the boundary layer for the Reissner Mindlin plate model, *SIAM J. Math. Anal.*, 27:486-514, 1999.
- [4] F. Auricchio and E. Sacco, Partial-mixed formulation and refined models for the analysis of composite laminates within FSDT *Composite Structures*, 46:103–113, 1999.
- [5] D. Caillerie, Thin Elastic and Periodic Plates, *Math. Meth. in the Appl. Sci.*, 6:159-191, 1984.
- [6] P.G. Ciarlet, *Mathematical Elasticity, volume II: Theory of Plates*, Studies in Mathematics and its Applications, 27. North-Holland Publishing Co., Amsterdam, 1997.
- [7] P.G. Ciarlet and Ph. Destuynder, A justification of the two dimensional linear plate model. *Journal de Mécanique* 18, No.2 (1979), 315–344.
- [8] M. Dauge and I. Gruais, Asymptotics of arbitrary order for a thin elastic clamped plate, I. Optimal error estimates, *Asymptotic Anal.*, 13:167-197, 1996.
- [9] P. Destuynder, *Sur une Justification des Modeles de Plaques et de Coques par les Methodes Asymptotiques*. Ph.D. thesis, Université Pierre et Marie Curie - Paris, 1980.
- [10] R. Durán and E. Liberman, On mixed finite element methods for the Reissner–Mindlin Plate Model, *Mathematics of Computation*, 198:561–573, 1992.

- [11] K.H. Lo, R.M. Christensen and E.M. Wu, A high-Order Theory of Plate Deformation, *J. Appl. Mech.*, 46:663–676, 1977.
- [12] J.C. Paumier and A. Raoult, Asymptotic consistency of the polynomial approximation in the linearized plate theory application to the Reissner–Mindlin model, *Sonderforschungsbereich*, 164, 1996 (preprint).
- [13] J. Sanchez Hubert and E. Sanchez Palencia, *Introduction aux méthodes asymptotiques et à l’homogénéisation*. Masson, Paris, 1992.
- [14] O.V. Motygin and S.A. Nazarov, Justification of the Kirchhoff hypotheses and error estimation for two-dimensional models of anisotropic and inhomogeneous plates, including laminated plates *IMA Journal of Applied Mathematics*, 65:1–28, 2000

APPENDIX A

In this appendix we present the arguments necessary to obtain the model described in Subsection 3.3. Throughout this derivation, we assume that (22), (23) and (24) hold. Also, we consider that p^m and p^b are linearly independent. When these polynomials are linearly dependent, the computations are analogous and simpler. During the computations, we need that

$$(52) \quad \underline{\underline{A}}\underline{\underline{\tau}} = \begin{pmatrix} \underline{\underline{A}}\underline{\underline{\tau}} - \frac{\nu}{E}\tau_{33}\underline{\underline{\delta}} & \frac{1+\nu}{E}\underline{\underline{\tau}} \\ \frac{1+\nu}{E}\underline{\underline{\tau}}^T & \frac{\tau_{33}}{E} - \frac{\nu}{E}\text{tr}(\underline{\underline{\tau}}) \end{pmatrix}, \quad \text{and} \quad \underline{\underline{A}}\underline{\underline{B}}\underline{\underline{s}} = \begin{pmatrix} \underline{\underline{s}} - \frac{\nu}{E}s_{33}\underline{\underline{\delta}} & \frac{1+\nu}{E}\underline{\underline{s}} \\ \frac{1+\nu}{E}\underline{\underline{s}}^T & \frac{s_{33}}{E} - \frac{\nu}{E}\text{tr}(\underline{\underline{A}}^{-1}\underline{\underline{s}}) \end{pmatrix}.$$

The equations (30), together with the boundary conditions for $\underline{\underline{\eta}}$ and $\underline{\underline{\phi}}$ in (27), are obtained from the first equation of (25), by considering

$$\underline{\underline{t}}(\underline{\underline{x}}) = \begin{pmatrix} \underline{\underline{t}}^m(\underline{\underline{x}}) + x_3\underline{\underline{t}}^b(\underline{\underline{x}}) & 0 \\ 0 & 0 \end{pmatrix},$$

where $\underline{\underline{t}}^m, \underline{\underline{t}}^b \in H^1(\Omega)$ are arbitrary.

By considering next

$$(53) \quad \underline{\underline{t}}(\underline{\underline{x}}) = \begin{pmatrix} 0 & \underline{\underline{t}}^m(\underline{\underline{x}})p^m(x_3) \\ [\underline{\underline{t}}^m(\underline{\underline{x}})p^m(x_3)]^T & 0 \end{pmatrix}, \quad \underline{\underline{t}}(\underline{\underline{x}}) = \begin{pmatrix} 0 & \underline{\underline{t}}^b(\underline{\underline{x}})p^b(x_3) \\ [\underline{\underline{t}}^b(\underline{\underline{x}})p^b(x_3)]^T & 0 \end{pmatrix},$$

in the first equation of (25), we conclude that

$$\begin{cases} \int_{-\varepsilon}^{\varepsilon} \frac{2(1+\nu)}{E}(\underline{\underline{s}}^m p^m + \underline{\underline{s}}^b p^b)p^m dx_3 = \int_{-\varepsilon}^{\varepsilon} (-\underline{\underline{\phi}} + \underline{\underline{\nabla}}\omega)p^m dx_3 - \varepsilon^2 \int_{-\varepsilon}^{\varepsilon} \frac{2(1+\nu)}{E}p^l p^m dx_3, \\ \int_{-\varepsilon}^{\varepsilon} \frac{2(1+\nu)}{E}(\underline{\underline{s}}^m p^m + \underline{\underline{s}}^b p^b)p^b dx_3 = \int_{-\varepsilon}^{\varepsilon} (-\underline{\underline{\phi}} + \underline{\underline{\nabla}}\omega)p^b dx_3 - \varepsilon^2 \int_{-\varepsilon}^{\varepsilon} \frac{2(1+\nu)}{E}p^l p^b dx_3. \end{cases}$$

Equation (31) expresses the solution of the above algebraic system. It follows also that $\omega = 0$ on $\partial\Omega$.

Next, we consider the second equation of (25). By taking variations of test functions of the form $\underline{\underline{v}}(\underline{\underline{x}}) = (\underline{\underline{v}}(\underline{\underline{x}}), 0)$, we find

$$\underline{\underline{\text{div}}} \int_{-\varepsilon}^{\varepsilon} \underline{\underline{A}}^{-1}(\underline{\underline{s}}^m + \varepsilon^{-1}x_3\underline{\underline{s}}^b) dx_3 = - \int_{-\varepsilon}^{\varepsilon} \underline{\underline{f}}^\varepsilon dx_3 - 2\underline{\underline{q}}^{\varepsilon,\varepsilon}.$$

Using the first equation of (30), we obtain the first equation of (27). By considering $\underline{v}(\underline{x}) = (x_3 \underline{v}(\underline{x}), 0)$ we find

$$(54) \quad \operatorname{div} \int_{-\varepsilon}^{\varepsilon} x_3 \underline{A}^{-1}(\underline{s}^m + \varepsilon^{-1} x_3 \underline{s}^b) dx_3 - \int_{-\varepsilon}^{\varepsilon} (\underline{s}^m p^m + \underline{s}^b p^b) dx_3 = - \int_{-\varepsilon}^{\varepsilon} x_3 \underline{f}^\varepsilon dx_3 - \varepsilon^2 \int_{-\varepsilon}^{\varepsilon} x_3 \frac{\partial \underline{p}^l}{\partial x_3} dx_3,$$

where we integrated by parts the second integral above. Using (31) we obtain that

$$(55) \quad \underline{s}^m I^0(p^m) + \underline{s}^b I^0(p^b) = C^S(-\underline{\phi} + \nabla \omega) - \varepsilon^2 \underline{l}_5.$$

Therefore, the second equation of (27) follows from (54), the second equation of (30) and from (55).

To obtain the final equilibrium equation, we consider $\underline{v}(\underline{x}) = (0, 0, v_3(\underline{x}))$. Hence, it follows that

$$\operatorname{div} \int_{-\varepsilon}^{\varepsilon} (\underline{s}^m p^m + \underline{s}^b p^b) dx_3 = -\varepsilon^3 l_4 - \varepsilon^2 \operatorname{div} \int_{-\varepsilon}^{\varepsilon} \underline{p}^l dx_3.$$

Using then (55), we obtain the third equation of (27).

APPENDIX B

As we mention in the Introduction, to find out the suitable profiles for the shear stress, and also to derive error estimates, we look at the asymptotic expansion for the exact solution $\underline{u}^\varepsilon$. We do not give here a complete description of the expansion, but we only mention the basic ideas and some final results. This expansion for heterogeneous plates generalize the work of Dauge and Gruais [8]. See also [14], where the asymptotic for a general elasticity problem is investigated. The first step to obtain an asymptotic expansion for $\underline{u}^\varepsilon$ is, as usual, to introduce the change of variable $(\underline{u}, x_3) \rightarrow (\underline{u}, \hat{x}_3) = (\underline{u}, \varepsilon^{-1} x_3)$, which maps the domain $P^\varepsilon = \Omega \times (-\varepsilon, \varepsilon)$ onto $\Omega \times (-1, 1)$. Thus, the 3D elasticity problem

$$(56) \quad \begin{cases} -\operatorname{div} \underline{A}^{-1} \underline{e}(\underline{u}^\varepsilon) = \underline{f}^\varepsilon & \text{in } P^\varepsilon, \\ \underline{u}^\varepsilon = 0 & \text{on } \partial P_L^\varepsilon, \\ \underline{A}^{-1} \underline{e}(\underline{u}^\varepsilon) \underline{n} = \underline{g}^\varepsilon & \text{on } \partial P_\pm^\varepsilon, \end{cases}$$

can be accordingly transformed into a problem (depending on the parameter ε) defined on the fixed domain $\Omega \times (-1, 1)$. By assuming that

- there exist ε -independent functions $\underline{f} : \Omega \times (-1, 1) \rightarrow \mathbb{R}^3$, and $\underline{g} : \Omega \times \{-1, 1\} \rightarrow \mathbb{R}^3$ such that

$$(57) \quad \begin{cases} \underline{f}(\underline{x}, \hat{x}_3) = \varepsilon^{-1} \underline{f}^\varepsilon(\underline{x}, \varepsilon \hat{x}_3), & f_3(\underline{x}, \hat{x}_3) = \varepsilon^{-2} f_3^\varepsilon(\underline{x}, \varepsilon \hat{x}_3), \\ \underline{g}(\underline{x}, \hat{x}_3) = \varepsilon^{-2} \underline{g}^\varepsilon(\underline{x}, \varepsilon \hat{x}_3), & g_3(\underline{x}, \hat{x}_3) = \varepsilon^{-3} g_3^\varepsilon(\underline{x}, \varepsilon \hat{x}_3), \end{cases}$$

for $\hat{x}_3 \in (-1, 1)$;

- the functions

$$\hat{\nu}(\hat{x}_3) = \nu(\varepsilon \hat{x}_3), \quad \hat{E}(\hat{x}_3) = E(\varepsilon \hat{x}_3) \quad \text{for } \hat{x}_3 \in (-1, 1),$$

are independent of ε ;

an asymptotic expansion for $\underline{u}^\varepsilon$ reads as follows

$$(58) \quad \underline{u}^\varepsilon(\underline{x}) \sim \begin{pmatrix} \varepsilon \zeta^1(\underline{x}) - x_3 \nabla \zeta_3^0(\underline{x}) \\ \zeta_3^0(\underline{x}) \end{pmatrix} + \varepsilon \begin{pmatrix} \varepsilon \zeta^2(\underline{x}) - x_3 \nabla \zeta_3^1(\underline{x}) \\ \zeta_3^1(\underline{x}) \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 0 \\ \dot{u}_3^2(\underline{x}, \varepsilon^{-1}x_3) \end{pmatrix} \\ + \varepsilon^2 \underline{w}^2(\underline{x}, \varepsilon^{-1}\rho, \varepsilon^{-1}x_3) + \varepsilon^2 \begin{pmatrix} \varepsilon \zeta^3(\underline{x}) - x_3 \nabla \zeta_3^2(\underline{x}) \\ \zeta_3^2(\underline{x}) \end{pmatrix} + \varepsilon^3 \dot{u}_3^3(\underline{x}, \varepsilon^{-1}x_3) + \dots$$

Above, $\zeta^k : \Omega \rightarrow \mathbb{R}^2$, $\zeta_3^k : \Omega \rightarrow \mathbb{R}$ and $\dot{u}^k : \Omega \times (-1, 1) \rightarrow \mathbb{R}^3$ are functions independent of ε . Moreover, the boundary correctors \underline{w}^k , needed to capture the boundary layers of the 3D solution, are functions which decay exponentially to zero with $\varepsilon^{-1}\rho$, where ρ is the distance of a point $\underline{x} \in \Omega$ from the boundary $\partial\Omega$. Inserting the formal expansion (58) into (56), one recognizes that the functions ζ^{k+1} and ζ_3^k can be determined by solving 2D partial differential equations on Ω . In particular, the functions ζ^1 and ζ_3^0 entering in the leading term

$$(59) \quad \underline{u}_L(\underline{x}) = \begin{pmatrix} \varepsilon \zeta^1(\underline{x}) - x_3 \nabla \zeta_3^0(\underline{x}) \\ \zeta_3^0(\underline{x}) \end{pmatrix}$$

of the expansion (58) solve the problem

$$(60) \quad \begin{cases} \varepsilon \operatorname{div} \left[\underset{\approx}{I}^0(A^{-1}) \underset{\approx}{e}(\zeta^1) \right] - \varepsilon \operatorname{div} \left[\underset{\approx}{I}^1(A^{-1}) \underset{\approx}{e}(\nabla \zeta_3^0) \right] = -\underset{\approx}{I}^0(f^\varepsilon) - 2\varepsilon^{-1} \underset{\approx}{g}^{\varepsilon, \varepsilon} & \text{in } \Omega, \\ \varepsilon \operatorname{div} \operatorname{div} \left[\underset{\approx}{I}^1(A^{-1}) \underset{\approx}{e}(\zeta^1) \right] - \varepsilon \operatorname{div} \operatorname{div} \left[\underset{\approx}{I}^2(A^{-1}) \underset{\approx}{e}(\nabla \zeta_3^0) \right] = -\varepsilon^{-1} \underset{\approx}{I}^0(f_3^\varepsilon) \\ \quad - \underset{\approx}{I}^1(\operatorname{div} f^\varepsilon) - 2\varepsilon^{-1} \operatorname{div} \underset{\approx}{g}^{\varepsilon, \varepsilon} - 2\varepsilon^{-2} \underset{\approx}{g}_3^{\varepsilon, \varepsilon} & \text{in } \Omega, \\ \zeta^1 = 0, \quad \zeta_3^0 = \frac{\partial \zeta_3^0}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore the functions \dot{u}^k solve the following Neumann problem in each vertical fiber

$$(61) \quad \int_{-1}^1 \dot{u}^k(\underline{x}, \hat{x}_3) d\hat{x}_3 = 0.$$

The asymptotic expansion for the stress comes from formal substitution of (58) in the constitutive equation $\underline{\underline{\sigma}}^\varepsilon = \underset{\approx}{A}^{-1} \underset{\approx}{e}(\underline{u}^\varepsilon)$. Hence

$$\underline{\underline{\sigma}}^\varepsilon(\underline{x}) \sim \varepsilon \begin{pmatrix} \underset{\approx}{\sigma}^1 & 0 \\ 0 & 0 \end{pmatrix}(\underline{x}, \varepsilon^{-1}x_3) + \varepsilon \underset{\approx}{\Xi}^1(\underline{x}, \varepsilon^{-1}\rho, \varepsilon^{-1}x_3) + \varepsilon^2 \begin{pmatrix} \underset{\approx}{\sigma}^2 & \underset{\approx}{\sigma}^2 \\ \underset{\approx}{\sigma}^{2T} & 0 \end{pmatrix}(\underline{x}, \varepsilon^{-1}x_3) \\ + \varepsilon^2 \underset{\approx}{\Xi}^2(\underline{x}, \varepsilon^{-1}\rho, \varepsilon^{-1}x_3) + \varepsilon^3 \begin{pmatrix} \underset{\approx}{\sigma}^3 & \underset{\approx}{\sigma}^3 \\ \underset{\approx}{\sigma}^{3T} & \underset{\approx}{\sigma}_{33}^3 \end{pmatrix}(\underline{x}, \varepsilon^{-1}x_3) + \dots,$$

where $\underline{\Xi}^k$ correspond to boundary layer terms, and $\underline{\sigma}^k$ are defined in $\Omega \times (-1, 1)$. It can be shown that the components of $\underline{\sigma}^k$ are determined by

$$(62) \quad \begin{cases} \underline{\sigma}^k(\underline{x}, \hat{x}_3) = A^{-1} \underset{\cong}{e}(\hat{u}^k) + A^{-1} \underset{\cong}{e}(\zeta^k - \hat{x}_3 \nabla \zeta_3^{k-1}) + \frac{\nu}{1-\nu} \sigma_{33}^k, \\ \underline{\sigma}^k(\underline{x}, \hat{x}_3) = \int_{-1}^{\hat{x}_3} \left(-\operatorname{div} \underline{\sigma}^{k-1} - \delta_{k,2} \underline{f} \right) d\xi - \delta_{k,2} g(\underline{x}, -1), \\ \sigma_{33}^k(\underline{x}, \hat{x}_3) = \int_{-1}^{\hat{x}_3} \left(-\operatorname{div} \sigma^{k-1} - \delta_{k,3} f_3 \right) d\xi - \delta_{k,3} g_3(\underline{x}, -1), \end{cases}$$

where $\delta_{k,l}$ denotes the Kronecker symbol. In particular, from (62) it follows

$$(63) \quad \begin{cases} \underline{\sigma}^1(\underline{x}, \hat{x}_3) = A^{-1} \underset{\cong}{e}(\zeta^1 - \hat{x}_3 \nabla \zeta_3^0), \\ \underline{\sigma}^2(\underline{x}, \hat{x}_3) = \int_{-1}^{\hat{x}_3} \left(-\operatorname{div} A^{-1} \underset{\cong}{e}(\zeta^1 - \xi \nabla \zeta_3^0) - \underline{f} \right) d\xi - g(\underline{x}, -1), \\ \sigma_{33}^3(\underline{x}, \hat{x}_3) = \int_{-1}^{\hat{x}_3} \left(-\operatorname{div} \underline{\sigma}^2 - f_3 \right) d\xi - g_3(\underline{x}, -1). \end{cases}$$

Although the first term of the asymptotic expansion of $\underline{\sigma}^\varepsilon$ is given by (63), the first equation of (60) can be used to provide an alternative expression of $\underline{\sigma}^2$. More precisely, after an easy but cumbersome calculation, we find that

$$(64) \quad \underline{\sigma}^2(\underline{x}, \hat{x}_3) = \nabla \operatorname{div} \zeta^1(\underline{x}) p^m(\varepsilon \hat{x}_3) + \nabla \Delta \zeta_3^0(\underline{x}) p^b(\varepsilon \hat{x}_3) + p^l(\underline{x}, \varepsilon \hat{x}_3),$$

where (cf. (21))

$$\begin{cases} p^m(x_3) = \frac{I^0\left(\frac{E\nu}{1-\nu^2}\right)}{I^0\left(\frac{E}{1+\nu}\right)} \varepsilon^{-1} \int_{-\varepsilon}^{x_3} \frac{E}{1+\nu} d\xi - \varepsilon^{-1} \int_{-\varepsilon}^{x_3} \frac{E\nu}{1-\nu^2} d\xi, \\ p^b(x_3) = \frac{I^1\left(\frac{E}{1-\nu^2}\right)}{I^0\left(\frac{E}{1+\nu}\right)} \varepsilon^{-1} \int_{-\varepsilon}^{x_3} \frac{E}{1+\nu} d\xi - \varepsilon^{-2} \int_{-\varepsilon}^{x_3} \frac{E}{1-\nu^2} \xi d\xi, \\ p^l(\underline{x}, x_3) = \frac{\varepsilon^{-3}}{I^0\left(\frac{E}{1+\nu}\right)} \left(\int_{-\varepsilon}^{\varepsilon} \underline{f}^\varepsilon d\xi + 2g^{\varepsilon,\varepsilon} \right) \int_{-\varepsilon}^{x_3} \frac{E}{1+\nu} d\xi - \varepsilon^{-2} \int_{-\varepsilon}^{x_3} \underline{f}^\varepsilon d\xi - \varepsilon^{-2} g^\varepsilon(\underline{x}, -\varepsilon). \end{cases}$$

Even though the expansion (58) is formal, it is possible to derive asymptotic error estimates using the technique of [8]. Considering the error function (cf. also (59))

$$(65) \quad \begin{aligned} \underline{u}^{\text{er}}(\underline{x}) &= \underline{u}^\varepsilon(\underline{x}) - \begin{pmatrix} \varepsilon \zeta^1(\underline{x}) - x_3 \nabla \zeta_3^0(\underline{x}) \\ \zeta_3^0(\underline{x}) \end{pmatrix} = \underline{u}^\varepsilon(\underline{x}) - \underline{u}_L(\underline{x}), \\ \underline{\sigma}^{\text{er}}(\underline{x}) &= \underline{\sigma}^\varepsilon(\underline{x}) - \begin{pmatrix} \varepsilon \underline{\sigma}^1 & \varepsilon^2 \underline{\sigma}^2 \\ \varepsilon^2 \underline{\sigma}^{2T} & \varepsilon^3 \sigma_{33}^3 \end{pmatrix}(\underline{x}, \varepsilon^{-1} x_3), \end{aligned}$$

one has the following result.

Theorem 7. *Given \underline{u}^{er} and $\underline{\sigma}^{er}$ as in (65), there exists a constant $C = C(\Omega, f, g)$ independent of ε , such that*

$$(66) \quad \|\underline{u}^{er}\|_{H^1(P^\varepsilon)} \leq C\varepsilon^2, \quad \|u_3^{er}\|_{H^1(P^\varepsilon)} \leq C\varepsilon^{3/2},$$

$$(67) \quad \|\underline{\sigma}^{er}\|_{L^2(P^\varepsilon)} \leq C\varepsilon^2, \quad \|\underline{\sigma}^{er}\|_{L^2(P^\varepsilon)} \leq C\varepsilon^2, \quad \|\sigma_{33}^{er}\|_{L^2(P^\varepsilon)} \leq C\varepsilon^2.$$

Moreover, let Ω_0 be a domain such that $\bar{\Omega}_0 \subset \Omega$, and let $P_0^\varepsilon = \Omega_0 \times (-\varepsilon, \varepsilon)$. Then there exists a constant $C = C(\Omega, \Omega_0, f, g)$ independent of ε , such that

$$(68) \quad \|\underline{u}^{er}\|_{H^1(P_0^\varepsilon)} \leq C\varepsilon^{5/2},$$

$$(69) \quad \|\underline{\sigma}^{er}\|_{L^2(P_0^\varepsilon)} \leq C\varepsilon^{7/2}, \quad \|\sigma_{33}^{er}\|_{L^2(P_0^\varepsilon)} \leq C\varepsilon^{9/2}.$$

□

APPENDIX C

In this Appendix we briefly report some results about the asymptotic expansion for the solution of our Reissner–Mindlin model (cf. (27)–(29)). We closely follow the arguments of Arnold and Falk [3], in which a detailed asymptotic analysis is provided for the bending problem of an isotropic and homogeneous plate.

The first step is to apply the divergence operator to the second equation of (27) and subtract the third equation of (27) from the result, to obtain

$$(70) \quad \Delta \operatorname{div} \phi = \frac{1}{I^2(D)} [\varepsilon^{-1} I^1(D) \Delta \operatorname{div} \eta - \operatorname{div} l_2 + l_4 - \varepsilon^2 \operatorname{div} l_3],$$

where $D = E/(1 - \nu^2)$. Applying the Laplace operator to the third equation of (27) and using (70), we get

$$(71) \quad \varepsilon I^2(D) \Delta^2 \omega - I^1(D) \Delta \operatorname{div} \eta = \varepsilon(l_4 - \operatorname{div} l_2) - \varepsilon^3 \operatorname{div} l_3 \\ + \frac{I^2(D)}{CS} \varepsilon^3 [-\Delta \operatorname{div} I^0(p^l) + \Delta \operatorname{div} l_5 + \Delta l_4].$$

Above, the function p^l , the constant C^S and the l_i 's are defined by (21), (28) and (29). Next, we assume the following formal expansions

$$(72) \quad \begin{cases} \omega(\underline{x}) \sim \omega^0(\underline{x}) + \varepsilon \omega^1(\underline{x}) + \varepsilon^2 \omega^2(\underline{x}) + \dots, \\ \eta(\underline{x}) \sim \varepsilon \eta^1(\underline{x}) + \varepsilon^2 \eta^2(\underline{x}) + \dots, \\ \phi(\underline{x}) \sim \phi^0(\underline{x}) + \varepsilon \phi^1(\underline{x}) + \varepsilon^2 \phi^2(\underline{x}) + \varepsilon^2 \Phi^2(\underline{x}, \varepsilon^{-1} \rho) + \varepsilon^3 \phi^3(\underline{x}) + \varepsilon^3 \Phi^3(\underline{x}, \varepsilon^{-1} \rho) + \dots, \end{cases}$$

where Φ^i are boundary corrector terms. Note that there is no boundary layer for ω or η .

We also remark that, due to the choice (22), the expansion (72) implies that for the model displacement solution $\underline{u}^{RM}(\underline{x})$ we are supposing that

$$(73) \quad \underline{u}^{RM}(\underline{x}) = \underline{u}_L^{RM}(\underline{x}) + \text{higher order terms}$$

where

$$(74) \quad \underline{u}_L^{RM}(\underline{x}) = \begin{pmatrix} \varepsilon \eta^1(\underline{x}) - x_3 \phi^0(\underline{x}) \\ \omega^0(\underline{x}) \end{pmatrix}.$$

Disregarding the boundary correctors for ϕ , formally substituting the above expansions in (27) and matching the terms with the same power of ε , we find that ω^i , η^i and ϕ^i solve

$$(75) \quad \left\{ \begin{array}{l} \operatorname{div} \left[I^0(A^{-1}) \underline{\underline{e}}(\underline{\underline{\eta}}^{i+1}) \right] - \operatorname{div} \left[I^1(A^{-1}) \underline{\underline{e}}(\underline{\underline{\phi}}^i) \right] = \delta_{i,0} \underline{\underline{l}}_0 + \delta_{i,2} \underline{\underline{l}}_1, \\ I^1(D) \Delta \operatorname{div} \underline{\underline{\eta}}^{i+1} - I^2(D) \Delta^2 \omega^i = \delta_{i,0} (\operatorname{div} \underline{\underline{l}}_2 - \underline{\underline{l}}_4) + \delta_{i,2} \operatorname{div} \underline{\underline{l}}_3 \\ \quad - \delta_{i,2} \frac{I^2(D)}{C^S} \left[-\Delta \operatorname{div} I^0(\underline{\underline{p}}^l) + \Delta \operatorname{div} \underline{\underline{l}}_5 + \Delta \underline{\underline{l}}_4 \right], \\ C^S(\underline{\underline{\phi}}^i - \nabla \omega^i) = -\operatorname{div} \left[I^1(A^{-1}) \underline{\underline{e}}(\underline{\underline{\eta}}^{i-1}) \right] + \operatorname{div} \left[I^2(A^{-1}) \underline{\underline{e}}(\underline{\underline{\phi}}^{i-2}) \right] \\ \quad + \delta_{i,2} \left[I^0(\underline{\underline{p}}^l) - \underline{\underline{l}}_5 + \underline{\underline{l}}_2 \right] + \delta_{i,4} \underline{\underline{l}}_3. \end{array} \right.$$

In particular, by taking $i = 0$ in (75), we have that η^1 , ϕ^0 , and ω^0 satisfy

$$(76) \quad \left\{ \begin{array}{l} \operatorname{div} \left[I^0(A^{-1}) \underline{\underline{e}}(\underline{\underline{\eta}}^1) \right] - \operatorname{div} \left[I^1(A^{-1}) \underline{\underline{e}}(\underline{\underline{\phi}}^0) \right] = \underline{\underline{l}}_0, \\ I^1(D) \Delta \operatorname{div} \underline{\underline{\eta}}^1 - I^2(D) \Delta^2 \omega^0 = \operatorname{div} \underline{\underline{l}}_2 - \underline{\underline{l}}_4 \\ \underline{\underline{\phi}}^0 - \nabla \omega^0 = 0 \end{array} \right.$$

Recalling that $D = E/(1 - \nu^2)$ and (cf. (9))

$$\underline{\underline{A}}^{-1} \underline{\underline{\tau}} \approx \frac{E}{1 + \nu} \underline{\underline{\tau}} + \frac{E\nu}{1 - \nu^2} \operatorname{tr}(\underline{\underline{\tau}}) \underline{\underline{\delta}} \quad \forall \underline{\underline{\tau}},$$

it is easily seen that (76) can be alternatively written as

$$(77) \quad \left\{ \begin{array}{l} \operatorname{div} \left[I^0(A^{-1}) \underline{\underline{e}}(\underline{\underline{\eta}}^1) \right] - \operatorname{div} \left[I^1(A^{-1}) \underline{\underline{e}}(\underline{\underline{\phi}}^0) \right] = \underline{\underline{l}}_0, \\ \operatorname{div} \operatorname{div} \left[I^1(A^{-1}) \underline{\underline{e}}(\underline{\underline{\eta}}^1) \right] - \operatorname{div} \operatorname{div} \left[I^2(A^{-1}) \underline{\underline{e}}(\nabla \omega^0) \right] = \operatorname{div} \underline{\underline{l}}_2 - \underline{\underline{l}}_4 \\ \underline{\underline{\phi}}^0 - \nabla \omega^0 = 0 \end{array} \right.$$

Using the techniques of [3], it is possible to estimate the difference between the leading terms in (72) and the exact Reissner-Mindlin solution, as stated in the theorem below.

Theorem 8. *Let η , ϕ and ω be the solution of (27). Assume that η^1 , ϕ^0 , and ω^0 solve (76). Then, for every positive integer k , there exists a constant $C = C(\Omega, \underline{\underline{f}}, \underline{\underline{g}}, k)$ independent of ε such that*

$$(78) \quad \varepsilon^{-1} \|\eta - \varepsilon \eta^1\|_{H^k(\Omega)} + \|\omega - \omega^0\|_{H^k(\Omega)} + \|\phi - \phi^0\|_{H^1(\Omega)} \leq C\varepsilon.$$

APPENDIX D

In this Appendix we present proofs of the convergence results of Subsection 4.2.

Proof of Theorem 3. We recall that we wish to prove the following estimate:

$$\frac{\|\underline{\underline{\sigma}}^\varepsilon - \underline{\underline{\sigma}}^{RM}\|_{L^2(P^\varepsilon)}}{\|\underline{\underline{\sigma}}^\varepsilon\|_{L^2(P^\varepsilon)}} \leq C\varepsilon^{1/2}.$$

We also recall that (cf. (15) and (32))

$$\underset{\approx}{\sigma}^\varepsilon = A^{-1} \underset{\approx}{e}(u^\varepsilon) + \frac{\nu}{1-\nu} \underset{\approx}{\sigma}_{33}^\varepsilon, \quad \underset{\approx}{\sigma}^{RM} = A^{-1} \underset{\approx}{s}^{RM} + \frac{\nu}{1-\nu} \underset{\approx}{\sigma}_{33}^{RM}.$$

Thus, from the triangle inequality, we obtain

$$(79) \quad \begin{aligned} \|\underset{\approx}{\sigma}^\varepsilon - \underset{\approx}{\sigma}^{RM}\|_{L^2(P^\varepsilon)} &\leq c \left(\|\underset{\approx}{e}(u^\varepsilon) - \underset{\approx}{s}^{RM}\|_{L^2(P^\varepsilon)} + \|\underset{\approx}{\sigma}_{33}^\varepsilon\|_{L^2(P^\varepsilon)} + \|\underset{\approx}{\sigma}_{33}^{RM}\|_{L^2(P^\varepsilon)} \right) \\ &\leq c \left(\|\underset{\approx}{e}(u^\varepsilon) - \underset{\approx}{e}(\varepsilon \zeta^1 - x_3 \nabla \zeta_3^0)\|_{L^2(P^\varepsilon)} + \|\underset{\approx}{s}^{RM} - \underset{\approx}{e}(\varepsilon \zeta^1 - x_3 \nabla \zeta_3^0)\|_{L^2(P^\varepsilon)} \right. \\ &\quad \left. + \|\underset{\approx}{\sigma}_{33}^\varepsilon\|_{L^2(P^\varepsilon)} + \|\underset{\approx}{\sigma}_{33}^{RM}\|_{L^2(P^\varepsilon)} \right). \end{aligned}$$

From the two-dimensional Korn's inequality in Ω and the first estimate of (67), we have

$$(80) \quad \|\underset{\approx}{e}(u^\varepsilon) - \underset{\approx}{e}(\varepsilon \zeta^1 - x_3 \nabla \zeta_3^0)\|_{L^2(P^\varepsilon)} \leq c\varepsilon^2.$$

From (67) of Theorem 7, the estimate $\|\underset{\approx}{\sigma}_{33}^\varepsilon\|_{L^2(P^\varepsilon)} \leq c\varepsilon^{1/2}$, it follows that

$$(81) \quad \|\underset{\approx}{\sigma}_{33}^\varepsilon\|_{L^2(P^\varepsilon)} \leq \|\underset{\approx}{\sigma}_{33}^\varepsilon - \varepsilon^3 \underset{\approx}{\sigma}_{33}^3\|_{L^2(P^\varepsilon)} + \varepsilon^3 \|\underset{\approx}{\sigma}_{33}^3\|_{L^2(P^\varepsilon)} \leq c\varepsilon^2.$$

Next, from the definition of $\underset{\approx}{\sigma}_{33}^{RM}$ in (32), (10), and the scaling assumption (7), we have

$$(82) \quad \|\underset{\approx}{\sigma}_{33}^{RM}\|_{L^2(P^\varepsilon)} \leq c\varepsilon^2.$$

To bound the term $\|\underset{\approx}{s}^{RM} - \underset{\approx}{e}(\varepsilon \eta^1 - x_3 \nabla \zeta_3^0)\|_{L^2(P^\varepsilon)}$, we first recall that (cf. (23))

$$\underset{\approx}{s}^{RM} = \underset{\approx}{s}^m + \varepsilon^{-1} x_3 \underset{\approx}{s}^b,$$

where $(\underset{\approx}{s}^m, \underset{\approx}{s}^b)$ is the solution of system (30). Therefore, since Lemma 1 implies $\zeta^1 = \eta^1$ and $\nabla \zeta_3^0 = \phi^0$, we get

$$(83) \quad \begin{aligned} \|\underset{\approx}{s}^{RM} - \underset{\approx}{e}(\varepsilon \zeta^1 - x_3 \nabla \zeta_3^0)\|_{L^2(P^\varepsilon)} &= \|\underset{\approx}{s}^{RM} - \underset{\approx}{e}(\varepsilon \eta^1 - x_3 \phi^0)\|_{L^2(P^\varepsilon)} \\ &\leq \|\underset{\approx}{s}^m - \varepsilon \underset{\approx}{e}(\eta^1)\|_{L^2(P^\varepsilon)} + \|\varepsilon^{-1} x_3 (\underset{\approx}{s}^b + \varepsilon \underset{\approx}{e}(\phi^0))\|_{L^2(P^\varepsilon)}. \end{aligned}$$

Defining $\underset{\approx}{s}^{er,m} := \underset{\approx}{s}^m - \varepsilon \underset{\approx}{e}(\eta^1)$ and $\underset{\approx}{s}^{er,b} := \underset{\approx}{s}^b + \varepsilon \underset{\approx}{e}(\phi^0)$, we see from (30) that the couple $(\underset{\approx}{s}^{er,m}, \underset{\approx}{s}^{er,b})$ solves

$$\begin{cases} I^0(A^{-1}) \underset{\approx}{s}^{er,m} + I^1(A^{-1}) \underset{\approx}{s}^{er,b} = I^0(A^{-1}) \underset{\approx}{e}(\eta - \varepsilon \eta^1) - \varepsilon I^1(A^{-1}) \underset{\approx}{e}(\phi - \phi^0) \\ \quad + I^0\left(\frac{\nu}{1-\nu}\right) g_3^{\varepsilon,o} \underset{\approx}{\delta} + I^1\left(\frac{\nu}{1-\nu}\right) g_3^{\varepsilon,\varepsilon} \underset{\approx}{\delta}, \\ I^1(A^{-1}) \underset{\approx}{s}^{er,m} + I^2(A^{-1}) \underset{\approx}{s}^{er,b} = I^1(A^{-1}) \underset{\approx}{e}(\eta - \varepsilon \eta^1) - \varepsilon I^2(A^{-1}) \underset{\approx}{e}(\phi - \phi^0) \\ \quad + I^1\left(\frac{\nu}{1-\nu}\right) g_3^{\varepsilon,o} \underset{\approx}{\delta} + I^2\left(\frac{\nu}{1-\nu}\right) g_3^{\varepsilon,\varepsilon} \underset{\approx}{\delta}. \end{cases}$$

Therefore, for every $x \in \Omega$, we have

$$|\underset{\approx}{s}^{er,m}| + |\underset{\approx}{s}^{er,b}| \leq c \left(|\underset{\approx}{e}(\eta - \varepsilon \eta^1)| + \varepsilon |\underset{\approx}{e}(\phi - \phi^0)| + |g_3^{\varepsilon,o}| + |g_3^{\varepsilon,\varepsilon}| \right),$$

where $|\cdot|$ denotes the usual algebraic norm for tensors. Hence, using (7) and Theorem 8 we get

$$(84) \quad \|\underset{\sim}{s}^{er,m}\|_{L^2(P^\varepsilon)} + \|\underset{\sim}{s}^{er,b}\|_{L^2(P^\varepsilon)} \leq \varepsilon^{5/2}.$$

Collecting (79)–(84) we obtain

$$(85) \quad \|\underset{\sim}{\sigma}^\varepsilon - \underset{\sim}{\sigma}^{RM}\|_{L^2(P^\varepsilon)} \leq \varepsilon^2.$$

Moreover, proceeding as in (50) we find the following lower bound for $\underset{\sim}{\sigma}^\varepsilon$

$$(86) \quad \|\underset{\sim}{\sigma}^\varepsilon\|_{L^2(P^\varepsilon)} \geq \varepsilon \|\underset{\sim}{\sigma}^1\|_{L^2(P^\varepsilon)} - \|\underset{\sim}{\sigma}^\varepsilon - \varepsilon \underset{\sim}{\sigma}^1\|_{L^2(P^\varepsilon)} \geq c\varepsilon^{3/2},$$

for ε sufficiently small. The theorem now follows from (85) and (86). \square

Proof of Theorem 4. We wish to obtain

$$\frac{\|\underset{\sim}{I}^0(\underset{\sim}{\sigma}^\varepsilon - \underset{\sim}{\sigma}^{RM})\|_{L^2(\Omega_0)}}{\|\underset{\sim}{I}^0(\underset{\sim}{\sigma}^\varepsilon)\|_{L^2(\Omega_0)}} \leq C\varepsilon.$$

We first recall that (cf. (32) and (24))

$$\underset{\sim}{\sigma}^{RM} = \varepsilon^2 \underset{\sim}{p}^l(\underline{x}) + \underset{\sim}{s}^m(\underline{x}) p^m(x_3) + \underset{\sim}{s}^b(\underline{x}) p^b(x_3).$$

We focus on the more difficult case of p^m and p^b linearly independent (otherwise the computations become easier).

From the triangle inequality it follows that

$$\|\underset{\sim}{I}^0(\underset{\sim}{\sigma}^\varepsilon - \underset{\sim}{\sigma}^{RM})\|_{L^2(\Omega_0)} \leq \|\underset{\sim}{I}^0(\underset{\sim}{\sigma}^{er})\|_{L^2(\Omega_0)} + \|\underset{\sim}{I}^0(\underset{\sim}{\sigma}^{RM} - \varepsilon^2 \underset{\sim}{\sigma}^2)\|_{L^2(\Omega_0)},$$

where $\underset{\sim}{\sigma}^{er}$ is defined in (65) and $\underset{\sim}{\sigma}^2(\underline{x}) = \underset{\sim}{\sigma}^2(\underline{x}, \varepsilon^{-1}x_3)$. Using Cauchy-Schwarz inequality, we find that

$$(87) \quad \|\varepsilon \underset{\sim}{I}^0(\underset{\sim}{\sigma}^{er})\|_{L^2(\Omega_0)} \leq (2\varepsilon)^{1/2} \left\| \left(\int_{-\varepsilon}^{\varepsilon} |\underset{\sim}{\sigma}^{er}|^2 dx_3 \right)^{1/2} \right\|_{L^2(\Omega_0)} = (2\varepsilon)^{1/2} \|\underset{\sim}{\sigma}^{er}\|_{L^2(P_0^\varepsilon)} \leq c\varepsilon^4,$$

where Theorem 7 is used to obtain the last inequality (cf. (69)). Using (54) and (30), we have that

$$(88) \quad \begin{aligned} \underset{\sim}{I}^0(\underset{\sim}{\sigma}^{RM}) &= \varepsilon \operatorname{div} \underset{\sim}{I}^1(A^{-1}) \underset{\sim}{s}^m + \varepsilon \underset{\sim}{I}^2(A^{-1}) \underset{\sim}{s}^b + \varepsilon \underset{\sim}{I}^1(\underline{f}^\varepsilon) + 2g^{\varepsilon,0} \\ &= \varepsilon \operatorname{div} \underset{\sim}{I}^1(A^{-1}) \underset{\sim}{e}(\underline{\eta}) - \varepsilon^2 \operatorname{div} \underset{\sim}{I}^2(A^{-1}) \underset{\sim}{e}(\underline{\phi}) + \varepsilon \underset{\sim}{I}^1\left(\frac{\nu}{1-\nu}\right) \underset{\sim}{\nabla} g_3^{\varepsilon,0} + \varepsilon \underset{\sim}{I}^2\left(\frac{\nu}{1-\nu}\right) \underset{\sim}{\nabla} g_3^{\varepsilon,\varepsilon} \\ &\quad + \varepsilon \underset{\sim}{I}^1(\underline{f}^\varepsilon) + 2g^{\varepsilon,0}. \end{aligned}$$

Using (63) and integrating by parts, we gather that

$$\begin{aligned}
(89) \quad & \int_{-\varepsilon}^{\varepsilon} \tilde{\sigma}^2(\cdot, \varepsilon^{-1}x_3) dx_3 \\
&= \int_{-\varepsilon}^{\varepsilon} \left(x'_3 \int_{-1}^{\varepsilon^{-1}x_3} \left(-\operatorname{div} A^{-1} \underline{e}(\zeta^1 - \xi \nabla \zeta_3^0) - \underline{f} \right) d\xi - g(x, -1) \right) dx_3 \\
&= \int_{-\varepsilon}^{\varepsilon} \varepsilon^{-1}x_3 \left[\operatorname{div} A^{-1} \underline{e}(\zeta^1 - \varepsilon^{-1}x_3 \nabla \zeta_3^0) + \underline{f}(\cdot, \varepsilon^{-1}x_3) \right] dx_3 - 2\varepsilon g(x, -1) \\
&\quad + \varepsilon \int_{-1}^1 \left(-\operatorname{div} A^{-1} \underline{e}(\zeta^1 - \xi \nabla \zeta_3^0) - \underline{f} \right) d\xi.
\end{aligned}$$

Substituting the first equation of (8) in (89), we obtain

$$(90) \quad \tilde{I}^0(\tilde{\sigma}^2) = \operatorname{div} I^1(A^{-1}) \underline{e}(\zeta^1) + \operatorname{div} I^2(A^{-1}) \underline{e}(\nabla \zeta_3^0) + I^1(\underline{f}) + 2\varepsilon^{-2}g^{\varepsilon, \circ}.$$

Therefore, from (88) and (90) we get

$$\begin{aligned}
(91) \quad & \|\tilde{I}^0(\sigma^{RM} - \varepsilon^2 \tilde{\sigma}^2)\|_{L^2(\Omega_0)} \\
&\leq \|\varepsilon \operatorname{div} I^1(A^{-1}) \underline{e}(\eta - \varepsilon \zeta^1) - \varepsilon^2 \operatorname{div} I^2(A^{-1}) \underline{e}(\phi - \nabla \zeta_3^0)\|_{L^2(\Omega_0)} + c\varepsilon \|g_3^\varepsilon\|_{H^1(\Omega_0)} \\
&\leq c\varepsilon \|\eta - \varepsilon \zeta^1\|_{H^2(\Omega_0)} + c\varepsilon^2 \|\phi - \nabla \zeta_3^0\|_{H^2(\Omega_0)} + c\varepsilon \|g_3^\varepsilon\|_{H^1(\Omega_0)} \leq c\varepsilon^3,
\end{aligned}$$

where the final inequality follows from Theorem 8 and hypothesis (7). Combining estimates (87) and (91), we conclude that

$$\|\tilde{I}^0(\sigma^\varepsilon - \sigma^{RM})\|_{L^2(\Omega_0)} \leq c\varepsilon^3.$$

The result follows since from Theorem 7 and the definition of $\tilde{\sigma}^2$, it holds

$$\|\tilde{I}^0(\sigma^\varepsilon)\|_{L^2(\Omega_0)} \geq c\varepsilon^2.$$

□

Proof of Corollary 5. Our aim is to prove that

$$\frac{\|\sigma^\varepsilon - \sigma^{RM}\|_{L^2(P_0^\varepsilon)}}{\|\sigma^\varepsilon\|_{L^2(P_0^\varepsilon)}} \leq C\varepsilon.$$

As before, let $\tilde{\sigma}^2(x) = \sigma^2(x, \varepsilon^{-1}x_3)$. From the triangle inequality and Theorem 7, we have

$$\|\sigma^\varepsilon - \sigma^{RM}\|_{L^2(P_0^\varepsilon)} \leq \|\sigma^\varepsilon - \varepsilon^2 \tilde{\sigma}^2\|_{L^2(P_0^\varepsilon)} + \|\sigma^{RM} - \varepsilon^2 \tilde{\sigma}^2\|_{L^2(P_0^\varepsilon)} \leq \|\sigma^{RM} - \varepsilon^2 \tilde{\sigma}^2\|_{L^2(P_0^\varepsilon)} + c\varepsilon^{7/2}.$$

In the pure bending case, σ^{RM} simplifies as

$$\sigma^{RM} = \tilde{s}^b p^b + \varepsilon^2 p^l.$$

Hence, from the definition of $\tilde{\sigma}^2$ and (64), we obtain

$$\|\tilde{I}^0(\sigma^{RM} - \varepsilon^2 \tilde{\sigma}^2)\|_{L^2(\Omega_0)} = \|(s^b - \varepsilon^2 \nabla \Delta \zeta_3^0) \tilde{I}^0(p^b)\|_{L^2(\Omega_0)} = \|s^b - \varepsilon^2 \nabla \Delta \zeta_3^0\|_{L^2(\Omega_0)} |\tilde{I}^0(p^b)|.$$

Using (91) we get

$$\|\xi^b - \varepsilon^2 \nabla \Delta \zeta_3^0\|_{L^2(\Omega_0)} \leq c\varepsilon^3.$$

Thus, it follows that

$$\|\sigma^{RM} - \varepsilon^2 \bar{\sigma}^2\|_{L^2(P_0^\varepsilon)} = \|p^b\|_{L^2(-\varepsilon, \varepsilon)} \|\xi^b - \varepsilon^2 \nabla \Delta \zeta_3^0\|_{L^2(\Omega_0)} \leq c\varepsilon^{7/2},$$

and the result is a consequence of $\|\sigma^\varepsilon\|_{L^2(P_0^\varepsilon)} \geq c\varepsilon^{5/2}$. \square

Proof of Corollary 6. We recall that we are considering the case of p^m and p^b linearly dependent. Therefore, let $a \in \mathbb{R}$ such that $p^m = ap^b$. Then, using the same notation of the proof of Theorem 5, we have that

$$\bar{\sigma}^2 = (a \nabla \operatorname{div} \zeta^1 + \nabla \Delta \zeta_3^0)p^b.$$

It is enough now to proceed as in the proof of Theorem 5. \square

DIPARTIMENTO DI MECCANICA STRUTTURALE, UNIVERSITÀ DI PAVIA, VIA FERRATA 1, 27100 PAVIA, ITALY

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PAVIA, VIA FERRATA 1, 27100 PAVIA, ITALY

DEPARTAMENTO DE MATEMÁTICA APLICADA E COMPUTACIONAL, LNCC, AV. GETÚLIO VARGAS 333, PETRÓPOLIS - RJ, BRAZIL