Element diameter free stability parameters for stabilized methods applied to fluids

by

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Abstract

Stability parameters for stabilized methods in fluids are suggested. The computation of the largest eigenvalue of a generalized eigenvalue problem replaces controversial definitions of element diameters and inverse estimate constants, used heretofore to compute these stability parameters. The design is employed in the advective-diffusive model, incompressible Navier-Stokes equations and the Stokes problem.

1. Introduction

Stabilized finite element methods are formed adding to the standard Galerkin method perturbation terms, which are functions of the Euler-Lagrange equations evaluated elementwise, so that consistency is preserved. The perturbation terms are constructed to enhance stability of the original Galerkin formulation, allowing convergence of a variety of simple finite element interpolations in various applications. For background the reader is referred to [2-3,5-15] and references therein.

One of the questions that has often raised some controversy is on "how much of the perturbation term one has to add to obtain the desirable effects of additional stability with high accuracy". This question can be alternatively stated as "how to design the stability parameters that achieve our goals". The answers to these questions are often associated to the convergence analysis of the particular method in a particular model. In these analyses, it becomes clear that, to obtain optimal convergence rates, these parameters are frequently mesh-dependent.

Realizing that the additional terms should be mesh-dependent has caused various reactions, mainly from "pure" analysts that had to swallow notions of "upwinding" and "numerical dissipation" without really digesting them... Questions such as "what the mesh parameter h is in these methods" has become commonplace and perfectly reasonable for specialists that do not deal with the intricacies related to the numerical simulations of flows and other complex models.

In recent developments of stabilized methods for flows, the design of the stability parameters has been readdressed [6,7]. A particular feature of the stability parameters proposed in these works is the inclusion of inverse estimate constants as part of their definition. As a consequence, desirable features of the stability parameters are preserved for high interpolations, without the restrictions that should only affect diffusion dominated regions. However there are two drawbacks in these designs: i) the need to compute the inverse estimate constant, ii) and the usual dependence on the mesh parameter h. Recently, Harari and Hughes [11] have made various computations of inverse estimate constants and mesh parameters. It turns out to be too cumbersome to evaluate the inverse estimate constant in certain situations. Only for rather simple mesh structures (such as uniform mesh, or rectangles only, etc.) one can do such computations explicitly. To do these computations an associated eigenvalue problem is used.

In this note we use an associated eigenvalue problem to *define* the stability parameters, so that *no explicit* computations of inverse estimate constants are needed, nor the computation of mesh parameters. This second feature is possibly the most interesting: from the presentation one can design stabilized finite element methods without recourse to mesh parameters!

In the next section, we consider these designs of the stability parameters to the simple advective-diffusive model, and then we extend it to the incompressible Navier-Stokes equations, including the Stokes problem as the diffusive limit when the advection term is zero.

2. Designing Stability Parameters

2.1 The advective-diffusive problem

As a first model problem, let us consider the advective-diffusive problem of finding u such that

$$\mathbf{a} \cdot \nabla u - \kappa \Delta u = f \qquad \text{in} \;\; \Omega, \tag{1}$$

$$u = 0 \qquad \text{on } \Gamma \tag{2}$$

where $\mathbf{a}(\mathbf{x})$ is the given velocity field with $\nabla \cdot \mathbf{a} = 0$, κ is the diffusivity and $f(\mathbf{x})$ is a source function. The problem is defined on a open bounded domain $\Omega \subset I\!R^N, N = 2, 3$ with a polygonal or polyhedral boundary Γ. The homogeneous boundary condition (2) suffices for our discussion, and can be simply generalized for more complex situations such as non-homogenous Dirichlet combined with Neumann boundary conditions.

To introduce the finite element methods, consider a partition $\mathcal{C}_h$ of $\overline{\Omega}$ into elements consisting of triangles (tetrahedrons in $I\!R^3$) or convex quadrilaterals (hexahedrons) performed in the usual way (i.e., no overlapping is allowed between any two elements of the partition; the union of all domains K reproduces $\overline{\Omega}$, etc.). Quasiuniformity is *not* assumed. We also employ the following notation:

$$R_m(K) = \begin{cases} P_m(K) & \text{if } K \text{ is a triangle or tetrahedron,} \\ Q_m(K) & \text{if } K \text{ is a quadrilateral or hexahedron.} \end{cases}$$

where for each integer $m \geq 0$, P_m and Q_m have the usual meaning.

The scalar field u is approximated in the following standard finite element space:

$$V_h = \{ v \in H^1_0(\Omega) \,|\, v_{|K} \in R_k(K),\ K \in \mathcal{C}_h \} \,, \tag{3}$$

where $H^1_0(\Omega)$ is the Sobolev space of functions with square-integrable value and derivatives in Ω with zero on the boundary Γ, and the H^1-norm will be denoted by $\|\cdot\|_1$.

The stabilized finite element method we wish to consider is the Galerkin-least-squares method studied for this model in [14]: find $u_h \in V_h$ such that

$$B(u_h, v) = F(v) \qquad v \in V_h \tag{4}$$

with

$$B(u,v) = (\mathbf{a}\cdot\nabla u, v) + (\kappa\nabla u,\nabla v) + \sum_{K\in\mathcal{C}_h}(\mathbf{a}\cdot\nabla u - \kappa\Delta u, \tau(\mathbf{a}\cdot\nabla v - \kappa\Delta v))_K \quad (5)$$

and

$$F(v) = (f,v) + \sum_{K\in\mathcal{C}_h}(f,\tau(\mathbf{a}\cdot\nabla v - \kappa\Delta v))_K \quad (6)$$

where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$-inner product, with $L^2(\Omega)$ being the space of square-integrable functions in Ω; $(\cdot,\cdot)_K$ denotes the $L^2(K)$-inner product. We will use $\|\cdot\|_0$ and $\|\cdot\|_{0,K}$ to denote the $L^2(\Omega)$- and $L^2(K)$- norms, respectively.

The stability parameter τ in (5) and (6) has a new design for $k \geq 2$ given by:

$$\tau = \frac{2}{\sqrt{\lambda_K}|\mathbf{a}(\mathbf{x})|_p}\xi(\mathrm{Pe}_K(\mathbf{x})) \quad (7)$$

$$\mathrm{Pe}_K(\mathbf{x}) = \frac{|\mathbf{a}(\mathbf{x})|_p}{4\sqrt{\lambda_K}\,\kappa(\mathbf{x})} \quad (8)$$

$$\xi(\mathrm{Pe}_K(\mathbf{x})) = \begin{cases} \mathrm{Pe}_K(\mathbf{x}) & ,\ 0 \leq \mathrm{Pe}_K(\mathbf{x}) < 1 \\ 1 & ,\ \mathrm{Pe}_K(\mathbf{x}) \geq 1 \end{cases} \quad (9)$$

$$\lambda_K = \max_{0\neq v\in R_k(K)/\boldsymbol{R}} \frac{\|\Delta v\|^2_{0,K}}{\|\nabla v\|^2_{0,K}} \quad ,K\in\mathcal{C}_h \quad (10)$$

$$|\mathbf{a}(\mathbf{x})|_p = \begin{cases} \left(\sum_{i=1}^N |a_i(\mathbf{x})|^p\right)^{1/p} & ,\quad 1\leq p<\infty \\ \max\limits_{i=1,N}|a_i(\mathbf{x})| & ,\quad p=\infty \end{cases} \quad (11)$$

Remarks

1. The design given above excludes the linear interpolation case, $k = 1$. It is only valid for high order interpolations.

2. The formulae (7)-(11) resembles the definition given by eqs (26)-(31) of [7]. In that reference an inverse estimate constant was used to *define* the method.

Here, instead, we *define* a parameter λ_K by eq.(10), which ("secretly speaking at this moment") carries the inverse estimate constant and the element diameter values implicitly.

3. Contrary to the usual definition of the Peclet number Pe_K, we consider in eq.(8) a replacement of the mesh parameter h_K (and in the case of reference [7] the inverse estimate constant value as well) by $\lambda_K^{-1/2}$. As discussed below in Remark 5, λ_K takes into account both the mesh parameter h_K and the value of the inverse estimate constant, implicit in its definition. At the same time λ_K is well defined for any type of distortion of the element, as long as we keep within the scope of the definition of a 'regular' element as given in, e.g., Ciarlet [4]. The main advantage of formulae (7)-(11) is that there are no extraneous dubious parameters in the definition of τ. The sources of potential controversy have been all condensed in the definition of λ_K, which in turn is uniquely determined once the geometry of K is set and the degree of the polynomial k is selected.

4. The parameter λ_K is calculated by computing the largest eigenvalue of the following generalized eigenvalue problem defined for each K: Find $w_h \in R_k(K)/\mathbb{R}$ and λ such that

$$(\Delta w_h, \Delta v) - \lambda(\nabla w_h, \nabla v) = 0 \qquad \forall\, v \in R_k(K)/\mathbb{R} \tag{12}$$

In practice to simulate the quotient space $R_k(K)/\mathbb{R}$, we fix a degree of freedom to zero and solve for the remaining ones. The largest eigenvalue is computed using the power method as described in, e.g., Bathe [1].

5. The standard inverse estimate at the element level associated to this model

reads: $\exists\, C_k > 0$, independent of the element diameter h_K such that

$$C_k \sum_{K \in \mathcal{C}_h} h_K^2 \|\Delta v\|_{0,K}^2 \leq \|\nabla v\|_0^2\,, \quad \forall v \in V_h \tag{13}$$

See [4] for a proof. Now by definition of λ_K in eq.(10), it follows from (13) that

$$\lambda_K^{-1} = C_k h_K^2 \tag{14}$$

This is the link between the λ_K parameter with the inverse estimate constant C_k and the mesh parameter h_K. Contrary to previous designs depending on the inverse constant as in [7], here the use of λ_K is free from the nonstandard definitions of C_k and h_K as suggested in [11].

The convergence analysis of GLS with this design of τ can be done similarly as in [7]. The main difference here is that inverse estimates are not needed. Instead, by definition of λ_K from eq.(10) we have

$$\lambda_K \|\nabla v\|_{0,K}^2 \geq \|\Delta v\|_{0,K}^2, \qquad K \in \mathcal{C}_h \qquad \forall\, v \in V_h \tag{15}$$

and this is used to establish stability as follows. First note that from (7)-(9), for $\mathrm{Pe}_K \geq 1$:

$$\tau = \frac{2}{\sqrt{\lambda_K}|\mathbf{a}(\mathbf{x})|_p} \frac{|\mathbf{a}(\mathbf{x})|_p}{4\sqrt{\lambda_K}\,\kappa} \frac{1}{\mathrm{Pe}_K} \leq \frac{1}{2\lambda_K \kappa(\mathbf{x})} \tag{16}$$

Since by definition the equality sign of (16) holds for $\mathrm{Pe}_K < 1$ therefore it follows that the bound (16) is valid for all $\mathrm{Pe}_K \geq 0$.

Now, by definition (eq.(5)), and using that $\nabla \cdot \mathbf{a} = 0$ and κ is supposed constant:

$$
\begin{aligned}
B(v,v) =&(\mathbf{a}\cdot\nabla v, v) + \kappa\|\nabla v\|_0^2 + \|\tau^{1/2}\,\mathbf{a}\cdot\nabla v\|_0^2 - 2\sum_{K\in\mathcal{C}_h}(\mathbf{a}\nabla v, \tau\,\kappa\Delta v)_K \\
&+ \sum_{K\in\mathcal{C}_h}\|\tau^{1/2}\,\kappa\Delta v\|_{0,K}^2 \\
\geq&\kappa\|\nabla v\|_0^2 + \frac{1}{2}\|\tau^{1/2}\,\mathbf{a}\cdot\nabla v\|_0^2 - \sum_{K\in\mathcal{C}_h}\|\tau^{1/2}\,\kappa\Delta v\|_{0,K}^2
\end{aligned}
\tag{17}
$$

Note that from (15)-(16) the last term in (17) can be estimated as follows

$$
\begin{aligned}
\sum_{K\in\mathcal{C}_h}\|\tau^{1/2}\,\kappa\Delta v\|_{0,K}^2 =& \sum_{K\in\mathcal{C}_h}\|(\tau\,\kappa)^{1/2}\kappa^{1/2}\Delta v\|_{0,K}^2 \\
\leq& \sum_{K\in\mathcal{C}_h}\frac{\kappa}{2\lambda_K}\|\Delta v\|_{0,K}^2 \qquad \text{(by (16))} \\
\leq& \frac{\kappa}{2}\|\nabla v\|_0^2 \qquad \text{(by (15))}
\end{aligned}
$$

Therefore combining this estimate with (17) implies

$$
B(v,v) \geq \frac{1}{2}(\kappa\|\nabla v\|_0^2 + \|\tau^{1/2}\,\mathbf{a}\cdot\nabla v\|_0^2) \tag{18}
$$

The following convergence of u_h solution of (4)-(6) to u solution of (1)-(2) follows in the norm (18):

$$
\begin{aligned}
&\kappa\|\nabla(u_h-u)\|_0^2 + \|\tau^{1/2}\,\mathbf{a}\cdot\nabla(u_h-u)\|_0^2 \\
&\qquad\leq C\sum_{K\in\mathcal{C}_h} h_K^{2k}|u|_{k+1,K}^2\Big(\mathrm{H}(\mathrm{Pe}_K-1)h_K\sup_{\mathbf{x}\in K}|\mathbf{a}|_p + \mathrm{H}(1-\mathrm{Pe}_K)\kappa\Big)
\end{aligned}
\tag{19}
$$

where $\mathrm{H}(\,\cdot\,)$ is the Heaviside function given by

$$
\mathrm{H}(x-y) = \begin{cases} 0, & x<y; \\ 1, & x>y. \end{cases} \tag{20}
$$

To establish (19), besides (18) we need an interpolation estimate for this particular design of τ that can be obtained as Lemma 3.2 of [7] using the relation between

λ_K with C_k and h_K given by eq.(14). This is the only instance that we need an inverse estimate in the analysis: to obtain the rates of convergence. Otherwise, the entire analysis goes through without ever needing (14) or (13).

2.2 The incompressible Navier-Stokes equations

Let us consider the steady state incompressible Navier-Stokes given by:

$$(\nabla \mathbf{u})\mathbf{u} - 2\nu \nabla \cdot \boldsymbol{\varepsilon}(\mathbf{u}) + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{21}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{22}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma, \tag{23}$$

where $\mathbf{u}$ is the velocity, p is the pressure, ν is the viscosity, $\boldsymbol{\varepsilon}(\mathbf{u})$ is the symmetric part of the velocity gradient and $\mathbf{f}$ is the body force.

Employing a partition $\mathcal{C}_h$ of the domain Ω as dicussed in section 2.1, we may introduce the following finite element spaces:

$$\mathbf{V}_h = \left\{ \mathbf{v} \in H_0^1(\Omega)^N \,|\, \mathbf{v}_{|K} \in R_k(K)^N, \; K \in \mathcal{C}_h \right\}, \tag{24}$$

$$P_h = \left\{ p \in \mathcal{C}^0(\Omega) \cap L_0^2(\Omega) \,|\, p_{|K} \in R_l(K), \; K \in \mathcal{C}_h \right\}, \tag{25}$$

The stabilized finite element method we wish to consider is the "minus" formulation studied in [6] that can be written as: Find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in P_h$ such that

$$B(\mathbf{u}_h, p_h; \mathbf{v}, q) = F(\mathbf{v}, q), \qquad \forall\, (\mathbf{v}, q) \in \mathbf{V}_h \times P_h, \tag{26}$$

with

$$\begin{aligned} B(\mathbf{u}, p; \mathbf{v}, q) &= ((\nabla \mathbf{u})\mathbf{u}, \mathbf{v}) + (2\nu\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v})) - (\nabla \cdot \mathbf{v}, p) + (\nabla \cdot \mathbf{u}, q) \\ &+ (\nabla \cdot \mathbf{u}, \delta \nabla \cdot \mathbf{v}) \\ &+ \sum_{K \in \mathcal{C}_h} \big((\nabla \mathbf{u})\mathbf{u} + \nabla p - 2\nu \nabla \cdot \boldsymbol{\varepsilon}(\mathbf{u}), \tau((\nabla \mathbf{v})\mathbf{u} + \nabla q - 2\nu \nabla \cdot \boldsymbol{\varepsilon}(\mathbf{v})) \big)_K \end{aligned} \tag{27}$$

and

$$F(\mathbf{v},q) = (\mathbf{f},\mathbf{v}) + \sum_{K\in\mathcal{C}_h} \big(\mathbf{f}, \tau((\nabla\mathbf{v})\mathbf{u} + \nabla q - 2\nu\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v}))\big)_K \tag{28}$$

Similarly to the previous section, we wish to consider the following desing of the stability parameters τ and δ for $k \geq 2$:

$$\delta = \frac{|\mathbf{u}(\mathbf{x})|_p}{\sqrt{\lambda_K}} \xi(\mathrm{Re}_K(\mathbf{x})) \tag{29}$$

$$\tau = \frac{\xi(\mathrm{Re}_K(\mathbf{x}))}{\sqrt{\lambda_K}|\mathbf{u}(\mathbf{x})|_p} \tag{30}$$

$$\mathrm{Re}_K(\mathbf{x}) = \frac{|\mathbf{u}(\mathbf{x})|_p}{4\sqrt{\lambda_K}\nu(\mathbf{x})} \tag{31}$$

$$\xi(\mathrm{Re}_K(\mathbf{x})) = \begin{cases} \mathrm{Re}_K(\mathbf{x}) & , 0 \leq \mathrm{Re}_K(\mathbf{x}) < 1 \\ 1 & , \mathrm{Re}_K(\mathbf{x}) \geq 1 \end{cases} \tag{32}$$

$$\lambda_K = \max_{0\neq\mathbf{v}\in(R_k(K)/I\!\!R)^N} \frac{\|\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v})\|^2_{0,K}}{\|\boldsymbol{\varepsilon}(\mathbf{v})\|^2_{0,K}} \quad , K \in \mathcal{C}_h \tag{33}$$

$$|\mathbf{u}(\mathbf{x})|_p = \begin{cases} \left(\sum_{i=1}^N |u_i(\mathbf{x})|^p\right)^{1/p} & , \quad 1 \leq p < \infty \\ \max\limits_{i=1,N} |u_i(\mathbf{x})| & , \quad p = \infty \end{cases} \tag{34}$$

Remarks

1. The design given above excludes linear velocity interpolations. However, pressures may be linearly interpolated or higher, i.e., $l \geq 1$.

2. Similarly to the previous section, the parameter λ_K is computed as the largest eigenvalue of the following generalized eigenvalue problem defined for each K: Find $\mathbf{w}_h \in (R_k(K)/I\!\!R)^N$ and λ_K such that

$$(\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{w}_h), \nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v})) - \lambda_K(\nabla\mathbf{w}_h, \nabla\mathbf{v}) = 0 \qquad \forall\, \mathbf{v} \in (R_k(K)/I\!\!R)^N \tag{35}$$

   This problem is solved for the largest eigenvalue by the power method.

3. For this model the relevant inverse estimate reads: $\exists C_k > 0$, independent of the element diameter h_K, such that

$$C_k \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla \cdot \boldsymbol{\varepsilon}(\mathbf{v})\|_{0,K}^2 \leq \|\boldsymbol{\varepsilon}(\mathbf{v})\|_0^2 \ , \quad \forall \mathbf{v} \in \mathbf{V}_h \tag{36}$$

Again, as in eq.(14), by definition of λ_K in (33) combined with (36) yields

$$\lambda_K^{-1} = C_k h_K^2 \tag{37}$$

The same link of λ_K with C_k and h_K holds in this model.

4. We wish to reiterate that the present design of the stability parameters (29)-(34) is free from controversial definitions of the element diameter h_K and from how to compute inverse estimate constants C_k. Once a mesh is set-up and the finite element approximation polynomial is selected, then problem (35) can be solved once and for all to determine each parameter λ_K in the entire mesh, i.e., for $K \in \mathcal{C}_h$. This can be done in a pre-processor before entering the loop of the non-linear algorithm (e.g. Newton-like) to solve the nonlinear set of discrete equations associated to (26)-(28).

5. Convergence analysis taking into account the present design of the stability parameters, can be performed for a linearized model similarly as in ref.[6]. As pointed out in section 2.1, inverse estimates are no longer needed to establish stability and carry out the entire analysis, up to the point where interpolation estimates results are needed to characterize the rates of convergence. The analysis considerations for this case are similar to what is described in section 2.1, which combined with the analysis presented in [6] yields a similar convergence result.

Finally, if we consider the Stokes problem given by

$$-2\mu\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u})+\nabla p=\mathbf{f}\qquad \text{in}\ \ \Omega, \tag{38}$$

$$\nabla\cdot\mathbf{u}=0\qquad \text{in}\ \ \Omega, \tag{39}$$

$$\mathbf{u}=\mathbf{0}\qquad \text{on}\ \ \Gamma \tag{40}$$

and approximate velocity and pressure by the standard finite element spaces given by (24) and (25), the method given by eqs. (26) to (28) reduces to: Find $\mathbf{u}_h\in\mathbf{V}_h$ and $p_h\in P_h$ such that

$$B(\mathbf{u}_h,p_h;\mathbf{v},q)=F(\mathbf{v},q)\qquad (\mathbf{v},q)\in\mathbf{V}_h\times P_h \tag{41}$$

with

$$\begin{aligned}B(\mathbf{u},p;\mathbf{v},q)&=(2\mu\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v}))-(\nabla\cdot\mathbf{v},p)+(\nabla\cdot\mathbf{u},q)\\&+\sum_{K\in\mathcal{C}_h}\big(\nabla p-2\mu\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u}),\tau(\nabla q-2\mu\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v}))\big)_K\end{aligned} \tag{42}$$

and

$$F(\mathbf{v},q)=(\mathbf{f},\mathbf{v})+\sum_{K\in\mathcal{C}_h}\big(\mathbf{f},\tau(\nabla q-2\mu\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v}))\big)_K \tag{43}$$

with

$$\tau=\frac{1}{4\lambda_K\mu} \tag{44}$$

$$\lambda_K=\max_{0\neq\mathbf{v}\in(R_k(K)/\boldsymbol{R})^N}\frac{\|\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v})\|^2_{0,K}}{\|\boldsymbol{\varepsilon}(\mathbf{v})\|^2_{0,K}},\qquad K\in\mathcal{C}_h \tag{45}$$

Remarks

1. The stabilized method (41)-(43) was proposed in [5] as a modification of the method introduced in [12-13]. It is clear from the stability analysis of this method that any positive parameter τ renders this method stable. However,

from numerical experience, better accuracy is obtained for a choice near the design given in (44).

2. The design of the stability parameter τ is obtained by taking the limit as $\mathrm{Re}_K \to 0$ in (30)-(34). It can be viewed as the diffusive limit of the general situation of advective-diffusive incompressible flows governed by the Navier-Stokes equations.

3. Concluding remarks

Guidelines are given for designing stability parameters for stabilized methods in fluids with the following features:

i) The parameters do not depend on element diameters (nor mesh parameters).

ii) The parameters do not depend on inverse estimate constant calculations.

iii) The parameters are obtained in association with the solution of a generalized eigenvalue problem, which gives a measure on the combination of mesh parameter and inverse estimate constants. This eigenvalue can be obtained for any type of regular element (including distorted ones).

The presented guidelines encompass previous works [6,7] simplifying the requirements of computing inverse estimate constants. It is interesting to see how a finite element analysis tool such as the inverse estimate was included in the design of these parameters in [6,7] and here is invited to be in the background, so that by simply including the solution of a generalized eigenvalue problem, the same features discussed in [6,7] are achieved here, without the drawback of having to compute inverse estimate constants and the mesh size (or element diameters).

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