A new discontinuous Galerkin method for the Reissner–Mindlin plate model

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Abstract. *Recently, a hp interior penalty discontinuous Galerkin finite element method for the biharmonic equation was introduced and analysed by the first and third authors. We now extend such ideas in a nontrivial way for the Reissner–Mindlin plate model. The extension is such that, as the plate thickness tends to zero, we recover the method for the biharmonic problem. Our present scheme does not introduce shear as an extra unknown, and does not need reduced integration techniques. We present here an* a priori *error analysis of these methods and prove error bounds in h which are uniform with respect to the plate thickness. Numerical tests are presented.*

Keywords: *Reissner–Mindlin, Discontinuous Galerkin*

1. Introduction

The Reissner–Mindlin model for plates is not only a good model for linearly elastic plates, but also it brings in computational challenges that require ingenious numerical methods. The reason is that the system depends in a nontrivially on ε, the half-thickness of the plate. As the plate become thinner, the Reissner–Mindlin solution approaches the Kirchhoff–Love, biharmonic solution. Thus, for small ε, naive numerical schemes fail, since they do not approximate well solutions of fourth order problems in general. This is described as a *numerical locking*.

This is all well known and there are in the literature finite element schemes that avoid locking altogether, see (Falk, 2008) and references therein for a comprehensive review. Recently some authors started to take advantage of the flexibility of discontinuous Galerkin (DG) finite element methods to design new, locking free, plate models (Arnold et al., 2005; Brezzi and Marini, 2003; Chinosi et al., 2006; Lovadina, 2005). We follow the same philosophy.

The discontinuous Galerkin method for fourth order elliptic equation was introduced and analyzed by (Baker, 1977). A hp version of interior penalty discontinuous Galerkin methods have been considered in (Mozolevski and Bösing, 2007; Mozolevski and Süli, 2003; Mozolevski et al., 2007; Süli and Mozolevski, 2007), where the authors analyse and obtain *a priori* error bounds for variants of the method. Such DG, hp-scheme for the biharmonic equation was our motivation in this present work. We propose here a method for the Reissner–Mindlin system that, as ε tends to zero, "converges" to the scheme for the biharmonic. We prove convergence in a natural energy norm, and provide numerical tests that confirm our predictions.

Let Ω be a convex and polygonal two-dimensional domain with boundary $\partial\Omega$. Consider a homogeneous and isotropic linearly elastic plate occupying the three-dimensional domain $\Omega \times (-\epsilon, \epsilon)$. Assume that such a plate is clamped on its lateral side, and under a transverse load of density per unity area $\epsilon^3 g$ that is symmetric with respect to its middle surface. Under such pure bending regime, there are two popular two-dimension models for the plate's displacement.

In the Kirchhoff–Love model, the displacement at $(\boldsymbol{x}, x_3) \in \Omega \times (-\epsilon, \epsilon)$ is approximated by $\big(-x_3 \, \boldsymbol{\nabla}\, \psi(\boldsymbol{x}), \psi(\boldsymbol{x})\big)$, where

$$\begin{aligned} D\,\Delta^2\,\psi &= g \quad \text{in } \Omega, \\ \psi = \frac{\partial \psi}{\partial n} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

and $D = 4\mu(\mu+\lambda)/[3(2\mu+\lambda)]$. Here, μ and λ are the Lamé coefficients.

The simplest Reissner–Mindlin model approximation (Alessandrini, 1996) is given by $\big(-x_3\boldsymbol{\theta}(\boldsymbol{x}), \omega(\boldsymbol{x})\big)$, where

$$\begin{aligned} -\,\mathbf{div}\,\mathcal{C}\,e(\boldsymbol{\theta}) + \epsilon^{-2}\mu(\boldsymbol{\theta} - \boldsymbol{\nabla}\,\omega) &= 0 \quad \text{in } \Omega, \\ \epsilon^{-2}\mu\,\mathrm{div}(\boldsymbol{\theta} - \boldsymbol{\nabla}\,\omega) &= g \quad \text{in } \Omega, \\ \boldsymbol{\theta} = 0, \quad \omega &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2}$$

We denote by $e(\boldsymbol{\theta})$ the symmetric part of the gradient of $\boldsymbol{\theta}$, and

$$\mathcal{C}\,e(\boldsymbol{\theta}) = \frac{1}{3}\,[\,2\mu\,e(\boldsymbol{\theta}) + \lambda^*\,\mathrm{div}\,\boldsymbol{\theta}\,I\,],$$

with $\lambda^* = 2\mu\lambda/(2\mu+\lambda)$, and I is the identity matrix. Let Λ_0, Λ_1 be positive constants such that

$$\Lambda_0|\,e(\boldsymbol{\theta})|^2 \le |\,\mathcal{C}\,e(\boldsymbol{\theta}) : e(\boldsymbol{\theta})| \le \Lambda_1|\,e(\boldsymbol{\theta})|^2, \tag{3}$$

where $\tau : \sigma = \sum_{i,j=1}^{2} \tau_{ij}\sigma_{ij}$ denote the inner product between two matrices τ and σ, and $|\tau| = (\tau : \tau)^{1/2}$.

In the weak formulation, $\boldsymbol{\theta} \in \mathring{\boldsymbol{H}}^1(\Omega)$ and $\omega \in \mathring{H}^1(\Omega)$ are such that

$$\begin{aligned} a(\boldsymbol{\theta}, \boldsymbol{\eta}) + \epsilon^{-2}\mu(\boldsymbol{\theta} - \boldsymbol{\nabla}\,\omega, \boldsymbol{\eta}) = 0 \quad &\text{for all } \boldsymbol{\eta} \in \mathring{\boldsymbol{H}}^1(\Omega), \\ -\epsilon^{-2}\mu(\boldsymbol{\theta} - \boldsymbol{\nabla}\,\omega, \boldsymbol{\nabla}\,\nu) = (g, \nu) \quad &\text{for all } \nu \in \mathring{H}^1(\Omega), \end{aligned} \tag{4}$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$ and $\boldsymbol{L}^2(\Omega)$, and

$$a(\boldsymbol{\theta}, \boldsymbol{\eta}) = \int_\Omega \mathcal{C}\, e(\boldsymbol{\theta}) : e(\boldsymbol{\eta})\, d\boldsymbol{x}.$$

Note that the Poincaré's and Korn's inequalities hold, i.e., there exists an ϵ-independent constant c such that

$$\|\boldsymbol{\eta}\|_{1,\Omega}^2 \le c a(\boldsymbol{\eta}, \boldsymbol{\eta}), \quad \|\omega\|_{0,\Omega} \le c\|\boldsymbol{\nabla}\,\omega\|_{0,\Omega} \qquad \text{for all } (\boldsymbol{\eta}, \omega) \in \big(\mathring{\boldsymbol{H}}^1(\Omega) \times \mathring{H}^1(\Omega)\big).$$

The existence and uniqueness of solutions for Reissner–Mindlin follow since $(\boldsymbol{\theta}, \omega)$ is the unique minimum of the functional

$$a(\boldsymbol{\theta}, \boldsymbol{\theta}) + \epsilon^{-2}\mu(\boldsymbol{\theta} - \boldsymbol{\nabla}\,\omega, \boldsymbol{\theta} - \boldsymbol{\nabla}\,\omega) - (g, \omega)$$

in $\big(\mathring{\boldsymbol{H}}^1(\Omega), \mathring{H}^1(\Omega)\big)$.

The relation between the Kirchhoff–Love and Reissner–Mindlin models becomes clear since, as $\epsilon \to 0$, the sequence of solutions $(\boldsymbol{\theta}, \omega)$ converges to $(\boldsymbol{\nabla}\,\psi, \psi)$, where ψ solves (1), and minimizes

$$a(\boldsymbol{\nabla}\,\nu, \boldsymbol{\nabla}\,\nu) - (g, \nu)$$

in $\mathring{H}^2(\Omega)$. This is an instance of a more general result of (Chenais and Paumier, 1994).

The contents of this paper are as follows. In the next section, we introduce a formulation for the Reissner–Mindlin system, and in Section 3., we define our scheme and state continuity and coercivity results. Sections 4. and 5. contain the convergence and numerical results.

We now introduce some notation used in this paper. For a given open set $\mathfrak{D}$, the set $L^2(\mathfrak{D})$ contains the square integrable functions in $\mathfrak{D}$. For a non-negative t, $H^t(\mathfrak{D})$ is the corresponding Sobolev space of order t. The notation for its inner product, norm and semi-norm is $(\cdot, \cdot)_{t,\mathfrak{D}}$, $\|\cdot\|_{t,\mathfrak{D}}$ and $|\cdot|_{t,\mathfrak{D}}$. We write vectors, and vector spaces in bold, and we denote by c a generic constant (not necessarily the same in all occurrences) which is independent of ε.

2. Weak formulation in broken Sobolev space

Let $\mathcal{K}_h = \{K\}$ be a shape-regular partition of Ω into non-overlapping triangles. The number h_K denotes the diameter of an element $K \in \mathcal{K}_h$, and h is the maximum of h_K, for all $K \in \mathcal{K}_h$. Let $\mathcal{E}_h$ be the set of all open faces e of all elements in $\mathcal{K}_h$, and h_e the length of e. The set $\mathcal{E}_h$ will be divided into two subsets, $\mathcal{E}_h^\circ$ (the set of interior faces) and $\mathcal{E}_h^\partial$ (the set of boundary faces), defined by

$$\mathcal{E}_h^\circ = \{e \in \mathcal{E}_h : \ e \subset \Omega\}, \qquad \mathcal{E}_h^\partial = \{e \in \mathcal{E}_h : \ e \subset \partial\Omega\}.$$

In addition, we define $\Gamma^\circ = \{x \in e : \ e \in \mathcal{E}_h^\circ\}$ and $\Gamma = \Gamma^\circ \cup \partial\Omega$. Let

$$H^t(\mathcal{K}_h) = \{v \in L^2(\Omega) : \ v_{|K} \in H^t(K), \ \text{for all } K \in \mathcal{K}_h\}$$

be the space of piecewise Sobolev H^t-functions and denote its inner product, norm and semi–norm by $(\cdot,\cdot)_{t,h}$, $\|\cdot\|_{t,h}$ and $|\cdot|_{t,h}$. For simplicity denote $\boldsymbol{H}^t(\mathcal{K}_h) = H^t(\mathcal{K}_h) \times H^t(\mathcal{K}_h)$. Similarly, for any open subset $\gamma \subset \Gamma$ let denote by $(\cdot,\cdot)_\gamma$ and $\|\cdot\|_\gamma$ the inner product and norm in $L^2(\gamma)$.

For any $K \in \mathcal{K}_h$ let $\mathbf{n}_K$ be the outer normal to the boundary ∂K. Let K^- and K^+ be two distinct elements of $\mathcal{K}_h$ sharing the edge $e = K^- \bigcap K^+ \in \mathcal{E}_h^\circ$. We define the jump of $\phi \in H^1(\mathcal{K}_h)$ by

$$[\phi] = \phi^- \mathbf{n}^- + \phi^+ \mathbf{n}^+,$$

where $\phi^\pm = \phi|_{K^\pm}$ and $\mathbf{n}^\pm = \mathbf{n}_{K^\pm}$. For an vector function $\boldsymbol{\theta} \in \boldsymbol{H}^1(\mathcal{K}_h)$, define

$$[\boldsymbol{\theta}] = \boldsymbol{\theta}^- \cdot \mathbf{n}^- + \boldsymbol{\theta}^+ \cdot \mathbf{n}^+, \qquad [\![\boldsymbol{\theta}]\!] = \boldsymbol{\theta}^- \odot \mathbf{n}^- + \boldsymbol{\theta}^+ \odot \mathbf{n}^+,$$

where $\boldsymbol{\theta} \odot \mathbf{n} = (\boldsymbol{\theta}\mathbf{n}^T + \mathbf{n}\boldsymbol{\theta}^T)/2$. Note that the jump of a scalar function is a vector, and for a vector function $\boldsymbol{\theta}$, the jump $[\boldsymbol{\theta}]$ is a scalar, while the jump $[\![\boldsymbol{\theta}]\!]$ is a symmetric matrix. The average of scalar or vector function χ is defined by

$$\{\chi\} = \frac{1}{2}(\chi^- + \chi^+).$$

On a boundary faces $e \in \mathcal{E}_h^\partial \bigcap \partial K$ with outer normal $\mathbf{n}$, define the jumps and averages as

$$[\phi] = \phi|_K \mathbf{n}, \qquad [\boldsymbol{\theta}] = \boldsymbol{\theta}|_K \cdot \mathbf{n}, \qquad [\![\boldsymbol{\theta}]\!] = \boldsymbol{\theta}|_K \odot \mathbf{n}, \qquad \{\chi\} = \chi|_K.$$

Then, the following identities hold (Arnold et al., 2005) for smooth $\boldsymbol{\theta}$ and τ:

$$\sum_{K \in \mathcal{K}_h} \int_{\partial K} \boldsymbol{\theta} \cdot \mathbf{n}_K v = \sum_{e \in \mathcal{E}_h} \int_e \{\boldsymbol{\theta}\} \cdot [v], \tag{5}$$

$$\sum_{K \in \mathcal{K}_h} \int_{\partial K} \tau \mathbf{n}_K \cdot \boldsymbol{\eta} = \sum_{e \in \mathcal{E}_h} \int_e \{\tau\} : [\![\boldsymbol{\eta}]\!]. \tag{6}$$

From the shape-regularity, there exists a constant c such that on any face $e \in \mathcal{E}_h \bigcap \partial K$

$$h_e \leq h_K \leq c h_e.$$

Thus, the following multiplicative trace inequality holds (Thomée, 1997).

Lemma 1 *For a shape regular partition $\mathcal{K}_h$, there exists a constant c such that*

$$\|v\|_{0,\partial K}^2 \leq c\left(\frac{1}{h_K}\|v\|_{0,K}^2 + h_k |v|_{1,K}^2\right) \quad \textit{for all } v \in H^1(K), \textit{ and all } K \in \mathcal{K}_h. \tag{7}$$

For the biharmonic equation (1), the following symmetric discontinuous Galerkin formulation (Süli and Mozolevski, 2007) defines $\psi_h \in H^4(\mathcal{K}_h)$ such that

$$B_b(\psi_h, \phi) = (g, \phi) \quad \text{for all } \phi \in H^4(\mathcal{K}_h), \tag{8}$$

where the bilinear form $B_b(\psi, \phi) = B_{\mathcal{K}_h}(\psi, \phi) + B_\Gamma(\psi, \phi) + B_s(\psi, \phi)$, and

$$\begin{aligned}
B_{\mathcal{K}_h}(\psi, \phi) &= (\Delta \psi, \Delta \phi)_h, \\
B_\Gamma(\psi, \phi) &= \sum_{e \in \mathcal{E}_h} \big[(\{\boldsymbol{\nabla} \Delta \psi\}, [\phi])_e + ([\psi], \{\boldsymbol{\nabla} \Delta \phi\})_e - (\{\Delta \psi\}, [\boldsymbol{\nabla} \phi])_e \\
&\qquad - ([\boldsymbol{\nabla} \psi], \{\Delta \phi\})_e\big], \\
B_s(\psi, \phi) &= \sum_{e \in \mathcal{E}_h} \big[\alpha_e([\psi], [\phi])_e + \beta_e([\boldsymbol{\nabla} \psi], [\boldsymbol{\nabla} \phi])_e\big].
\end{aligned}$$

The positive stabilization parameters α and β with values α_e, β_e for $e \in \mathcal{E}_h$, are fixed at Lemma 3, to weakly impose the boundary conditions and inter-element continuity, and to stabilize the method.

We next derive a discontinuous Galerkin formulation for the Reissner–Mindlin problem that "recovers" (8) as $\varepsilon \to 0$. For smooth $(\boldsymbol{\theta}, \omega)$, multiplying both sides of the first equation in (2) by $\boldsymbol{\eta} \in \boldsymbol{H}^3(\mathcal{K}_h)$ and integrating by parts over an element K, we get

$$a_K(\boldsymbol{\theta}, \boldsymbol{\eta}) + \epsilon^{-2}\mu(\boldsymbol{\theta} - \boldsymbol{\nabla}\,\omega, \boldsymbol{\eta})_K - (\mathcal{C}\, e(\boldsymbol{\theta})\mathbf{n}, \boldsymbol{\eta})_{\partial K} = 0, \tag{9}$$

where $a_K(\boldsymbol{\theta}, \boldsymbol{\eta}) = \int_K \mathcal{C}\, e(\boldsymbol{\theta}) : e(\boldsymbol{\eta})\, d\boldsymbol{x}$. In the same way, from the second equation in (2), for any $\nu \in H^1(\mathcal{K}_h)$ we obtain

$$-\epsilon^{-2}\mu(\boldsymbol{\theta} - \boldsymbol{\nabla}\,\omega, \boldsymbol{\nabla}\,\nu)_K + \epsilon^{-2}\mu((\boldsymbol{\theta} - \boldsymbol{\nabla}\,\omega)\cdot \mathbf{n}, \nu)_{\partial K} = (g, \nu)_K.$$

To eliminate $\boldsymbol{\theta} - \boldsymbol{\nabla}\,\omega$ in the second term of the above equation, we use the first equation in (2), and

$$-\epsilon^{-2}\mu(\boldsymbol{\theta} - \boldsymbol{\nabla}\,\omega, \boldsymbol{\nabla}\,\nu)_K + (\mathbf{div}\,\mathcal{C}\, e(\boldsymbol{\theta})\cdot \mathbf{n}, \nu)_{\partial K} = (g, \nu)_K. \tag{10}$$

Summing (9), (10) over the elements, and from (5), (6), it follows that

$$\begin{aligned}
&a_h(\boldsymbol{\theta}, \boldsymbol{\eta}) + \epsilon^{-2}\mu(\boldsymbol{\theta} - \boldsymbol{\nabla}\,\omega, \boldsymbol{\eta})_h + \sum_{e\in\mathcal{E}_h} -(\mathcal{C}\{e(\boldsymbol{\theta})\}, [\![\boldsymbol{\eta}]\!])_e = 0,\\
&-\epsilon^{-2}\mu(\boldsymbol{\theta} - \boldsymbol{\nabla}\,\omega, \boldsymbol{\nabla}\,\nu)_h + \sum_{e\in\mathcal{E}_h} (\{\mathbf{div}\,\mathcal{C}\, e(\boldsymbol{\theta})\}, [\nu])_e = (g, \nu),
\end{aligned}$$

where $a_h(\boldsymbol{\theta}, \boldsymbol{\eta}) = (\mathcal{C}\, e(\boldsymbol{\theta}) : e(\boldsymbol{\eta}))_h$. Adding the symmetrization and penalization terms,

$$\begin{aligned}
a_h(\boldsymbol{\theta}, \boldsymbol{\eta}) + \epsilon^{-2}\mu(\boldsymbol{\theta} - \boldsymbol{\nabla}\,\omega, \boldsymbol{\eta})_h + &\sum_{e\in\mathcal{E}_h} -(\mathcal{C}\{e(\boldsymbol{\theta})\}, [\![\boldsymbol{\eta}]\!])_e - ([\![\boldsymbol{\theta}]\!], \mathcal{C}\{e(\boldsymbol{\eta})\})_e\\
&+ ([\omega], \{\mathbf{div}\,\mathcal{C}\, e(\boldsymbol{\eta})\})_e + \beta_e([\![\boldsymbol{\theta}]\!], [\![\boldsymbol{\eta}]\!])_e = 0,\\
-\epsilon^{-2}\mu(\boldsymbol{\theta} - \boldsymbol{\nabla}\,\omega, \boldsymbol{\nabla}\,\nu)_h + &\sum_{e\in\mathcal{E}_h} (\{\mathbf{div}\,\mathcal{C}\, e(\boldsymbol{\theta})\}, [\nu])_e + \alpha_e([\omega], [\nu])_e = (g, \nu).
\end{aligned} \tag{11}$$

The system (11) corresponds to the critical point of

$$\begin{aligned}
\frac{1}{2}a_h(\boldsymbol{\eta}, \boldsymbol{\eta}) + \sum_{e\in\mathcal{E}_h} \Big[&-(\{\mathcal{C}\, e(\boldsymbol{\eta})\}, [\![\boldsymbol{\eta}]\!])_e + \frac{\beta_e}{2}([\![\boldsymbol{\eta}]\!], [\![\boldsymbol{\eta}]\!])_e + ([\nu], \{\mathbf{div}\,\mathcal{C}\, e(\boldsymbol{\eta})\})_e\\
&+ \frac{\alpha_e}{2}([\nu], [\nu])_e\Big] + \epsilon^{-2}\mu(\boldsymbol{\eta} - \boldsymbol{\nabla}\,\nu, \boldsymbol{\eta} - \boldsymbol{\nabla}\,\nu)_h - (g, \nu).
\end{aligned}$$

As $\varepsilon \to 0$,

$$\begin{aligned}
\frac{1}{2}a_h(\boldsymbol{\nabla}\,\nu, \boldsymbol{\nabla}\,\nu) + \sum_{e\in\mathcal{E}_h} \Big[&-(\{\mathcal{C}\, e(\boldsymbol{\nabla}\,\nu)\}, [\![\boldsymbol{\nabla}\,\nu]\!])_e + \frac{\beta_e}{2}([\![\boldsymbol{\nabla}\,\nu]\!], [\![\boldsymbol{\nabla}\,\nu]\!])_e\\
&+ ([\nu], \{\mathbf{div}\,\mathcal{C}\, e(\boldsymbol{\nabla}\,\nu)\})_e + \frac{\alpha_e}{2}([\nu], [\nu])_e\Big] - (g, \nu).
\end{aligned}$$

The variational formulation characterizing the minimum of this functional is

$$\begin{aligned}
a_h(\boldsymbol{\nabla}\,\omega, \boldsymbol{\nabla}\,\nu) + \sum_{e\in\mathcal{E}_h} \Big[&-(\{\mathcal{C}\, e(\boldsymbol{\nabla}\,\omega)\}, [\![\boldsymbol{\nabla}\,\nu]\!])_e - ([\![\boldsymbol{\nabla}\,\omega]\!], \{\mathcal{C}\, e(\boldsymbol{\nabla}\,\nu)\})_e\\
+ \beta_e([\![\boldsymbol{\nabla}\,\omega]\!], [\![\boldsymbol{\nabla}\,\nu]\!])_e &+ (\{\mathbf{div}\,\mathcal{C}\, e(\boldsymbol{\nabla}\,\omega)\}, [\nu])_e + ([\omega], \{\mathbf{div}\,\mathcal{C}\, e(\boldsymbol{\nabla}\,\nu)\})_e + \alpha_e([\omega], [\nu])_e\Big]\\
&= (g, \nu).
\end{aligned} \tag{12}$$

After integrating by parts, it follows that (12) recovers as $\varepsilon \to 0$, a variant of the formulation for the biharmonic equation from (Süli and I. Mozolevski, 2007).

We now consider a more general, possibly non-symmetric, formulation, depending on λ_1, $\lambda_2 \in [-1, 1]$. Let $(\boldsymbol{\theta}, \omega) \in \boldsymbol{H}^3(\mathcal{K}_h) \times H^1(\mathcal{K}_h)$ satisfy

$$\mathcal{A}(\boldsymbol{\theta}, \omega; \boldsymbol{\eta}, \nu) = (g, \nu), \quad \text{for all } (\boldsymbol{\eta}, \nu) \in \boldsymbol{H}^3(\mathcal{K}_h) \times H^1(\mathcal{K}_h), \tag{13}$$

where

$$\begin{aligned}\mathcal{A}(\boldsymbol{\theta}, \omega; \boldsymbol{\eta}, \nu) = a_h(\boldsymbol{\theta}, \boldsymbol{\eta}) + \epsilon^{-2}\mu(\boldsymbol{\theta} - \boldsymbol{\nabla}\, \omega, \boldsymbol{\eta} - \boldsymbol{\nabla}\, \nu)_h + \lambda_1 \mathcal{A}_1(\boldsymbol{\eta}, \omega) + \mathcal{A}_1(\boldsymbol{\theta}, \nu) \\ - \mathcal{A}_2(\boldsymbol{\theta}, \boldsymbol{\eta}) - \lambda_2 \mathcal{A}_2(\boldsymbol{\eta}, \boldsymbol{\theta}) + \mathcal{A}_\alpha(\omega, \nu) + \mathcal{A}_\beta(\boldsymbol{\theta}, \boldsymbol{\eta}),\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}_1(\boldsymbol{\eta}, \omega) &= \sum_{e \in \mathcal{E}_h} ([\omega], \{\mathbf{div}\, \mathcal{C}\, e(\boldsymbol{\eta})\})_e, \qquad \mathcal{A}_2(\boldsymbol{\theta}, \boldsymbol{\eta}) = \sum_{e \in \mathcal{E}_h} (\mathcal{C}\{e(\boldsymbol{\theta})\}, [\![\boldsymbol{\eta}]\!])_e, \\ \mathcal{A}_\alpha(\omega, \nu) &= \sum_{e \in \mathcal{E}_h} \alpha_e([\omega], [\nu])_e, \qquad \mathcal{A}_\beta(\boldsymbol{\theta}, \boldsymbol{\eta}) = \sum_{e \in \mathcal{E}_h} \beta_e([\![\boldsymbol{\theta}]\!], [\![\boldsymbol{\eta}]\!])_e.\end{aligned}$$

In case $\lambda_1 = \lambda_2 = 1$, the above formulation is symmetric.

3. Discontinuous Galerkin finite element method

Let $\mathcal{P}_p(K)$ be the space of polynomials with total degree less or equal to p in $K \in \mathcal{K}_h$. We introduce the global discontinuous finite element space

$$\mathcal{S}^{p,h}(\mathcal{K}_h) = \{v \in L^2(\Omega) : \, v_{|K} \in \mathcal{P}_p(K) \quad \text{for all } K \in \mathcal{K}_h\}.$$

To formulate the method let $p \geq 2$, and the finite element spaces $\boldsymbol{\Theta}_h = \mathcal{S}^{(p-1),h}(\mathcal{K}_h) \times \mathcal{S}^{(p-1),h}(\mathcal{K}_h)$, and $W_h = \mathcal{S}^{p,h}(\mathcal{K}_h)$. We define $(\boldsymbol{\theta}_h, \omega_h) \in \boldsymbol{\Theta}_h \times W_h$ such that

$$\mathcal{A}(\boldsymbol{\theta}_h, \omega_h; \boldsymbol{\eta}, \nu) = (g, \nu), \quad \text{for all } (\boldsymbol{\eta}, \nu) \in \boldsymbol{\Theta}_h \times W_h. \tag{14}$$

Note that this formulation is consistent with Reissner–Mindlin problems (2) that admit sufficiently smooth solutions. In this case, the Galerkin orthogonality

$$\mathcal{A}(\boldsymbol{\theta} - \boldsymbol{\theta}_h, \omega - \omega_h; \boldsymbol{\eta}, \nu) = 0 \quad \text{for all } (\boldsymbol{\eta}, \nu) \in \boldsymbol{\Theta}_h \times W_h \tag{15}$$

holds.

Consider the following norm for $(\boldsymbol{\eta}, \nu) \in \boldsymbol{H}^3(\mathcal{K}_h) \times H^1(\mathcal{K}_h)$:

$$\begin{aligned}|\!|\!| \boldsymbol{\eta}, \nu |\!|\!|^2 = \|\, e(\boldsymbol{\eta})\|^2_{0,h} + \epsilon^{-2}\|\boldsymbol{\eta} - \boldsymbol{\nabla}\, \nu\|^2_{0,h} + \|\sqrt{\alpha}[\nu]\|^2_\Gamma + \|\sqrt{\beta}[\![\boldsymbol{\eta}]\!]\|^2_\Gamma \\ + \left\| \frac{1}{\sqrt{\alpha}}\{\mathbf{div}\, \mathcal{C}\, e(\boldsymbol{\eta})\} \right\|^2_\Gamma + \left\| \frac{1}{\sqrt{\beta}}\{\mathcal{C}\, e(\boldsymbol{\eta})\} \right\|^2_\Gamma,\end{aligned}$$

for $p \geq 3$, and

$$|\!|\!| \boldsymbol{\eta}, \nu |\!|\!|^2 = \|\, e(\boldsymbol{\eta})\|^2_{0,h} + \epsilon^{-2}\|\boldsymbol{\eta} - \boldsymbol{\nabla}\, \nu\|^2_{0,h} + \|\sqrt{\alpha}[\nu]\|^2_\Gamma + \|\sqrt{\beta}[\![\boldsymbol{\eta}]\!]\|^2_\Gamma + \left\| \frac{1}{\sqrt{\beta}}\{\mathcal{C}\, e(\boldsymbol{\eta})\} \right\|^2_\Gamma,$$

for $p = 2$. The bilinear form $\mathcal{A}$ is continuous in $(\boldsymbol{H}^3(\mathcal{K}_h) \times H^1(\mathcal{K}_h))^2$ in respect to this norm.

Lemma 2 *For a shape regular partition $\mathcal{K}_h$, there exists a positive constant c, such that for all $\big((\boldsymbol{\theta},\omega);(\boldsymbol{\eta},\nu)\big) \in \big(\boldsymbol{H}^3(\mathcal{K}_h)\times H^1(\mathcal{K}_h)\big)^2$,*

$$|\mathcal{A}(\boldsymbol{\theta},\omega;\boldsymbol{\eta},\nu)| \leq c|\!|\!|\boldsymbol{\theta},\omega|\!|\!|\,|\!|\!|\boldsymbol{\eta},\nu|\!|\!|, \tag{16}$$

where c is independent of h_K, $K \in \mathcal{K}_h$.

Let us prove now the coercivity of bilinear form $\mathcal{A}$ in discrete space.

Lemma 3 *Let $\mathcal{K}_h$ be a shape regular partition, and assume that* (3) *holds. Then there exist positive constants $\hat{\sigma}_\alpha$, $\hat{\sigma}_\beta$ such that if $\sigma_\alpha \geq \hat{\sigma}_\alpha$, $\sigma_\beta \geq \hat{\sigma}_\beta$, and*

$$\alpha_e = \frac{\sigma_\alpha}{h_e^3}, \quad \beta_e = \frac{\sigma_\beta}{h_e} \qquad \text{for } e \in \mathcal{E}_h, \tag{17}$$

then there exists a positive constant ζ such that

$$\mathcal{A}(\boldsymbol{\theta},\omega;\boldsymbol{\theta},\omega) \geq \zeta|\!|\!|\boldsymbol{\theta},\omega|\!|\!|^2 \quad \text{for all } (\boldsymbol{\theta},\omega) \in \boldsymbol{\Theta}_h \times W_h. \tag{18}$$

4. A priori error analysis

From the continuity and coercivity of the bilinear form $\mathcal{A}$ in discrete spaces, the error analysis follows from standard techniques. Let $(\boldsymbol{\theta},\omega)$ denote the exact solution of (2), and $(\boldsymbol{\theta}_h,\omega_h)$ its approximation, the solution of (14). Next, let $(\boldsymbol{\theta}^i,\omega^i)$ be an interpolant of $(\boldsymbol{\theta},\omega)$ in $\boldsymbol{\Theta}_h \times W_h$. We start by decomposing the approximation errors as follows:

$$\boldsymbol{\theta} - \boldsymbol{\theta}_h = (\boldsymbol{\theta} - \boldsymbol{\theta}^i) + (\boldsymbol{\theta}^i - \boldsymbol{\theta}_h) \equiv e^i_{\boldsymbol{\theta}} - \xi_{\boldsymbol{\theta}}, \tag{19}$$

$$\omega - \omega_h = (\omega - \omega^i) + (\omega^i - \omega_h) \equiv e^i_\omega - \xi_\omega. \tag{20}$$

From the continuity and coercivity of the bilinear form, and the Galerkin orthogonality (15) it follows that

$$\begin{aligned}
|\!|\!|\xi_{\boldsymbol{\theta}},\xi_\omega|\!|\!|^2 &\leq \zeta\mathcal{A}(\xi_{\boldsymbol{\theta}},\xi_\omega;\xi_{\boldsymbol{\theta}},\xi_\omega) = \zeta\mathcal{A}(e^i_{\boldsymbol{\theta}} - \boldsymbol{\theta} + \boldsymbol{\theta}_h, e^i_\omega - \omega + \omega_h;\xi_{\boldsymbol{\theta}},\xi_\omega)\\
&= \zeta\mathcal{A}(e^i_{\boldsymbol{\theta}}, e^i_\omega;\xi_{\boldsymbol{\theta}},\xi_\omega) - \zeta\mathcal{A}(\boldsymbol{\theta} - \boldsymbol{\theta}_h, \omega - \omega_h;\xi_{\boldsymbol{\theta}},\xi_\omega) = \zeta\mathcal{A}(e^i_{\boldsymbol{\theta}}, e^i_\omega;\xi_{\boldsymbol{\theta}},\xi_\omega)\\
&\leq c|\!|\!|e^i_{\boldsymbol{\theta}}, e^i_\omega|\!|\!|\,|\!|\!|\xi_{\boldsymbol{\theta}},\xi_\omega|\!|\!|,
\end{aligned}$$

and consequently, $|\!|\!|\xi_{\boldsymbol{\theta}},\xi_\omega|\!|\!| \leq c|\!|\!|e^i_{\boldsymbol{\theta}}, e^i_\omega|\!|\!|$. This means that

$$|\!|\!|\boldsymbol{\theta} - \boldsymbol{\theta}_h;\omega - \omega_h|\!|\!| \leq c|\!|\!|e^i_{\boldsymbol{\theta}}, e^i_\omega|\!|\!|, \tag{21}$$

and to estimate the error of the method it is enough to estimate the interpolation error.

Proceeding as in (Arnold et al., 2007) to choose appropriate interpolants, let us denote by π_W the projection of $L^2(\Omega)$ onto $W_h \cap H^1(\Omega)$. For $\omega \in H^{p+1}(\Omega)$, let $\omega^i = \pi_W\omega$. It follows then that for $0 \leq q \leq p+1$, there exists a constant c such that

$$\left\|\omega - \omega^i\right\|_{q,h} \leq ch^{p+1-q}\left\|\omega\right\|_{p+1,\Omega} \quad \text{for all } \omega \in H^{p+1}(\Omega). \tag{22}$$

Consider the rotated Brezzi–Douglas–Marini space BDM^R_{p-1} of degree $p-1$, i.e., the space of all piecewise polynomial vector fields of degree at most $p-1$ subject to inter-element continuity of the tangential components; obviously $\mathrm{BDM}^R_{p-1} \subset \boldsymbol{\Theta}_h$. Let $\boldsymbol{\pi}_{\boldsymbol{\Theta}}$ denotes the the natural projector of $\boldsymbol{H}^1(\Omega)$ into BDM^R_{p-1}. Note that $\boldsymbol{\nabla} W_h \subseteq \boldsymbol{\Theta}_h$, and the following commutativity property of the projectors follows from integration by parts:

$$\boldsymbol{\pi}_{\boldsymbol{\Theta}}\boldsymbol{\nabla}\,\omega = \boldsymbol{\nabla}\,\pi_W\omega. \tag{23}$$

So, let $\boldsymbol{\theta}^i = \boldsymbol{\pi}_{\boldsymbol{\Theta}}\boldsymbol{\theta}$ be the interpolator of $\boldsymbol{\theta} \in \boldsymbol{H}^1(\Omega)$. Defining $\boldsymbol{\gamma} = \epsilon^{-2}(\boldsymbol{\theta} - \boldsymbol{\nabla}\,\omega)$ as the shear stress vector, and $\boldsymbol{\gamma}^i = \epsilon^{-2}(\boldsymbol{\theta}^i - \boldsymbol{\nabla}\,\omega^i)$, it follows that

$$\boldsymbol{\pi}_{\boldsymbol{\Theta}}\boldsymbol{\gamma} = \epsilon^{-2}\boldsymbol{\pi}_{\boldsymbol{\Theta}}(\boldsymbol{\theta} - \boldsymbol{\nabla}\,\omega) = \epsilon^{-2}(\boldsymbol{\pi}_{\boldsymbol{\Theta}}\boldsymbol{\theta} - \boldsymbol{\nabla}\,\pi_W\omega) = \epsilon^{-2}(\boldsymbol{\theta}^i - \boldsymbol{\nabla}\,\omega^i) = \boldsymbol{\gamma}^i.$$

Thus, $\boldsymbol{\gamma}^i$ interpolates $\boldsymbol{\gamma}$, and with this key condition, the next results for interpolation error estimates holds (Arnold et al., 2007). For $0 \le s \le l$, and $1 \le l \le p$

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^i\|_{s,h} \le ch^{l-s}\|\boldsymbol{\theta}\|_{l,\Omega} \quad \text{for all } \boldsymbol{\theta} \in \boldsymbol{H}^l(\Omega), \tag{24}$$

$$\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^i\|_{s,h} \le ch^{l-s}\|\boldsymbol{\gamma}\|_{l,\Omega} \quad \text{for all } \boldsymbol{\gamma} \in \boldsymbol{H}^l(\Omega). \tag{25}$$

The main result of this paper is the following.

Theorem 4 *Let $\Omega \subset \mathbb{R}^2$ be an polygonal convex domain, and let $\mathcal{K}_h$ be a shape regular partition on Ω. Assume that the penalization parameters α and β are so that $\mathcal{A}(\cdot,\cdot;\cdot,\cdot)$ is coercive (according to Lemma 3). Assume also that the solution to* (2) *satisfy $(\boldsymbol{\theta}, \omega) \in \boldsymbol{H}^p(\Omega) \times H^{p+1}(\Omega)$, and that $p \ge 2$. Then $(\boldsymbol{\theta}_h, \omega_h) \in \boldsymbol{\Theta}_h \times W_h$, solution of discontinuous Galerkin finite element method* (14)*, satisfy*

$$|\!|\!|\boldsymbol{\theta} - \boldsymbol{\theta}_h; \omega - \omega_h|\!|\!|^2 \le ch^{2p-2}\left(\|\boldsymbol{\theta}\|_p^2 + \|\omega\|_{p+1}^2 + \epsilon^2\|\boldsymbol{\gamma}\|_{p-1}^2\right), \tag{26}$$

where c does not depend on h or ϵ.

Remark 5 *Note that estimate* (26) *holds any $\boldsymbol{\Theta}_h$ containing BDM_{p-1}^R (Arnold et al., 2007). In fact, in the proof of Theorem 4, it was enough that the projection $\boldsymbol{\pi}_{\boldsymbol{\Theta}}$ is well-defined and that* (23) *holds. Particular choices include the case where W_h has only continuous functions, and the case of equal interpolation degree for all the unknowns, i.e., $\boldsymbol{\Theta}_h = \mathcal{S}^{p,h}(\mathcal{K}_h)\times\mathcal{S}^{p,h}(\mathcal{K}_h)$. This is particularly useful since using equal order interpolation for all spaces might make the computational implementation easier.*

5. Numerical Results

We consider now some numerical tests that display the performance of our method. We start by adapting the solution given in (Chinosi et al., 2006), and it follows that

$$\begin{aligned}
&\omega_1(x,y) = \frac{1}{3}x^3(x-1)^3y^3(y-1)^3,\\
&\omega_2(x,y) = y^3(y-1)^3x(x-1)(5x^2-5x+1) + x^3(x-1)^3y(y-1)(5y^2-5y+1),\\
&\omega(x,y) = \omega_1(x,y) - \epsilon^2\frac{8(\mu+\lambda)}{3(2\mu+\lambda)}\omega_2(x,y),\\
&\theta_1(x,y) = y^3(y-1)^3x^2(x-1)^2(2x-1), \quad \theta_2(x,y) = x^3(x-1)^3y^2(y-1)^2(2y-1),
\end{aligned}$$

solves (2) in $\Omega = (0,1)\times(0,1)$ with

$$\begin{aligned}
g = \frac{4(\mu+\lambda)\mu}{3(2\mu+\lambda)}\{&12y(y-1)(5x^2-5x+1)[2y^2(y-1)^2 + x(x-1)(5y^2-5y+1)]\\
&+ 12x(x-1)(5y^2-5y+1)[2x^2(x-1)^2 + y(y-1)(5x^2-5x+1)]\}.
\end{aligned}$$

In our numerical simulations we set the Lamé coefficients $\lambda = \mu = 1$.

Implementing the DG method described above in the PZ environment (Devloo, 1997), we proceed to check the convergence of the scheme (14) for the symmetric ($\lambda_1 = \lambda_2 = 1$), and

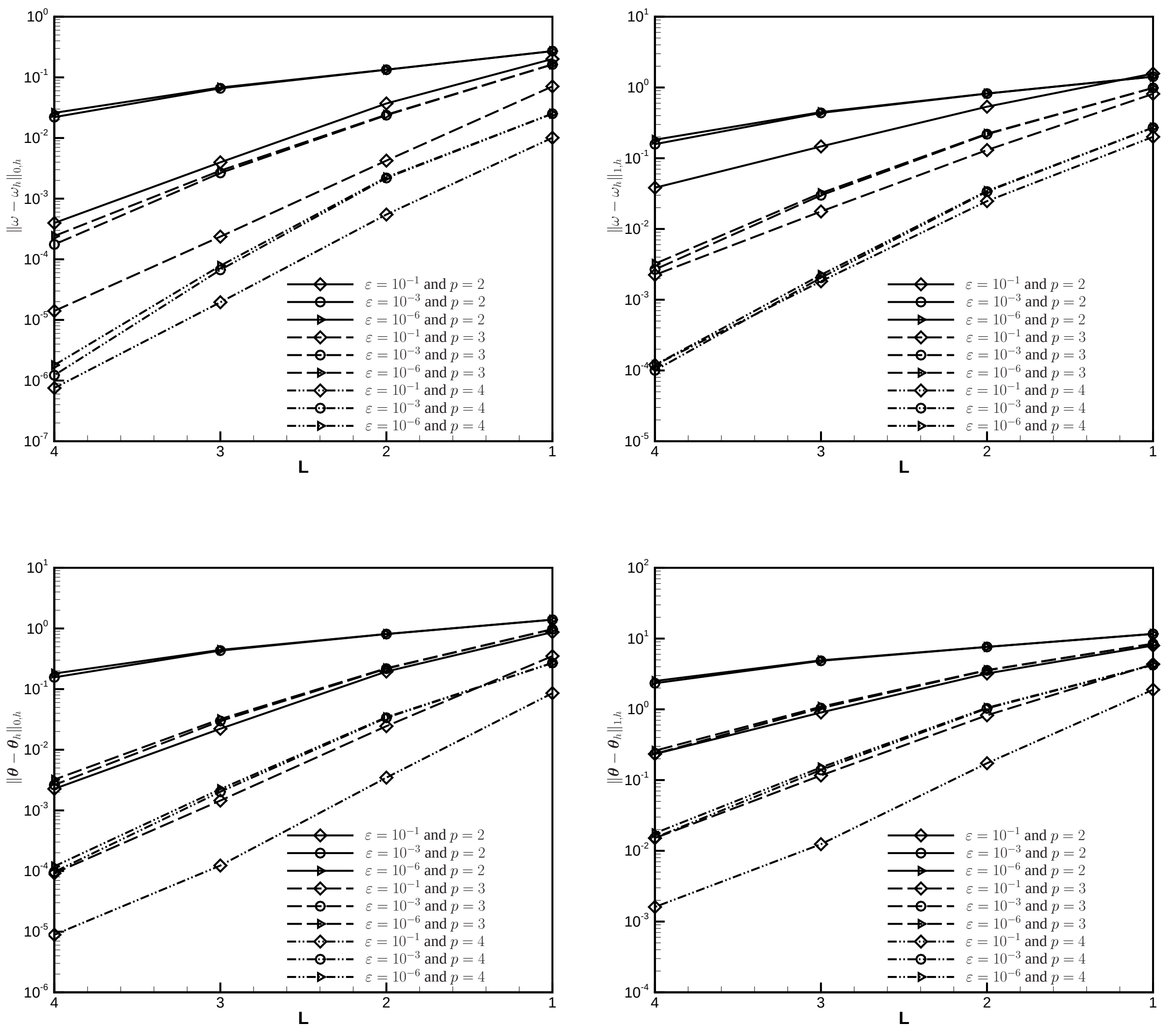

Figure 1: Errors in ω (top), and $\boldsymbol{\theta}$ (bottom), with respect to the refinement level L. At the left we consider the L_2 norm, and at the right the $H^1(\mathcal{T}_h)$ norm. All the results are for the symmetric formulation. We considered $p = 2, 3, 4$ and $\epsilon = 10^{-1}, 10^{-3}, 10^{-6}$.

non-symmetric ($\lambda_1 = \lambda_2 = -1$) cases. In both cases, we pick $\sigma_\alpha = \sigma_{\beta_e} = 10$. We used $\boldsymbol{\Theta}_h = \mathcal{S}^{p,h}(\mathcal{K}_h) \times \mathcal{S}^{p,h}(\mathcal{K}_h)$. As noted in Remark 5, the converge rates obtained in Theorem 4 are still valid under this choice.

We successively divide the domain using 2^{2L+1} triangles. Thus, if e_L denotes the error at the level of refinement L, the rate of convergence for such level is given by

$$r_L = \log\left(\frac{e_L}{e_{L-1}}\right) / \log(0.5).$$

Figure 1 shows the error of the symmetric method for the vertical displacement at the top, and for the rotation at the bottom, as a function of the refinement level for several approximation orders p and for different values of ϵ. The errors were in the L^2 norm at the left column, and the H^1 norm at the right column. The non-symmetric version of the method yields similar results. We observe that in the H^1 norm, the errors for all approximation orders exhibit similar behavior for ω and $\boldsymbol{\theta}$, confirming that in fact the constant in bound (26) does not depend on thickness ϵ. Note that, in the H^1 norm, the errors of approximation of vertical displacement is almost the same for all thickness, for a given approximation order, and the rotation error is less uniform in ϵ, since the method approximates better the rotation for thicker plates. The errors in the L^2 norm are significantly better than in the H^1 norm and exhibit a similar behavior in respect to ϵ. We stress that the results are locking free.

Table 1: Numerical convergence with the symmetric formulation and triangles

		e_ω with $L^2(\mathcal{T}_h)$			e_ω with $H^1(\mathcal{T}_h)$			$e_{\boldsymbol{\theta}}$ with $L^2(\mathcal{T}_h)$			$e_{\boldsymbol{\theta}}$ with $H^1(\mathcal{T}_h)$		
p	$L\backslash\varepsilon$	10^{-1}	10^{-3}	10^{-6}	10^{-1}	10^{-3}	10^{-6}	10^{-1}	10^{-3}	10^{-6}	10^{-1}	10^{-3}	10^{-6}
2	2	2.4	1.0	1.0	1.5	0.8	0.8	2.1	0.8	0.8	1.3	0.6	0.6
	3	3.2	1.0	1.0	1.9	0.9	0.9	3.1	0.9	0.9	1.8	0.6	0.6
	4	3.3	1.6	1.4	1.9	1.5	1.3	3.3	1.5	1.3	1.9	1.0	1.0
3	2	4.0	2.8	2.7	2.6	2.2	2.2	3.8	2.2	2.1	2.4	1.3	1.2
	3	4.2	3.2	3.1	2.9	2.9	2.8	4.1	2.9	2.8	2.8	1.8	1.7
	4	4.1	3.9	3.6	3.0	3.5	3.3	4.0	3.5	3.3	2.9	2.1	2.1
4	2	4.2	3.5	3.5	3.0	3.0	3.0	4.6	3.0	3.0	3.5	2.0	2.0
	3	4.8	5.0	4.8	3.8	4.0	3.9	4.8	4.1	3.9	3.8	2.9	2.8
	4	4.7	5.8	5.4	3.9	4.3	4.2	3.8	4.4	4.2	2.9	3.2	3.1

Table 2: Numerical convergence with the non-symmetric formulation and triangles

		e_ω with $L^2(\mathcal{T}_h)$			e_ω with $H^1(\mathcal{T}_h)$			$e_{\boldsymbol{\theta}}$ with $L^2(\mathcal{T}_h)$			$e_{\boldsymbol{\theta}}$ with $H^1(\mathcal{T}_h)$		
p	$L\backslash\varepsilon$	10^{-1}	10^{-3}	10^{-6}	10^{-1}	10^{-3}	10^{-6}	10^{-1}	10^{-3}	10^{-6}	10^{-1}	10^{-3}	10^{-6}
2	2	2.4	1.0	1.0	1.6	0.8	0.8	2.2	0.8	0.8	1.4	0.6	0.6
	3	3.0	1.0	1.0	1.9	0.9	0.9	2.9	0.9	0.9	1.8	0.7	0.6
	4	2.8	1.6	1.4	1.9	1.5	1.3	2.8	1.5	1.3	2.0	1.1	1.0
3	2	4.0	2.7	2.7	2.6	2.1	2.1	3.8	2.1	2.1	2.4	1.3	1.2
	3	4.1	3.0	2.9	2.9	2.8	2.7	4.1	2.8	2.7	2.8	1.8	1.7
	4	4.0	3.2	3.0	3.0	3.3	3.1	4.0	3.3	3.1	2.9	2.1	2.1
4	2	4.2	3.5	3.4	3.0	3.0	3.0	4.7	3.0	3.0	3.5	2.0	2.0
	3	4.8	4.9	4.7	3.8	4.1	4.0	4.9	4.1	4.0	3.8	2.9	2.8
	4	4.6	4.8	4.5	3.9	4.4	4.2	3.9	4.4	4.2	3.0	3.2	3.1

We now investigate the convergence rates for both the vertical displacements and rotations. Table 1 contains the results for the symmetric formulation and in Table 2, we display the convergence rates for the non-symmetric formulation. Since the norm $\|\cdot\|_{1,h}$ is bounded by above by a constant times $\| e(\cdot)\|_{0,h} + \|\beta\cdot\|_\Gamma$ (see Lemma 4.6 of (Arnold et al., 2005), from the theoretically predicted rate of convergence for energy norm follows that the rate of convergence of the error of rotation in the H^1 norm should to be $p-1$ in both formulations. This is clearly confirmed by our numerical experiments as can be seen in the last column of both tables 1 and 2 that contain the results for $e_{\boldsymbol{\theta}}$ in $H^1(\mathcal{K}_h)$ norm. The convergence order in L^2 norm for the vertical displacement is approximatively $p+1, p>2$ for symmetric version, that coincides with the similar results for biharmonic equation (Süli and I. Mozolevski, 2007). We remark also that for the non-symmetric version this rates are reduced when compared to the symmetric case. For all the other cases, both formulations display similar results for all p. For the sake of comparison, we include in Table 3 some numerical results for the symmetric formulation using uniform quadrilateral meshes. The convergence rates here are similar to the case of mesh with triangles.

Table 3: Numerical convergence with the symmetric formulation and rectangles

		e_ω em $L^2(\mathcal{T}_h)$			e_ω em $H^1(\mathcal{T}_h)$			$e_{\boldsymbol{\theta}}$ em $L^2(\mathcal{T}_h)$			$e_{\boldsymbol{\theta}}$ em $H^1(\mathcal{T}_h)$		
p	$r_L\backslash\varepsilon$	10^{-1}	10^{-3}	10^{-6}	10^{-1}	10^{-3}	10^{-6}	10^{-1}	10^{-3}	10^{-6}	10^{-1}	10^{-3}	10^{-6}
2	2	3.7	3.2	3.2	2.5	2.3	2.3	3.1	2.3	2.3	1.6	1.2	1.2
	3	3.2	1.8	1.8	1.9	1.5	1.5	2.9	1.5	1.5	1.9	0.6	0.6
	4	3.1	1.9	1.9	2.0	1.9	1.9	3.0	1.9	1.9	2.0	0.9	0.9
3	2	2.6	2.3	2.3	1.5	1.6	1.6	3.6	1.6	1.6	2.7	0.9	0.8
	3	3.9	3.9	3.9	2.8	2.8	2.8	4.0	2.8	2.8	3.0	1.9	1.9
	4	3.9	4.0	4.0	3.0	3.0	3.0	3.9	3.0	3.0	2.9	2.0	2.0
4	2	4.92	4.7	4.7	3.9	3.8	3.8	5.0	3.8	3.8	4.0	2.9	2.9
	3	4.8	4.8	4.8	4.0	4.0	4.0	4.1	4.0	4.0	3.3	3.0	3.0
	4	3.8	4.3	4.2	4.0	4.0	4.0	2.7	4.0	4.0	1.9	3.0	3.0

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