New Models for Heterogeneous Plates

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Abstract. *We investigate the problem of deriving provably good two-dimensional models for fiber reinforced plates. For simplicity we deal with the (scalar) steady state heat problem. The difficulty when dealing with such problems is the presence of two small parameters, the parameter related to the heterogeneity, and the plate thickness. The usual asymptotic based methods yield models that are reliable only when the two parameters hold a certain relation. Here we consider another venue, using a hierarchical approach to perform the dimension reduction. This leads to a multiscale two-dimensional problem that approximates well the original three-dimensional problem, regardless of the relative size of the parameters.*

Keywords: *dimension reduction, heterogeneous materials*

1. INTRODUCTION

We consider here problems posed in thin three-dimensional heterogeneous and thin plate. The interest in this problem comes from the fact that it is not clear what is a "reasonable" two-dimensional model that approximates in some sense the original problem. Our final goal is to investigate the elastic case, but for simplicity, and as a first step, we restrict ourselves to the steady state heat (Poisson) equation.

We consider here the presence of two distinct "small" parameters, the plate half-thickness δ, and the heterogeneity characteristic lenght ϵ. Depending whether $\epsilon \ll \delta$ or $\delta \ll \epsilon$, different two-dimensional models can be used. Of course this represents a drawback, since it would be nicer to have a single model that approximates both case. That is our goal.

Let the plate $P^\delta = \Omega \times (-\delta, \delta)$, where Ω is a two-dimensional "nice" domain. A typical point in P^δ is denoted by $\underline{x} = (\underset{\sim}{x}, x_3)$, where $\underset{\sim}{x} \in \Omega$, and $x_3 \in (-\delta, \delta)$. Let $P^\delta_\pm = \Omega \times \{-\delta, \delta\}$ be the plate's top and bottom, and $P^\delta_L = \partial\Omega \times [-\delta, \delta]$ its lateral side.

Let $u^{\delta\epsilon}_{3D}$ be the solution of the problem

$$\begin{gathered} -\operatorname{div}\left[\underline{\underline{a}}^\epsilon \underline{\nabla} u^{\delta\epsilon}_{3D}\right] = f^\delta \quad \text{in } P^\delta, \\ \left(\underline{\underline{a}}^\epsilon \underline{\nabla} u^{\delta\epsilon}_{3D}\right) \cdot \underline{n} = 0 \quad \text{on } \partial P^\delta_\pm, \qquad u^{\delta\epsilon}_{3D} = 0 \quad \text{on } \partial P^\delta_L. \end{gathered} \tag{1}$$

The 3×3 matrix $\underline{\underline{a}}^\epsilon$ is symmetric positive definite, and is given by

$$\underline{\underline{a}}^\epsilon(\underset{\sim}{x}, x_3) = \begin{pmatrix} \underset{\approx}{a}^\epsilon(\underset{\sim}{x}) & \underset{\sim}{a}^\epsilon(\underset{\sim}{x}) \\ \underset{\sim}{a}^\epsilon(\underset{\sim}{x})^T & a^\epsilon_{33}(\underset{\sim}{x}) \end{pmatrix}.$$

The 2×2 submatrix $\underset{\approx}{a}^\epsilon$ is also symmetric postive definite, and $\underset{\sim}{a}^\epsilon$ is a vector in $\mathbb{R}^2$. Note that we assume that $\underline{\underline{a}}^\epsilon$ is constant in the transverse direction but varies in the horizontal direction. That represents the case of a material reinforced with vertical fibers.

An usual way to deal with the small parameters ϵ and δ is to get rid of them via asymptotic methods, i.e., by taking the limits $\epsilon \to 0$ and $\delta \to 0$ in an appropriate fashion. The former is the *homogenization procedure* (Cioranescu and Donato, 1999), that works under convenient asumptions (for instance periodicity). The latter is a kind of *dimension reduction procedure*, where the original three-dimensional problem is described by its "two-dimensional limit."

One can expect that for quite small parameters, the asymptotic limits are good approximations of the original problems. The issue here is that in the presence of *two* parameters, it is not clear which limit to take first, and the order matters. Or simply put, under certain assumptions (Caillerie, 1981),

$$\text{if } u_{2D} \stackrel{\text{def}}{=} \lim_{\epsilon\to 0}\lim_{\delta\to 0} u^{\delta\epsilon}_{3D}, \quad \overline{u}_{2D} \stackrel{\text{def}}{=} \lim_{\delta\to 0}\lim_{\epsilon\to 0} u^{\delta\epsilon}_{3D}, \quad \text{then } u_{2D} \neq \overline{u}_{2D}.$$

Indeed, if $\underline{\underline{a}}(\underset{\sim}{x}, x_3) \stackrel{\text{def}}{=} \underline{\underline{a}}^\epsilon(\epsilon \underset{\sim}{x}, x_3)$ is periodic with period one in both horizontal directions, and ϵ-independent, and $f(\underset{\sim}{x}, \epsilon^{-1} x_3) \stackrel{\text{def}}{=} f^\delta(\underset{\sim}{x}, x_3)$ is ϵ-independent, then

$$\begin{gathered} -\operatorname{div}\left[\underset{\approx}{B}\,\underset{\sim}{\nabla}\, u_{2D}\right] = \frac{1}{2}\int_{-1}^{1} f\, d\hat{x}_3, \quad -\operatorname{div}\left[\overline{\underset{\approx}{B}}\,\underset{\sim}{\nabla}\, \overline{u}_{2D}\right] = \frac{1}{2}\int_{-1}^{1} f\, d\hat{x}_3 \qquad \text{in } \Omega, \\ u_{2D} = \overline{u}_{2D} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2}$$

where $\underset{\approx}{B}$ and $\overline{\underset{\approx}{B}}$ are constant, symmetric positive definite, *different* tensors:

$$
\begin{aligned}
B_{\alpha\beta} &= \int_Y A_{\alpha\beta} + \sum_{\gamma=1}^{2} A_{\alpha\gamma}\frac{\partial \chi_\beta}{\partial y_\gamma}\,dy_1dy_2, \qquad A_{\alpha\beta} = a_{\alpha\beta} - \frac{a_{\alpha 3}a_{3\beta}}{a_{33}},\\
\overline{B}_{\alpha\beta} &= \overline{A}_{\alpha\beta} - \frac{\overline{A}_{\alpha 3}\overline{A}_{3\beta}}{\overline{A}_{33}}, \qquad \overline{A}_{ij} = \int_Y a_{ij} + \sum_{\gamma=1}^{2} a_{i\gamma}\frac{\partial \overline{\chi}_j}{\partial y_\gamma}\,dy_1dy_2,
\end{aligned}
\tag{3}
$$

for $Y = (0,1)\times(0,1)$, and $\alpha,\beta = 1,2$ and $i,j = 1,2,3$. The above tensors depend on the cell problems

$$
\operatorname{div}\left[\underset{\approx}{A}\,\underset{\sim}{\nabla}\,\chi_\beta\right] = -\sum_{\alpha=1}^{2}\frac{\partial A_{\alpha\beta}}{\partial \hat{y}_\alpha}, \qquad \operatorname{div}\left[\underset{\approx}{a}\,\underset{\sim}{\nabla}\,\overline{\chi}_j\right] = -\sum_{\alpha=1}^{2}\frac{\partial a_{\alpha j}}{\partial y_\alpha} \qquad \text{in } Y, \tag{4}
$$

with periodic boundary conditions, for $\beta = 1,2$ and $j = 1,2,3$. Note that if $\underset{\sim}{a}^\epsilon = 0$ then $\underset{\approx}{B} = \overline{\underset{\approx}{B}}$, and the asymptotic limits commute.

To completely avoid such issue, we propose here the use of *hierarchical modeling* to derive provably good two-dimensional models for (1) (Alessandrini et al., 1999). The whole idea is to recast (1) in a variational form, defining $u_{3D}^{\delta\epsilon}$ as the argument that minimizes the potential energy, i.e.,

$$
u_{3D}^{\delta\epsilon} = \operatorname*{arg\,min}_{v\in V(P^\delta)}\left(\frac{1}{2}\int_{P^\delta} \underline{\underline{a}}^\epsilon \underline{\nabla} v\cdot \underline{\nabla}\, v\,d\underline{x} - \int_{P^\delta} f^\delta v\,d\underline{x}\right),
$$

where $V(P^\delta) = \{v \in H^1(P^\delta) : v = 0 \text{ in } \partial P_L^\delta\}$. Our model comes from minimizing the potential energy not in $V(P^\delta)$, but in the subspace

$$
V_1(P^\delta) = \{v \in V(P^\delta) : v(\underset{\sim}{x}, x_3) = v_0(\underset{\sim}{x}) + v_1(\underset{\sim}{x})x_3, \text{ for } v_0, v_1 \in H_0^1(\Omega)\} \tag{5}
$$

containing functions that vary linearly in the transverse direction. Finally, we define $\tilde{u}_{3D}^{\delta\epsilon} \in V_1(P^\delta)$ from

$$
\tilde{u}_{3D}^{\delta\epsilon} = \operatorname*{arg\,min}_{v\in V_1(P^\delta)}\left(\frac{1}{2}\int_{P^\delta} \underline{\underline{a}}^\epsilon \underline{\nabla} v\cdot \underline{\nabla}\, v\,d\underline{x} - \int_{P^\delta} f^\delta v\,d\underline{x}\right). \tag{6}
$$

It is a cumbersome but otherwise straightforward procedure to show that if we write

$$
\tilde{u}_{3D}^{\delta\epsilon}(\underset{\sim}{x}, x_3) = w_0^\epsilon(\underset{\sim}{x}) + w_1^\epsilon(\underset{\sim}{x})x_3, \tag{7}
$$

then

$$
\begin{gathered}
-2\delta \operatorname{div}_{\underset{\sim}{x}}\left[\underset{\approx}{a}^\epsilon \nabla_{\underset{\sim}{x}} w_0^\epsilon + \underset{\sim}{a}^\epsilon w_1^\epsilon\right] = \int_{-\delta}^{\delta} f\,dx_3 \quad \text{in } \Omega,\\
-\frac{2\delta^3}{3}\operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{a}^\epsilon \nabla_{\underset{\sim}{\hat{x}}} w_1^\epsilon\right] + 2\delta \underset{\sim}{a}^{\epsilon T}\nabla_{\underset{\sim}{\hat{x}}} w_0^\epsilon + 2\delta a_{33}^\epsilon w_1^\epsilon = \int_{-\delta}^{\delta} f x_3\,dx_3 \quad \text{in } \Omega,\\
w_0^\epsilon = w_1^\epsilon = 0 \quad \text{on } \partial\Omega.
\end{gathered}
\tag{8}
$$

Thus, to obtain our approximation for the original three-dimensional solution, one has to solve the coupled two-dimensional system (8), and then compute (7).

We remark that on our model derivation, there are no special assumptions (like periodicity or stochastic properties) on how the tensor $\underline{\underline{a}}^\epsilon$ behaves with respect to ϵ. The model is simply the Ritz–Galerkin projection of the original solution in the space of functions that are linear in the transverse direction. If the plate is thin, it is reasonable to expect that not much is lost from such approximation. Moreover, if one is interested in more accurate models, it is enough to consider polynomial dependance in the transverse direction. In the present case, the polynomial dependance is of order one.

The system (8) is not easy to solve since it still depends in a nontrivial way on ϵ and δ. Such dependance, that we perfer to call "richness," reflects the fact that the original problem also depends in a nontrivial fashion on the same parameters. Important information is lost if asymptotic limits are taken.

A nice feature of the new model is that it "captures" the correct asymptotic limits, i.e., the model is *asymptotically consistent*. In mathematical terms,

$$\lim_{\epsilon\to 0}\lim_{\delta\to 0}\tilde{u}_{3D}^{\delta\epsilon} = \lim_{\epsilon\to 0}\lim_{\delta\to 0} u_{3D}^{\delta\epsilon}, \qquad \lim_{\delta\to 0}\lim_{\epsilon\to 0}\tilde{u}_{3D}^{\delta\epsilon} = \lim_{\delta\to 0}\lim_{\epsilon\to 0} u_{3D}^{\delta\epsilon}.$$

So, no matter the characteristics of the material and the relative sizes of ϵ and δ, the two-dimensional model (7), (8) can be used. If one wants, under certain conditions it is possible to take a step further and homogenize the model (7). Otherwise, it is also possible to solve (8) directly. Note that this system is not trivial to solve since it combines singular perturbation features with the presence of the small parameter δ in front of the higher order derivatives (second equation), and highly oscillatory coefficients given by $\underset{\approx}{a}^\epsilon$, $\underset{\sim}{a}^\epsilon$, and a_{33}^ϵ. Naive numerical schemes are bound to fail, but there are alternatives like the Multiscale Finite Element Method (MsFEM), Heterogeneous Multiscale Method (HMM), and Residual Free Bubble Methods (RFB).

In the sections that follow, we perform the mathematical operations to obtain the results and equations we just presented. When dealing with the limits with respect to ϵ and δ, we proceed *formally*, i.e., we simply consider develop formal asympttic series with respect to ϵ and δ, and define the limit as the first terms of the expansions. We do so in Section 2. and 3.. In Section 4. we consider the hierarchical modeling.

2. Asymptotic limits: first $\delta \to 0$ and then $\epsilon \to 0$

In the present section, we reduce the dimension from a three-dimensional domain to a two-dimensional domain by considering $\delta \to 0$. We then homogenize the resulting problem by considering $\epsilon \to 0$.

2.1 Dimension reduction: $\delta \to 0$

The first step is to change coordinates and redefine (1) in an ϵ-independent domain. Let $P = \Omega \times (-1, 1)$, and the change of coordinates from P^δ to P given by

$$\underline{\hat{x}} = (\underset{\sim}{\hat{x}}, \hat{x}_3) = (\underset{\sim}{x}, \delta^{-1}x_3). \tag{9}$$

We also define

$$u_{3D}^\epsilon(\underline{\hat{x}}) = u_{3D}^{\delta\epsilon}(\underset{\sim}{\hat{x}}, \delta\hat{x}_3) = u_{3D}^{\delta\epsilon}(\underline{x}), \qquad f(\underline{\hat{x}}) = f^\delta(\underline{x}). \tag{10}$$

Using (9), we rewrite (1) as

$$\begin{aligned} -\operatorname{div}\left[\underset{\approx}{a}^\epsilon \underset{\sim}{\nabla} u_{3D}^\epsilon\right] - \delta^{-1}\operatorname{div}\left(\underset{\sim}{a}^\epsilon \frac{\partial u_{3D}^\epsilon}{\partial \hat{x}_3}\right) - \delta^{-1}\partial_3\left(\underset{\sim}{a}^\epsilon \cdot \underset{\sim}{\nabla} u_{3D}^\epsilon\right) - \delta^{-2}\partial_3\left(a_{33}^\epsilon \partial_3 u_{3D}^\epsilon\right) \\ = f \quad \text{in } P, \end{aligned}$$

$$\underset{\sim}{a}^{\epsilon T} \cdot \underset{\sim}{\nabla} u_{3D}^\epsilon + \delta^{-1} a_{33}^\epsilon \frac{\partial u_{3D}^\epsilon}{\partial \hat{x}_3} = 0 \quad \text{on } \Omega \times \{-1, 1\}.$$

(11)

Consider the asymptotic expansion of u^ϵ_{3D} with respect to δ,

$$u^\epsilon_{3D} \sim u^{\epsilon,0} + \delta u^{\epsilon,1} + \delta^2 u^{\epsilon,2} + \dots . \tag{12}$$

Substituting (12) in (11), we gather that

$$\begin{aligned}
&-\operatorname{div}\left[\underset{\approx}{a}^\epsilon \underset{\sim}{\nabla}\, u^{\epsilon,0}\right] - \delta \operatorname{div}\left[\underset{\approx}{a}^\epsilon \underset{\sim}{\nabla}\, u^{\epsilon,1}\right] - \delta^2 \operatorname{div}\left[\underset{\approx}{a}^\epsilon \underset{\sim}{\nabla}\, u^{\epsilon,2}\right] \\
&- \delta^{-1}\left[\operatorname{div}\left(\underset{\sim}{a}\frac{\partial u^{\epsilon,0}}{\partial \hat{x}_3}\right) + \frac{\partial}{\partial \hat{x}_3}\left(\underset{\sim}{a}\cdot \underset{\sim}{\nabla}\, u^{\epsilon,0}\right)\right] - \operatorname{div}\left(\underset{\sim}{a}\frac{\partial u^{\epsilon,1}}{\partial \hat{x}_3}\right) - \frac{\partial}{\partial \hat{x}_3}\left(\underset{\sim}{a}\cdot \underset{\sim}{\nabla}\, u^{\epsilon,1}\right) \\
&\qquad - \delta^{-2}\frac{\partial}{\partial \hat{x}_3}\left(a^\epsilon_{33}\frac{\partial u^{\epsilon,0}}{\partial \hat{x}_3}\right) - \delta^{-1}\frac{\partial}{\partial \hat{x}_3}\left(a^\epsilon_{33}\frac{\partial u^{\epsilon,1}}{\partial \hat{x}_3}\right) - \frac{\partial}{\partial \hat{x}_3}\left(a^\epsilon_{33}\frac{\partial u^{\epsilon,2}}{\partial \hat{x}_3}\right) + \dots = f,
\end{aligned}$$

in P, and

$$\underset{\sim}{a}^\epsilon \cdot \underset{\sim}{\nabla}\, u^{\epsilon,0} + \delta \underset{\sim}{a}^\epsilon \cdot \underset{\sim}{\nabla}\, u^{\epsilon,1} + \delta^2 \underset{\sim}{a}^\epsilon \cdot \underset{\sim}{\nabla}\, u^{\epsilon,2} + \delta^{-1} a^\epsilon_{33}\frac{\partial u^{\epsilon,0}}{\partial \hat{x}_3} + a^\epsilon_{33}\frac{\partial u^{\epsilon,1}}{\partial \hat{x}_3} + \delta a^\epsilon_{33}\frac{\partial u^{\epsilon 2}}{\partial \hat{x}_3} + \dots = 0,$$

on the top and bottom $\Omega \times \{-1, 1\}$ of the stretched plate. Grouping the terms with power δ^{-2}, and using that a^ϵ_{33} does not depend on $\hat{x}_3$, we gather that

$$\frac{\partial^2 u^{\epsilon,0}}{\partial \hat{x}_3^2} = 0 \quad \text{in } P, \tag{13}$$

and the boundary terms with power δ^{-1},

$$a^\epsilon_{33}\frac{\partial u^{\epsilon 0}}{\partial \hat{x}_3} = 0 \qquad \text{on } \Omega \times \{-1, 1\}. \tag{14}$$

From (13) (14) we gather that $u^{\epsilon,0}$ does not depend on $\hat{x}_3$.

Proceeding with the computation, now with terms with power δ^{-1},

$$\frac{\partial}{\partial \hat{x}_3}\left(\underset{\sim}{a}^\epsilon \cdot \underset{\sim}{\nabla}\, u^{\epsilon,0}\right) + \frac{\partial}{\partial \hat{x}_3}\left(a^\epsilon_{33}\frac{\partial u^{\epsilon 1}}{\partial \hat{x}_3}\right) = 0 \quad \text{in } P, \tag{15}$$

and boundary terms with power δ^0,

$$\underset{\sim}{a}^\epsilon \cdot \underset{\sim}{\nabla}\, u^{\epsilon,0} + a^\epsilon_{33}\frac{\partial u^{\epsilon,1}}{\partial \hat{x}_3} = 0 \quad \text{on } \Omega \times \{-1, 1\}. \tag{16}$$

It follows from (15), (16) that

$$\frac{\partial u^{\epsilon,1}}{\partial \hat{x}_3} = -\frac{1}{a^\epsilon_{33}}\left(\underset{\sim}{a}^\epsilon \cdot \underset{\sim}{\nabla}\, u^{\epsilon,0}\right). \tag{17}$$

From the terms with δ^0,

$$-\operatorname{div}\left(\underset{\approx}{a}^\epsilon \underset{\sim}{\nabla}\, u^{\epsilon,0}\right) - \operatorname{div}\left(\underset{\sim}{a}\frac{\partial u^{\epsilon,1}}{\partial \hat{x}_3}\right) - \frac{\partial}{\partial \hat{x}_3}\left(\underset{\sim}{a}\cdot \underset{\sim}{\nabla}\, u^{\epsilon,1}\right) - \frac{\partial}{\partial \hat{x}_3}\left(a^\epsilon_{33}\frac{\partial u^{\epsilon,2}}{\partial \hat{x}_3}\right) = f, \tag{18}$$

and boundary with power δ

$$\underset{\sim}{a}^\epsilon \cdot \underset{\sim}{\nabla}\, u^{\epsilon,1} + a^\epsilon_{33}\frac{\partial u^{\epsilon,2}}{\partial \hat{x}_3} = 0. \tag{19}$$

Integrating (18) in the vertical direction, and using the boundary conditions (19), we gather that

$$\operatorname{div}\left(\underset{\approx}{a}^\epsilon \underset{\sim}{\nabla} u^{\epsilon,0}\right) - \operatorname{div}\left(\underset{\sim}{a}^\epsilon \frac{1}{a_{33}^\epsilon}\underset{\sim}{a}^\epsilon \cdot \underset{\sim}{\nabla} u^{\epsilon,0}\right) = -\frac{1}{2}\int_{-1}^{1} f\,d\hat{x}_3 \quad \text{in } \Omega,$$

i.e.,

$$\begin{aligned} -\operatorname{div}\left(\underset{\approx}{A}^\epsilon \underset{\sim}{\nabla} u^{\epsilon,0}\right) &= \frac{1}{2}\int_{-1}^{1} f\,d\hat{x}_3 \quad \text{in } \Omega, \\ u^{\epsilon,0} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{20}$$

where $\underset{\approx}{A}^\epsilon = \underset{\approx}{a}^\epsilon - \frac{1}{a_{33}^\epsilon}\underset{\sim}{a}^\epsilon \underset{\sim}{a}^{\epsilon T}$, compare with (3).

2.2 Homogenization: $\epsilon \to 0$

We next "formally" homogenize problem (20). The arguments are standard, and we repeat then here for the convenience of the reader. Assuming that $\underset{=}{a}^\epsilon$ is ϵ-periodic with "period" Y, then so is $\underset{\approx}{A}^\epsilon$. Let $\underset{\sim}{\hat{y}} = \epsilon^{-1}\underset{\sim}{\hat{x}}$ and define $\underset{\approx}{A}(\underset{\sim}{\hat{y}}) = \underset{\approx}{A}^\epsilon(\underset{\sim}{\hat{x}})$ (see (3)). We assume that $\underset{\approx}{A}$ is ϵ-independent. Let

$$u^{\epsilon,0}(\underset{\sim}{x}) \sim u_{2D}^0(\underset{\sim}{x}, \epsilon^{-1}\underset{\sim}{\hat{x}}) + \epsilon u_{2D}^1(\underset{\sim}{x}, \epsilon^{-1}\underset{\sim}{\hat{x}}) + \epsilon^2 u_{2D}^2(\underset{\sim}{x}, \epsilon^{-1}\underset{\sim}{\hat{x}}) + \ldots, \tag{21}$$

where we assume that each term of the expansion is periodic with respect to $\underset{\sim}{\hat{y}}$.

Using the chain rule, substituting (21) in (20), and formally grouping terms with same power of ϵ, we have

$$\begin{aligned} &-\epsilon^{-2}\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{A}\nabla_{\underset{\sim}{\hat{y}}} u_{2D}^0\right] - \epsilon^{-1}\left(\operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{A}\nabla_{\underset{\sim}{\hat{y}}} u_{2D}^0\right] + \operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{A}\nabla_{\underset{\sim}{\hat{x}}} u_{2D}^0\right]\right) - \operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{A}\nabla_{\underset{\sim}{\hat{x}}} u_{2D}^0\right] \\ &-\epsilon^{-1}\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{A}(\underset{\sim}{\hat{y}})\nabla_{\underset{\sim}{\hat{y}}} u_{2D}^1\right] - \operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{A}\nabla_{\underset{\sim}{\hat{y}}} u_{2D}^1\right] - \operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{A}(\underset{\sim}{\hat{y}})\nabla_{\underset{\sim}{\hat{x}}} u_{2D}^1\right] - \epsilon\operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{A}\nabla_{\underset{\sim}{\hat{x}}} u_{2D}^1\right] \\ &\qquad - \operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{A}\nabla_{\underset{\sim}{\hat{y}}} u_{2D}^2\right] - \ldots\Big|_{\underset{\sim}{\hat{y}}=\epsilon^{-1}\underset{\sim}{\hat{x}}} = \frac{1}{2}\int_{-1}^{1} f\,d\hat{x}_3. \end{aligned}$$

Grouping the terms with power ϵ^{-2},

$$\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{A}\nabla_{\underset{\sim}{\hat{y}}} u_{2D}^0\right] = 0 \quad \text{in } \Omega \times Y.$$

Thus, u_{2D}^0 does not depend on $\underset{\sim}{\hat{y}}$, i.e.,

$$u_{2D}^0(\underset{\sim}{\hat{x}}, \underset{\sim}{\hat{y}}) = \overline{u}_{2D}(\underset{\sim}{\hat{x}}),$$

for some $\overline{u}_{2D}$. Working now with the power ϵ^{-1},

$$\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{A}\nabla_{\underset{\sim}{\hat{y}}} u_{2D}^1\right] = -\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\frac{\partial A_{\alpha\beta}}{\partial \hat{y}_\alpha}\frac{\partial \overline{u}_{2D}}{\partial \hat{x}_\beta}, \tag{22}$$

and power ϵ^0,

$$-\operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{A}\nabla_{\underset{\sim}{\hat{x}}}\overline{u}_{2D}\right] - \operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{A}\nabla_{\underset{\sim}{\hat{y}}} u_{2D}^1\right] - \operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{A}\nabla_{\underset{\sim}{\hat{x}}} u_{2D}^1\right] - \operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{A}\nabla_{\underset{\sim}{\hat{y}}} u_{2D}^2\right] = \int_{-1}^{1} f\,d\hat{x}_3. \tag{23}$$

Note that

$$\int_Y \operatorname{div}_{\underset{\sim}{\hat{y}}} \left[\underset{\approx}{A} \nabla_{\underset{\sim}{\hat{x}}} u^1_{2D} + \underset{\approx}{A} \nabla_{\underset{\sim}{\hat{y}}} u^2_{2D} \right] \, d\hat{y}_1 d\hat{y}_2 = 0.$$

Thus, integrating (23) in Y it follows that

$$- \operatorname{div}_{\underset{\sim}{\hat{x}}} \left[\underset{\approx}{A} \nabla_{\underset{\sim}{\hat{x}}} \overline{u}_{2D} + \underset{\approx}{A} \nabla_{\underset{\sim}{\hat{y}}} u^1_{2D} \right] = \int_{-1}^{1} f \, d\hat{x}_3. \tag{24}$$

From the first set of cell problems (4), it follows from pure substitution that $u^1_{2D}(\underset{\sim}{\hat{x}}, \underset{\sim}{\hat{y}}) = \sum_{\beta=1}^{2} \chi_\beta(\underset{\sim}{\hat{y}}) \partial \overline{u}_{2D}(\underset{\sim}{\hat{x}}) / \partial \hat{x}_\beta$ solves (22). Using (24), we obtain the first problem of (2).

3. Asymptotic limits: first $\epsilon \to 0$ and then $\delta \to 0$

We now consider asymtptotic limits as in Section 2., but in the reverse order. Most arguments here are not only pretty standard, but are already pressented in the previous scetion. We skip then most of the details.

3.1 Homogenization: $\epsilon \to 0$

Consider the usual periodicity assumptions, i.e., $\underset{=}{a}^\epsilon(\underset{\sim}{x}) = \underset{=}{a}(\epsilon^{-1}\underset{\sim}{x})$, where $\underset{=}{a}$ is periodic with respect to Y, and

$$u^{\delta\epsilon}_{3D}(\underset{-}{x}) \sim u^{\delta 0}_{3D}(\underset{-}{x}, \epsilon^{-1}\underset{\sim}{x}, 0) + \epsilon u^{\delta 1}_{3D}(\underset{-}{x}, \epsilon^{-1}\underset{\sim}{x}, 0) + \epsilon^2 u^{\delta 2}_{3D}(\underset{-}{x}, \epsilon^{-1}\underset{\sim}{x}, 0) + \dots. \tag{25}$$

Substituting the above expansion in (1), and proceeding as before, we gather that $u^{\delta 0}_{3D}$ does not depend of $\underset{-}{y}$. Thus, there is a function $\overline{u}^\delta_{3D} : P^\delta \to \mathbb{R}$ such that

$$u^{\delta 0}_{3D}(\underset{-}{x}, \underset{-}{y}) = \overline{u}^\delta_{3D}(\underset{-}{x}).$$

Again, proceeding as before, we gather that

$$\begin{gathered} - \operatorname{div} \left[\overline{\underset{=}{A}} \, \underset{\sim}{\nabla} \, \overline{u}^\delta_{3D} \right] = f^\delta \quad \text{in } P^\delta, \\ \overline{u}^\delta_{3D} = 0 \quad \text{on } \partial P^\delta_L, \qquad \left(\overline{\underset{=}{A}} \nabla \overline{u}^\delta_{3D} \right) \cdot \underset{-}{n} = 0 \quad \text{on } \partial P^\delta_\pm, \end{gathered} \tag{26}$$

where $\overline{\underset{=}{A}}$ is as in (3).

Problem (26) is the three-dimensional limit problem as of (1) as $\epsilon \to 0$. Next, we take $\delta \to 0$ to reduce the dimension of (26).

3.2 Dimension reduction: $\delta \to 0$

The procedure to take $\delta \to 0$ in (26) is exactly the same as the considered in Subsection 2.1, just replacing $\underset{=}{a}^\epsilon$ by $\overline{\underset{=}{A}}$. Thus, if we consider the expansion

$$\overline{u}^\delta_{3D} \sim \overline{u}^0 + \delta \overline{u}^1 + \delta^2 \overline{u}^2 + \dots, \tag{27}$$

and arguing as in Subsection 2.1, we gather that $\overline{u}^0 = \overline{u}_{2D}$, defined in (2).

4. Dimension reduction via Hierarchical Modeling

In this section we reduce the dimension of the original problem using hierarchical modeling. In such approach, as we already mentioned, the solution is projected in a subspace of functions that are polynomials in the transverse direction. After we find our two-dimensional model, we take asumptotic limits as before, and obtain the same problems as described in (2).

Considering the definition (6) we gather that

$$\int_{P^\delta} \underline{\underline{a}}^\epsilon \, \underline{\nabla} \, \tilde{u}_{3D}^{\delta\epsilon} \cdot \underline{\nabla} \, \tilde{v} \, d\underline{x} = \int_{P^\delta} f^\delta \tilde{v} d\underline{x} \quad \text{for all } \tilde{v} \in V_1(P^\delta). \tag{28}$$

To obtain the system (8), it is enough to consider (7), i.e., $\tilde{u}_{3D}^{\delta\epsilon}(\underline{x}) = w_0^\epsilon(\underset{\sim}{x}) + w_1^\epsilon(\underset{\sim}{x})x_3$, and use (28) with $\tilde{v}(\underset{\sim}{x}, x_3) = v_0(\underset{\sim}{x})$ and $\tilde{v}(\underset{\sim}{x}, x_3) = x_3 v_1(\underset{\sim}{x})$, for arbitrary $v_0, v_1 \in H_0^1(\Omega)$. The final step is to integrate in the transverse direction to obtain weak versions of the system (8).

4.1 Asymptotic limits: first $\delta \to 0$ and then $\epsilon \to 0$

We consider now the asymptotic limits of the problem (8). Formally taking the limit $\delta \to 0$ in (8), and denoting $w_0^{\epsilon,0} = \lim_{\delta\to 0} w_0^\epsilon$ and $w_1^{\epsilon,0} = \lim_{\delta\to 0} w_1^\epsilon$, we get

$$\begin{aligned} -\operatorname{div}\left(\underset{\approx}{a}^\epsilon \, \underset{\sim}{\nabla} \, w_0^{\epsilon,0} + \underset{\sim}{a}^\epsilon w_1^{\epsilon,0}\right) &= \frac{1}{2}\int_{-1}^1 f \, d\hat{x}_3, \\ \underset{\sim}{a}^\epsilon \cdot \underset{\sim}{\nabla} \, w_0^{\epsilon,0} + a_{33}^\epsilon w_1^{\epsilon,0} &= 0, \end{aligned} \tag{29}$$

in Ω. Thus

$$w_1^{\epsilon,0} = -\frac{1}{a_{33}^\epsilon}\underset{\sim}{a}^\epsilon \cdot \underset{\sim}{\nabla} \, w_0^{\epsilon,0}. \tag{30}$$

Using (30) in the first equation of (29), and comparing to (20), we see that $w_0^{\epsilon,0} = u^{\epsilon,0}$. Thus $\lim_{\delta\to 0} \tilde{u}^{\epsilon,\delta} = \lim_{\delta\to 0} u^{\epsilon,\delta}$. It is trivial then to conclude that

$$\lim_{\epsilon\to 0}\lim_{\delta\to 0} \tilde{u}^{\epsilon,\delta} = \lim_{\epsilon\to 0}\lim_{\delta\to 0} u^{\epsilon,\delta}.$$

4.2 Asymptotic limits: first $\epsilon \to 0$ and then $\delta \to 0$

As in Subsection 4.1, we consider the asymptotic limits for the hierarchical model, but in the reverse order. Let

$$w_0^\epsilon \sim w_0^0 + \epsilon w_0^1 + \epsilon^2 w_0^2 + \dots, \qquad w_1^\epsilon \sim w_1^0 + \epsilon w_1^1 + \epsilon^2 w_1^2 + \dots,$$

where again each term of the expansion depend on $\underset{\sim}{x}$ and $\underset{\sim}{\hat{y}} = \varepsilon^{-1}$. Using the chain rule, and substituting in the first equation of (8), we gather that

$$\begin{aligned} -\epsilon^{-2}\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}^\epsilon \nabla_{\underset{\sim}{\hat{y}}} w_0^\epsilon\right] - \epsilon^{-1}\operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{a}^\epsilon \nabla_{\underset{\sim}{\hat{y}}} w_0^\epsilon\right] - \epsilon^{-1}\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}^\epsilon \nabla_{\underset{\sim}{\hat{x}}} w_0^\epsilon\right] - \operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{a}^\epsilon \nabla_{\underset{\sim}{\hat{x}}} w_0^\epsilon\right] \\ - \epsilon^{-1}\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\sim}{a}^\epsilon w_1^\epsilon\right] - \operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\sim}{a}^\epsilon w_1^\epsilon\right] = \frac{1}{2}\int_{-1}^1 f \, d\hat{x}_3. \end{aligned}$$

Substituting the asymptotic expansions,

$$\begin{aligned} &-\epsilon^{-2}\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{y}}} w_0^0\right] - \epsilon^{-1}\operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{y}}} w_0^0\right] - \epsilon^{-1}\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{x}}} w_0^0\right] - \operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{x}}} w_0^0\right] \\ &-\epsilon^{-1}\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{y}}} w_0^1\right] - \operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{y}}} w_0^1\right] - \operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{x}}} w_0^1\right] - \epsilon\operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{x}}} w_0^1\right] \\ &-\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{y}}} w_0^2\right] - \epsilon^{-1}\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\sim}{a} w_1^0\right] - \operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\sim}{a} w_1^0\right] \\ &\qquad - \operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\sim}{a} w_1^1\right] - \epsilon\operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\sim}{a} w_1^1\right] - \dots = \frac{1}{2}\int_{-1}^1 f d\hat{x}_3. \end{aligned}$$

Putting together the terms with ϵ^{-2},

$$\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{y}}} w_0^0\right]=0,$$

thus w_0^0 does not depend on $\underset{\sim}{\hat{y}}$, i.e., $w_0^0(\underset{\sim}{\hat{x}},\underset{\sim}{\hat{y}})=\overline{w}_0(\underset{\sim}{\hat{x}})$ for some function $\overline{w}_0$.

Consider now the second equation of (8). Then

$$-\frac{2\delta^2}{3}\epsilon^{-2}\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}^{\epsilon}\nabla_{\underset{\sim}{\hat{y}}} w_1^{\epsilon}\right]-\frac{2\delta^2}{3}\epsilon^{-1}\operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{a}^{\epsilon}\nabla_{\underset{\sim}{\hat{y}}} w_1^{\epsilon}\right]-\frac{2\delta^2}{3}\epsilon^{-1}\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}^{\epsilon}\nabla_{\underset{\sim}{\hat{x}}} w_1^{\epsilon}\right]-$$
$$-\frac{2\delta^2}{3}\operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{a}^{\epsilon}\nabla_{\underset{\sim}{\hat{x}}} w_1^{\epsilon}\right]+2\epsilon^{-1}\underset{\sim}{a}^{\epsilon T}\nabla_{\underset{\sim}{\hat{y}}} w_0^{\epsilon}+2\underset{\sim}{a}^{\epsilon T}\nabla_{\underset{\sim}{\hat{x}}} w_0^{\epsilon}+2a_{33}^{\epsilon}w_1^{\epsilon}=\delta\int_{-1}^{1} f\hat{x}_3 d\hat{x}_3.$$

Using the asymptotics for w_0^{ϵ} and w_1^{ϵ},

$$\begin{aligned}
&-\frac{2\delta^2}{3}\epsilon^{-2}\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{y}}} w_1^0\right]-\frac{2\delta^2}{3}\epsilon^{-1}\operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{y}}} w_1^0\right]-\frac{2\delta^2}{3}\epsilon^{-1}\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{x}}} w_1^0\right]\\
&-\frac{2\delta^2}{3}\operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{x}}} w_1^0\right]-\frac{2\delta^2}{3}\epsilon^{-1}\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{y}}} w_1^1\right]-\frac{2\delta^2}{3}\operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{y}}} w_1^1\right]\\
&-\frac{2\delta^2}{3}\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{x}}} w_1^0\right]-\frac{2\delta^2}{3}\epsilon\operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{x}}} w_1^1\right]-\frac{2\delta^2}{3}\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{y}}} w_1^2\right]+2\epsilon^{-1}\underset{\sim}{a}^T\nabla_{\underset{\sim}{\hat{y}}} w_0^0\\
&\quad+2\underset{\sim}{a}^T\nabla_{\underset{\sim}{\hat{x}}} w_0^0+2\underset{\sim}{a}^T\nabla_{\underset{\sim}{\hat{y}}} w_0^1+2\epsilon\underset{\sim}{a}\nabla_{\underset{\sim}{\hat{x}}} w_0^1+2a_{33}w_1^0+2\epsilon a_{33}w_1^1+\cdots=\delta\int_{-1}^{1} f\hat{x}_3 d\hat{x}_3.
\end{aligned}$$

From the terms with power ϵ^{-2},

$$\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{y}}} w_1^0\right]=0,$$

and it follows then that w_1^0 is independent of $\underset{\sim}{\hat{y}}$, i.e., $w_1^0(\underset{\sim}{\hat{x}},\underset{\sim}{\hat{y}})=\overline{w}_1(\underset{\sim}{\hat{x}})$ for some $\overline{w}_1$.

From the ϵ^{-1} terms in both equations,

$$\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{y}}} w_0^1\right]=-\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{x}}}\overline{w}_0\right]-\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\sim}{a}\overline{w}_1\right], \tag{31}$$

$$\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{y}}} w_1^1\right]=-\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{x}}}\overline{w}_1\right]=-\sum_{\beta=1}^{2}\frac{\partial a_{\alpha\beta}}{\partial\hat{y}_\alpha}\frac{\partial\overline{w}_1}{\partial x_\beta}. \tag{32}$$

From the ϵ^0 terms in both equations, and from the periodicity,

$$-\operatorname{div}_{\underset{\sim}{\hat{x}}}\int_Y \underset{\approx}{a}\nabla_{\underset{\sim}{\hat{y}}} w_0^1+\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{x}}}\overline{w}_0+\underset{\sim}{a}\overline{w}_1\,d\underset{\sim}{\hat{y}}=\frac{1}{2}\int_{-1}^{1} f\,d\hat{x}_3, \tag{33}$$

$$\begin{aligned}
-\frac{2\delta^2}{3}\operatorname{div}_{\underset{\sim}{\hat{x}}}\int_Y \underset{\approx}{a}\nabla_{\underset{\sim}{\hat{x}}}\overline{w}_1+\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{y}}} w_1^1 d\underset{\sim}{\hat{y}}+2\int_Y \underset{\sim}{a}^T\nabla_{\underset{\sim}{\hat{y}}} w_0^1+\underset{\sim}{a}^T\nabla_{\underset{\sim}{\hat{x}}}\overline{w}_0+a_{33}\overline{w}_1\,d\underset{\sim}{\hat{y}}&\\
=\delta\int_{-1}^{1} f\hat{x}_3\,d\hat{x}_3.&
\end{aligned} \tag{34}$$

Recalling the definition (4) of $\overline{\chi}_j$ for $j=1,2,3$, we define

$$w_1^1(\underset{\sim}{\hat{x}},\underset{\sim}{\hat{y}})=\sum_{\beta=1}^{2}\chi_\beta\frac{\partial\overline{w}_1}{\partial x_\beta}. \tag{35}$$

which solves (32). Rewriting (31),

$$\operatorname{div}_{\underset{\sim}{\hat{y}}}\left[\underset{\approx}{a}\nabla_{\underset{\sim}{\hat{y}}}w_0^1\right] = -\sum_{\alpha,\beta=1}^{2}\frac{\partial a_{\alpha\beta}}{\partial \hat{y}_\alpha}\frac{\partial \overline{w}_0}{\partial x_\beta} - \sum_{\alpha=1}^{2}\frac{\partial a_{\alpha\beta}}{\partial \hat{y}_\alpha}\overline{w}_1.$$

Thus,

$$w_0^1(\underset{\sim}{\hat{x}}, \underset{\sim}{\hat{y}}) = \sum_{\beta=1}^{2}\chi_\beta\frac{\partial \overline{w}_0}{\partial x_\beta} + \chi_3\overline{w}_1. \tag{36}$$

solves (31).

Replacing (36) in (33), we get

$$-2\operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{\overline{A}}\nabla_{\underset{\sim}{\hat{x}}}\overline{w}_0\right] - 2\operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\sim}{\overline{A}}\overline{w}_1\right] = \int_{-1}^{1} f d\hat{x}_3, \tag{37}$$

where

$$\begin{aligned}\overline{A}_{ij} &= \frac{1}{Y_1Y_2}\int_Y a_{ij} + \sum_{\beta=1}^{2} a_{i\beta}\frac{\partial\chi_j}{\partial\hat{y}_\beta}\, d\underset{\sim}{\hat{y}} \quad \text{for } i,j=1,2,\\ \overline{A}_{i} &= \frac{1}{Y_1Y_2}\int_Y a_{i3} + \sum_{\beta=1}^{2} a_{i\beta}\frac{\partial\chi_3}{\partial\hat{y}_\beta}\, d\underset{\sim}{\hat{y}} \quad \text{for } i=1,2.\end{aligned} \tag{38}$$

Replacing (36), (35) in (34),

$$-\frac{2\delta^2}{3}\operatorname{div}_{\underset{\sim}{\hat{x}}}\left[\underset{\approx}{\overline{A}}\nabla_{\underset{\sim}{\hat{x}}}\overline{w}_1\right] + 2\underset{\sim}{\overline{A}}^T\nabla_{\underset{\sim}{\hat{x}}}\overline{w}_0 + 2\overline{A}_{33}\overline{w}_1 = \delta\int_{-1}^{1} f\hat{x}_3 d\hat{x}_3. \tag{39}$$

Taking $\delta \to 0$,

$$2\underset{\sim}{\overline{A}}^T\nabla_{\underset{\sim}{\hat{x}}}\overline{w}_0 + 2\overline{A}_{33}\overline{w}_1 = 0.$$

Thus

$$\overline{w}_1 = -\frac{1}{\overline{A}_{33}}\underset{\sim}{\overline{A}}^T\nabla_{\underset{\sim}{\hat{x}}}\overline{w}_0. \tag{40}$$

Substituting in (37), we get that $\overline{w}_0 = \overline{u}_{2D}$, defined by (2). In other words,

$$\lim_{\epsilon\to0}\lim_{\delta\to0}\tilde{u}_{3D}^{\delta\epsilon} = \lim_{\epsilon\to0}\lim_{\delta\to0}u_{3D}^{\delta\epsilon},$$

and the limits for the exact solution and our model behaves in the exactly same way.

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