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A NEW INTERIOR PENALTY DISCONTINUOUS GALERKIN METHOD FOR THE REISSNER–MINDLIN MODEL

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We introduce an interior penalty discontinuous Galerkin finite element method for the Reissner–Mindlin plate model that, as the plate's half-thickness ϵ tends to zero, recovers a hp interior penalty discontinuous Galerkin finite element methods for biharmonic equation. Our method does not introduce shear as an extra unknown, and does not need reduced integration techniques. We develop the *a priori* error analysis of these methods and prove error bounds that are optimal in h and uniform in ϵ. Numerical tests, that confirm our predictions, are provided.

Keywords: Reissner–Mindlin model; discontinuous Galerkin; interior penalty; *a priori* error.

AMS Subject Classification: 65N30; 65N15; 75K30

1. Introduction

The Reissner–Mindlin system of equations is one of the favorite playgrounds of numerical analysts. Indeed, such system is not only a good model for an important class of problems, elastic plates, but also it brings in computational challenges that require ingenious numerical methods. The matter is, despite being of second-order and elliptic, the system depends in a nontrivial manner on ϵ, the half-thickness of the plate. As ϵ goes to zero, the Reissner–Mindlin solution approaches the Kirchhoff–Love solution, which comes from a fourth-order partial differential equation. Thus, for ϵ positive but quite small, naive numerical schemes designed to solve Reissner–Mindlin fail, since in general they do not approximate well solutions of fourth-order problems. This is described as a *locking problem*.

This is all well known and sharply described in the introductory section of Ref. 7. There are in the literature finite element schemes that avoid locking altogether, for instance, see Refs. 2, 3, 8, 13, 18–20, 26–28 and 33; for a comprehensive review, see Ref. 24 and references therein. More recently some authors started to take advantage of the flexibility of discontinuous Galerkin (DG) finite element methods[5,6] to design new, locking free, plate models[4,14,16,29]; after the completion of our work, we also learned about Ref. 25. Our work fits in this realm.

Discontinuous Galerkin methods admit discontinuities of the elements in the discrete space, allowing the use of non-matching grids, approximations with varying polynomial order, and offer the possibility to implement weakly the desired smoothness of elements of the approximation space. This is particularly important when designing finite element methods for the biharmonic equation, since one can use high-order polynomial approximations without the necessity of fairly involved construction of C^1 continuous finite element spaces. Furthermore, the local elementwise conservation property of the DG methods is often desired in applications. The substantial flexibility of DG methods makes this approach quite suitable for a large range of computational problems such as linear and nonlinear hyperbolic PDEs, convection dominated diffusion PDEs and elliptic problems in general. A unified analysis of DG methods for the second-order elliptic equations can be found in Ref. 5, while a unified analysis encompassing both elliptic and hyperbolic equations in the framework of Friedrich's system can be found in Refs. 21–23.

The discontinuous Galerkin method for fourth-order elliptic equation was introduced and analyzed by Baker.[10] A *hp* version of interior penalty discontinuous Galerkin finite element methods have been considered and analyzed in Refs. 11, 30–32 and 35, where the authors present the stability analyses and *a priori* error bounds for symmetric, non-symmetric and semi-symmetric variants of the method. Such quite flexible DG, *hp*-scheme for the biharmonic equation was our main motivation in this present work. We propose here a method for the Reissner–Mindlin system that, as ϵ tends to zero, recovers the above-mentioned scheme for the biharmonic. We prove convergence in a natural energy norm, and provide numerical tests that confirm our predictions.

Let $\Omega \subset \mathbb{R}^2$ be a convex and polygonal domain with boundary $\partial\Omega$. Consider a homogeneous and isotropic linearly elastic plate occupying the three-dimensional domain $\Omega \times (-\epsilon, \epsilon)$. Assume that such a plate is clamped on its lateral side, and under a transverse load of density per unity area $\epsilon^3 g$ that is symmetric with respect to its middle surface. Under such pure bending regime, there are two popular two-dimensional models for the plate's displacement.

In the Kirchhoff–Love model, the displacement at $(\mathbf{x}, x_3) \in \Omega \times (-\epsilon, \epsilon)$ is approximated by $(-x_3 \boldsymbol{\nabla}\psi(\mathbf{x}), \psi(\mathbf{x}))$, where

$$\begin{aligned} D\Delta^2 \psi &= g \quad \text{in } \Omega, \\ \psi = \frac{\partial \psi}{\partial n} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

and $D = 4\mu(\mu+\lambda)/[3(2\mu+\lambda)]$. Here, μ and λ are the Lamé coefficients.

The simplest Reissner–Mindlin model approximation, as presented in Ref. 1 is $(-x_3\boldsymbol{\theta}(\mathbf{x}), \omega(\mathbf{x}))$, where

$$\begin{aligned} &-\mathbf{div}\,\mathcal{C}\,e(\boldsymbol{\theta}) + \epsilon^{-2}\mu(\boldsymbol{\theta} - \boldsymbol{\nabla}\omega) = 0 && \Omega, \\ &\epsilon^{-2}\mu\,\mathrm{div}(\boldsymbol{\theta} - \boldsymbol{\nabla}\omega) = g && \text{in } \Omega, \\ &\boldsymbol{\theta} = 0, \quad \omega = 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

We denote by $e(\boldsymbol{\theta})$ the symmetric part of the gradient of $\boldsymbol{\theta}$, and

$$\mathcal{C}\,e(\boldsymbol{\theta}) = \frac{1}{3}\,[\,2\mu\, e(\boldsymbol{\theta}) + \lambda^*\,\mathrm{div}\,\boldsymbol{\theta}\, I\,],$$

with $\lambda^* = 2\mu\,\lambda/(2\mu+\lambda)$, and I is the identity matrix. Let Λ_0, Λ_1 be positive constants such that

$$\Lambda_0 |e(\boldsymbol{\theta})|^2 \le |\mathcal{C}\,e(\boldsymbol{\theta}) : e(\boldsymbol{\theta})| \le \Lambda_1 |e(\boldsymbol{\theta})|^2, \tag{1.3}$$

where $\tau : \sigma = \sum_{i,j=1}^{2} \tau_{ij}\sigma_{ij}$ denote the inner product between two matrices τ and σ, and $|\tau| = (\tau:\tau)^{1/2}$.

In the weak formulation, $\boldsymbol{\theta} \in \mathring{\mathbf{H}}^1(\Omega)$ and $\omega \in \mathring{H}^1(\Omega)$ are such that

$$\begin{aligned} a(\boldsymbol{\theta}, \boldsymbol{\eta}) + \epsilon^{-2}\mu(\boldsymbol{\theta} - \boldsymbol{\nabla}\omega, \boldsymbol{\eta}) = 0 \quad & \text{for all } \boldsymbol{\eta} \in \mathring{\mathbf{H}}^1(\Omega), \\ -\epsilon^{-2}\mu(\boldsymbol{\theta} - \boldsymbol{\nabla}\omega, \boldsymbol{\nabla}\nu) = (g, \nu) \quad & \text{for all } \nu \in \mathring{H}^1(\Omega), \end{aligned}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$, and

$$a(\boldsymbol{\theta}, \boldsymbol{\eta}) = \int_\Omega \mathcal{C}\,e(\boldsymbol{\theta}) : e(\boldsymbol{\eta})\, d\mathbf{x}.$$

Note that the Poincaré's and Korn's inequalities hold, i.e. there exists an ϵ-independent constant c such that

$$\|\boldsymbol{\eta}\|_{1,\Omega}^2 \le c a(\boldsymbol{\eta}, \boldsymbol{\eta}), \quad \|\omega\|_{0,\Omega} \le c\|\boldsymbol{\nabla}\omega\|_{0,\Omega} \quad \text{for all } (\boldsymbol{\eta}, \omega) \in (\mathring{\mathbf{H}}^1(\Omega) \times \mathring{H}^1(\Omega)).$$

The existence and uniqueness of solutions for Reissner–Mindlin follow since from $\|\boldsymbol{\nabla}\omega\|_{0,\Omega} \le \|\boldsymbol{\theta} - \boldsymbol{\nabla}\omega\|_{0,\Omega} + \|\boldsymbol{\theta}\|_{0,\Omega}$, we gather that $(\boldsymbol{\theta}, \omega)$ is the unique minimum of the functional

$$a(\boldsymbol{\theta}, \boldsymbol{\theta}) + \epsilon^{-2}\mu(\boldsymbol{\theta} - \boldsymbol{\nabla}\omega, \boldsymbol{\theta} - \boldsymbol{\nabla}\omega) - (g, \omega)$$

in $(\mathring{\mathbf{H}}^1(\Omega), \mathring{H}^1(\Omega))$.

The relation between the Kirchhoff–Love and Reissner–Mindlin models becomes clear since, as $\epsilon \to 0$, the sequence of solutions $(\boldsymbol{\theta}, \omega)$ converges to $(\boldsymbol{\nabla}\psi, \psi)$, where ψ solves (1.1), and minimizes

$$a(\boldsymbol{\nabla}\nu, \boldsymbol{\nabla}\nu) - (g, \nu)$$

in $\overset{\circ}{H}{}^2(\Omega)$. This is an instance of a more general result of Ref. 15.

Next, we outline the contents of this paper. In the next section, we introduce a broken formulation for the Reissner–Mindlin system, and in Sec. 3, we define our finite element scheme and prove continuity and coercivity in an energy norm. Section 4 contains the convergence results, and Sec. 5 contains the numerical results.

We now briefly introduce and explain some basic notation that we use throughout this paper. As usual, if $\mathfrak{D}$ is an open set, then $L^2(\mathfrak{D})$ is the set of square integrable functions in $\mathfrak{D}$, and for a non-negative number t, $H^t(\mathfrak{D})$ is the corresponding Sobolev space of order t. The notation for its inner product, norm and semi-norm is $(\cdot,\cdot)_{t,\mathfrak{D}}$, $\|\cdot\|_{t,\mathfrak{D}}$ and $|\cdot|_{t,\mathfrak{D}}$. Let $\overset{\circ}{\mathbf{H}}{}^1(\Omega)$ be the space of functions in $\mathbf{H}^1(\Omega)$ vanishing on $\partial\Omega$. Similarly, $\overset{\circ}{\mathbf{H}}{}^2(\Omega)$ is the space of functions in $\mathbf{H}^2(\Omega)$ having their traces and their normal derivatives vanishing on $\partial\Omega$. We write vectors and vector spaces in bold.

2. Weak Formulation in Broken Sobolev Space

Let $\mathcal{K}_h = \{K\}$ be a shape-regular partition of Ω into non-overlapping triangles and let us assume for simplicity that this mesh does not include hanging nodes; all results below (with respective technical specification) are still valid for non-matching meshes. The number h_K denotes the diameter of an element $K \in \mathcal{K}_h$, and h is the maximum of h_K, for all $K \in \mathcal{K}_h$. Let $\mathcal{E}_h$ be the set of all open faces e of all elements in $\mathcal{K}_h$, and h_e the length of e. The set $\mathcal{E}_h$ will be divided into two subsets, $\mathcal{E}_h^\circ$ (the set of interior faces) and $\mathcal{E}_h^\partial$ (the set of boundary faces), defined by

$$\mathcal{E}_h^\circ = \{e \in \mathcal{E}_h : e \subset \Omega\}, \quad \mathcal{E}_h^\partial = \{e \in \mathcal{E}_h : e \subset \partial\Omega\}.$$

In addition, we define

$$\Gamma^\circ = \{x \in e : e \in \mathcal{E}_h^\circ\}$$

and $\Gamma = \Gamma^\circ \cup \partial\Omega$.

Let

$$H^t(\mathcal{K}_h) = \{v \in L^2(\Omega) : v_{|K} \in H^t(K), \text{ for all } K \in \mathcal{K}_h\}$$

be the space of piecewise Sobolev H^t-functions and denote its inner product, norm and semi-norm by $(\cdot,\cdot)_{t,h}$, $\|\cdot\|_{t,h}$ and $|\cdot|_{t,h}$ respectively. For simplicity denote by $\mathbf{H}^t(\mathcal{K}_h) = H^t(\mathcal{K}_h) \times H^t(\mathcal{K}_h)$ the respective broken Sobolev space of vector functions.

Similarly, for any open subset $\gamma \subset \Gamma$ let us denote by $(\cdot,\cdot)_\gamma$ and $\|\cdot\|_\gamma$ the inner product and the norm in the space $L^2(\gamma)$ respectively.

For any $K \in \mathcal{K}_h$ let $\mathbf{n}_K$ be the outer normal to the boundary ∂K. Let K^- and K^+ be two distinct elements of $\mathcal{K}_h$ sharing the edge $e = K^- \bigcap K^+ \in \mathcal{E}_h^\circ$. We define the jump of $\phi \in H^1(\mathcal{K}_h)$ by

$$[\phi] = \phi^-\mathbf{n}^- + \phi^+\mathbf{n}^+,$$

where $\phi^\pm = \phi|_{K^\pm}$ and $\mathbf{n}^\pm = \mathbf{n}_{K^\pm}$. For a vector function $\boldsymbol{\theta} \in \mathbf{H}^1(\mathcal{K}_h)$, define

$$[\boldsymbol{\theta}] = \boldsymbol{\theta}^- \cdot \mathbf{n}^- + \boldsymbol{\theta}^+ \cdot \mathbf{n}^+, \quad [\![\boldsymbol{\theta}]\!] = \boldsymbol{\theta}^- \odot \mathbf{n}^- + \boldsymbol{\theta}^+ \odot \mathbf{n}^+,$$

where $\boldsymbol{\theta} \odot \mathbf{n} = (\boldsymbol{\theta}\mathbf{n}^{\mathrm{T}} + \mathbf{n}\boldsymbol{\theta}^{\mathrm{T}})/2$. Note that the jump of a scalar function is a vector, and for a vector function $\boldsymbol{\theta}$, the jump $[\boldsymbol{\theta}]$ is a scalar, while the jump $[\![\boldsymbol{\theta}]\!]$ is a symmetric matrix. The average of scalar or vector function χ is defined by

$$\{\chi\} = \frac{1}{2}(\chi^- + \chi^+).$$

On a boundary face $e \in \mathcal{E}_h^\partial \bigcap \partial K$ with outer normal $\mathbf{n}$, define jumps and averages as

$$[\phi] = \phi|_K\mathbf{n}, \quad [\boldsymbol{\theta}] = \boldsymbol{\theta}|_K \cdot \mathbf{n}, \quad [\![\boldsymbol{\theta}]\!] = \boldsymbol{\theta}|_K \odot \mathbf{n}, \quad \{\chi\} = \chi|_K.$$

With such notation the following equalities hold[4]:

$$\sum_{K\in\mathcal{K}_h} \int_{\partial K} \boldsymbol{\theta} \cdot \mathbf{n}_K v = \sum_{e\in\mathcal{E}_h} \int_e \{\boldsymbol{\theta}\} \cdot [v], \tag{2.1}$$

$$\sum_{K\in\mathcal{K}_h} \int_{\partial K} \tau\mathbf{n}_K \cdot \boldsymbol{\eta} = \sum_{e\in\mathcal{E}_h} \int_e \{\tau\} : [\![\boldsymbol{\eta}]\!], \tag{2.2}$$

for sufficiently smooth vector $\boldsymbol{\theta}$ and symmetric tensor τ.

In what follows c denotes a generic constant (not necessarily the same in all occurrences) which is independent of the mesh-size h and the half-thickness ϵ. For instance, the shape-regularity implies that there exists a constant c such that on any face $e \in \mathcal{E}_h \bigcap \partial K$

$$h_e \le h_K \le ch_e.$$

Thus, the following multiplicative trace inequality holds[36]:

Lemma 1. *For a shape regular partition $\mathcal{K}_h$, there exists a constant c such that*

$$\|v\|_{0,\partial K}^2 \le c\left(\frac{1}{h_K}\|v\|_{0,K}^2 + h_k|v|_{1,K}^2\right) \quad \textit{for all } v \in H^1(K), \tag{2.3}$$

and for all $K \in \mathcal{K}_h$.

Let us suppose, for simplicity, that $D = 1$ in the biharmonic equation (1.1). Then the following symmetric discontinuous Galerkin formulation[35] defines $\psi_h \in H^4(\mathcal{K}_h)$ such that

$$B_b(\psi_h, \phi) = (g, \phi) \quad \text{for all } \phi \in H^4(\mathcal{K}_h), \tag{2.4}$$

where the bilinear form $B_b(\psi, \phi) = B_{\mathcal{K}_h}(\psi, \phi) + B_\Gamma(\psi, \phi) + B_s(\psi, \phi)$. The contributions from the elements are

$$B_{\mathcal{K}_h}(\psi, \phi) = (\Delta\psi, \Delta\phi)_h,$$

the consistency and symmetrization terms are

$$B_\Gamma(\psi,\phi) = \sum_{e\in\mathcal{E}_h}[(\{\boldsymbol{\nabla}\Delta\psi\},[\phi])_e + ([\psi],\{\boldsymbol{\nabla}\Delta\phi\})_e - (\{\Delta\psi\},[\boldsymbol{\nabla}\phi])_e - ([\boldsymbol{\nabla}\psi],\{\Delta\phi\})_e],$$

and the stabilization terms are

$$B_s(\psi,\phi) = \sum_{e\in\mathcal{E}_h}[\alpha_e([\psi],[\phi])_e + \beta_e([\boldsymbol{\nabla}\psi],[\boldsymbol{\nabla}\phi])_e].$$

The positive stabilization parameters α and β, which are defined at the mesh skeleton $\mathcal{E}_h$, taking the values α_e, β_e for $e\in\mathcal{E}_h$, are fixed further ahead (see Lemma 5) in order to weakly impose the boundary conditions and inter-element continuity, and also to guarantee stability to the method.

Our next goal is to derive a discontinuous Galerkin formulation for the Reissner–Mindlin problem that "recovers" (2.4) in the vanishing thickness limit. Assume that the solution $(\boldsymbol{\theta},\omega)$ is smooth, and multiplying both sides of the first equation in (1.2) by $\boldsymbol{\eta}\in\mathbf{H}^3(\mathcal{K}_h)$ and integrating by parts over an element K, we get

$$a_K(\boldsymbol{\theta},\boldsymbol{\eta}) + \epsilon^{-2}\mu(\boldsymbol{\theta}-\boldsymbol{\nabla}\omega,\boldsymbol{\eta})_K - (\mathcal{C}\,e(\boldsymbol{\theta})\mathbf{n},\boldsymbol{\eta})_{\partial K} = 0, \tag{2.5}$$

where $a_K(\boldsymbol{\theta},\boldsymbol{\eta}) = \int_K \mathcal{C}\,e(\boldsymbol{\theta}) : e(\boldsymbol{\eta})\,d\mathbf{x}$. In the same way, from the second equation in (1.2), for any $\nu\in H^1(\mathcal{K}_h)$, we obtain

$$-\epsilon^{-2}\mu(\boldsymbol{\theta}-\boldsymbol{\nabla}\omega,\boldsymbol{\nabla}\nu)_K + \epsilon^{-2}\mu((\boldsymbol{\theta}-\boldsymbol{\nabla}\omega)\cdot\mathbf{n},\nu)_{\partial K} = (g,\nu)_K.$$

To eliminate $\boldsymbol{\theta}-\boldsymbol{\nabla}\omega$ in the second term of the above equation, we use the first equation in (1.2), yielding

$$-\epsilon^{-2}\mu(\boldsymbol{\theta}-\boldsymbol{\nabla}\omega,\boldsymbol{\nabla}\nu)_K + (\mathbf{div}\,\mathcal{C}\,e(\boldsymbol{\theta})\cdot\mathbf{n},\nu)_{\partial K} = (g,\nu)_K. \tag{2.6}$$

Summing (2.5), (2.6) over all elements of the partition, and using (2.1), (2.2), we have

$$\begin{aligned} a_h(\boldsymbol{\theta},\boldsymbol{\eta}) + \epsilon^{-2}\mu(\boldsymbol{\theta}-\boldsymbol{\nabla}\omega,\boldsymbol{\eta})_h + \sum_{e\in\mathcal{E}_h} -(\mathcal{C}\{e(\boldsymbol{\theta})\},[\![\boldsymbol{\eta}]\!])_e &= 0,\\ -\epsilon^{-2}\mu(\boldsymbol{\theta}-\boldsymbol{\nabla}\omega,\boldsymbol{\nabla}\nu)_h + \sum_{e\in\mathcal{E}_h}(\{\mathbf{div}\,\mathcal{C}\,e(\boldsymbol{\theta})\},[\nu])_e &= (g,\nu), \end{aligned}$$

where $a_h(\boldsymbol{\theta},\boldsymbol{\eta}) = (\mathcal{C}\,e(\boldsymbol{\theta}) : e(\boldsymbol{\eta}))_h$. Finally, adding the symmetrization and penalization terms, we obtain

$$\begin{aligned} &a_h(\boldsymbol{\theta},\boldsymbol{\eta}) + \epsilon^{-2}\mu(\boldsymbol{\theta}-\boldsymbol{\nabla}\omega,\boldsymbol{\eta})_h + \sum_{e\in\mathcal{E}_h} - (\mathcal{C}\{e(\boldsymbol{\theta})\},[\![\boldsymbol{\eta}]\!])_e - ([\![\boldsymbol{\theta}]\!],\mathcal{C}\{e(\boldsymbol{\eta})\})_e\\ &\qquad + ([\omega],\{\mathbf{div}\,\mathcal{C}\,e(\boldsymbol{\eta})\})_e + \beta_e([\![\boldsymbol{\theta}]\!],[\![\boldsymbol{\eta}]\!])_e = 0,\\ &-\epsilon^{-2}\mu(\boldsymbol{\theta}-\boldsymbol{\nabla}\omega,\boldsymbol{\nabla}\nu)_h + \sum_{e\in\mathcal{E}_h}(\{\mathbf{div}\,\mathcal{C}\,e(\boldsymbol{\theta})\},[\nu])_e + \alpha_e([\omega],[\nu])_e = (g,\nu). \end{aligned} \tag{2.7}$$

Equations (2.7) correspond to the critical point of the functional

$$\frac{1}{2}a_h(\boldsymbol{\eta},\boldsymbol{\eta}) + \sum_{e\in\mathcal{E}_h}\Bigg[-(\{\mathcal{C}\,e(\boldsymbol{\eta})\},[\![\boldsymbol{\eta}]\!])_e + \frac{\beta_e}{2}([\![\boldsymbol{\eta}]\!],[\![\boldsymbol{\eta}]\!])_e + ([\nu],\{\mathbf{div}\,\mathcal{C}\,e(\boldsymbol{\eta})\})_e$$
$$+\frac{\alpha_e}{2}([\nu],[\nu])_e\Bigg] + \epsilon^{-2}\mu(\boldsymbol{\eta}-\boldsymbol{\nabla}\nu,\boldsymbol{\eta}-\boldsymbol{\nabla}\nu)_h - (g,\nu).$$

Calculating formally the limit of the last expression when $\epsilon\to 0$, we get

$$\frac{1}{2}a_h(\boldsymbol{\nabla}\nu,\boldsymbol{\nabla}\nu) + \sum_{e\in\mathcal{E}_h}\Bigg[-(\{\mathcal{C}\,e(\boldsymbol{\nabla}\nu)\},[\![\boldsymbol{\nabla}\nu]\!])_e + \frac{\beta_e}{2}([[\![\boldsymbol{\nabla}\nu]\!],[\![\boldsymbol{\nabla}\nu]\!])_e$$
$$+([\nu],\{\mathbf{div}\,\mathcal{C}\,e(\boldsymbol{\nabla}\nu)\})_e + \frac{\alpha_e}{2}([\nu],[\nu])_e\Bigg] - (g,\nu).$$

The variational formulation of the minimization problem for this functional is

$$\begin{aligned}
a_h(\boldsymbol{\nabla}\omega,\boldsymbol{\nabla}\nu) &+ \sum_{e\in\mathcal{E}_h}[-(\{\mathcal{C}\,e(\boldsymbol{\nabla}\omega)\},[\![\boldsymbol{\nabla}\nu]\!])_e - ([\![\boldsymbol{\nabla}\omega]\!],\{\mathcal{C}\,e(\boldsymbol{\nabla}\nu)\})_e\\
&+\ \beta_e([\![\boldsymbol{\nabla}\omega]\!],[\![\boldsymbol{\nabla}\nu]\!])_e + (\{\mathbf{div}\,\mathcal{C}\,e(\boldsymbol{\nabla}\omega)\},[\nu])_e + ([\omega],\{\mathbf{div}\,\mathcal{C}\,e(\boldsymbol{\nabla}\nu)\})_e\\
&+\ \alpha_e([\omega],[\nu])_e] = (g,\nu).
\end{aligned}\tag{2.8}$$

It follows from a piecewise integration by parts that the formulation (2.8), introduced in this paper, recovers in the limit $\epsilon\to 0$ a variant of the discontinuous Galerkin formulation for biharmonic equation from Ref. 35. The difference here is in the way the jump of the gradient of the displacement is penalized, cf. (2.4).

In this paper, we actually consider a more general, possibly nonsymmetric, formulation, depending on the values of the parameters $\lambda_1,\lambda_2\in[-1,1]$. Indeed, we have that $(\boldsymbol{\theta},\omega)\in\mathbf{H}^3(\mathcal{K}_h)\times H^1(\mathcal{K}_h)$ satisfy

$$\mathcal{A}(\boldsymbol{\theta},\omega;\boldsymbol{\eta},\nu) = (g,\nu),\quad \text{for all } (\boldsymbol{\eta},\nu)\in\mathbf{H}^3(\mathcal{K}_h)\times H^1(\mathcal{K}_h),\tag{2.9}$$

where

$$\begin{aligned}
\mathcal{A}(\boldsymbol{\theta},\omega;\boldsymbol{\eta},\nu) = a_h(\boldsymbol{\theta},\boldsymbol{\eta}) &+ \epsilon^{-2}\mu(\boldsymbol{\theta}-\boldsymbol{\nabla}\omega,\boldsymbol{\eta}-\boldsymbol{\nabla}\nu)_h + \lambda_1\mathcal{A}_1(\boldsymbol{\eta},\omega) + \mathcal{A}_1(\boldsymbol{\theta},\nu)\\
&- \mathcal{A}_2(\boldsymbol{\theta},\boldsymbol{\eta}) - \lambda_2\mathcal{A}_2(\boldsymbol{\eta},\boldsymbol{\theta}) + \mathcal{A}_\alpha(\omega,\nu) + \mathcal{A}_\beta(\boldsymbol{\theta},\boldsymbol{\eta})
\end{aligned}$$

and

$$\mathcal{A}_1(\boldsymbol{\eta},\omega) = \sum_{e\in\mathcal{E}_h}([\omega],\{\mathbf{div}\,\mathcal{C}\,e(\boldsymbol{\eta})\})_e,\quad \mathcal{A}_2(\boldsymbol{\theta},\boldsymbol{\eta}) = \sum_{e\in\mathcal{E}_h}(\mathcal{C}\{e(\boldsymbol{\theta})\},[\![\boldsymbol{\eta}]\!])_e,$$
$$\mathcal{A}_\alpha(\omega,\nu) = \sum_{e\in\mathcal{E}_h}\alpha_e([\omega],[\nu])_e,\quad \mathcal{A}_\beta(\boldsymbol{\theta},\boldsymbol{\eta}) = \sum_{e\in\mathcal{E}_h}\beta_e([\![\boldsymbol{\theta}]\!],[\![\boldsymbol{\eta}]\!])_e.$$

In case $\lambda_1=\lambda_2=1$, the above formulation is symmetric.

Remark 2. The nonsymmetric case $\lambda_1=\lambda_2=-1$ yields a trivially coercive scheme, independent of penalization parameters. On the other hand, the resulting formulation is not adjoint consistent, and therefore produces suboptimal error estimates in L^2

norm (while the optimal error estimate in energy norm is maintained). This is apparent in the numerical tests of Sec. 5.

3. Discontinuous Galerkin Finite Element Method

Let us denote by $\mathcal{P}_p(K)$ the space of polynomials with total degree less than or equal to p in $K \in \mathcal{K}_h$. We introduce the global discontinuous finite element space as

$$\mathcal{S}^{p,h}(\mathcal{K}_h) = \{v \in L^2(\Omega) : v_{|K} \in \mathcal{P}_p(K) \text{ for all } K \in \mathcal{K}_h\}.$$

To formulate our method let us choose $p \geq 2$, and the finite element spaces $\mathbf{\Theta}_h = \mathcal{S}^{(p-1),h}(\mathcal{K}_h) \times \mathcal{S}^{(p-1),h}(\mathcal{K}_h)$ to approximate $\boldsymbol{\theta}$, and $W_h = \mathcal{S}^{p,h}(\mathcal{K}_h)$ to approximate ω. We define $(\boldsymbol{\theta}_h, \omega_h) \in \mathbf{\Theta}_h \times W_h$ such that

$$\mathcal{A}(\boldsymbol{\theta}_h, \omega_h; \boldsymbol{\eta}, \nu) = (g, \nu), \quad \text{for all } (\boldsymbol{\eta}, \nu) \in \mathbf{\Theta}_h \times W_h. \tag{3.1}$$

Note that this formulation is consistent with Reissner–Mindlin problems (1.2) that admit sufficiently smooth solutions, for example $(\boldsymbol{\theta}, \omega) \in \mathbf{H}^3(\Omega) \times H^1(\Omega)$. In this case, the Galerkin orthogonality

$$\mathcal{A}(\boldsymbol{\theta} - \boldsymbol{\theta}_h, \omega - \omega_h; \boldsymbol{\eta}, \nu) = 0 \quad \text{for all } (\boldsymbol{\eta}, \nu) \in \mathbf{\Theta}_h \times W_h \tag{3.2}$$

holds.

Consider the following norm for $(\boldsymbol{\eta}, \nu) \in \mathbf{H}^3(\mathcal{K}_h) \times H^1(\mathcal{K}_h)$:

$$|||\boldsymbol{\eta}, \nu|||^2 = \|e(\boldsymbol{\eta})\|_{0,h}^2 + \epsilon^{-2}\|\boldsymbol{\eta} - \boldsymbol{\nabla}\nu\|_{0,h}^2 + \|\sqrt{\alpha}[\nu]\|_\Gamma^2 + \|\sqrt{\beta}[\![\boldsymbol{\eta}]\!]\|_\Gamma^2 + \left\|\frac{1}{\sqrt{\alpha}}\{\mathbf{div}\,\mathcal{C}\,e(\boldsymbol{\eta})\}\right\|_\Gamma^2 + \left\|\frac{1}{\sqrt{\beta}}\{\mathcal{C}\,e(\boldsymbol{\eta})\}\right\|_\Gamma^2$$

for $p \geq 3$, and

$$|||\boldsymbol{\eta}, \nu|||^2 = \|e(\boldsymbol{\eta})\|_{0,h}^2 + \epsilon^{-2}\|\boldsymbol{\eta} - \boldsymbol{\nabla}\nu\|_{0,h}^2 + \|\sqrt{\alpha}[\nu]\|_\Gamma^2 + \|\sqrt{\beta}[\![\boldsymbol{\eta}]\!]\|_\Gamma^2 + \left\|\frac{1}{\sqrt{\beta}}\{\mathcal{C}\,e(\boldsymbol{\eta})\}\right\|_\Gamma^2,$$

for $p = 2$.

It is readily seen that the bilinear form $\mathcal{A}$ is continuous in $(\mathbf{H}^3(\mathcal{K}_h) \times H^1(\mathcal{K}_h))^2$ with respect to this norm.

Lemma 3. *For a shape regular partition $\mathcal{K}_h$, there exists a positive constant c such that for all $((\boldsymbol{\theta}, \omega); (\boldsymbol{\eta}, \nu)) \in (\mathbf{H}^3(\mathcal{K}_h) \times H^1(\mathcal{K}_h))^2$,*

$$|\mathcal{A}(\boldsymbol{\theta}, \omega; \boldsymbol{\eta}, \nu)| \leq c|||\boldsymbol{\theta}, \omega|||\ |||\boldsymbol{\eta}, \nu|||,$$

where c is independent of h_K, $K \in \mathcal{K}_h$.

To show the coercivity of $\mathcal{A}$ in $(\mathbf{\Theta}_h \times W_h)^2$ we recall the following inverse inequalities.[34]

Lemma 4. *For a shape regular partition $\mathcal{K}_h$, there exist constants c_0 and c_1 such that*

$$\|v\|^2_{0,\partial K} \leq \frac{c_0}{h_K}\|v\|^2_{0,K} \quad \text{and} \quad \|\boldsymbol{\nabla} v\|^2_{0,\partial K} \leq \frac{c_1}{h_K^3}\|v\|^2_{0,K} \tag{3.3}$$

for all $v \in \mathcal{P}_p(K)$ and all $K \in \mathcal{K}_h$. The constants c_0, c_1 depend on the shape-regularity constant, and on the approximation order p, but not on the element diameter h_K.

Let us prove the coercivity of the bilinear form $\mathcal{A}$ in the discrete space. Note that the dependence of penalization parameters on h must be in accordance with the inverse inequalities above.

Lemma 5. *Let $\mathcal{K}_h$ be a shape regular partition, where the estimate (3.3) holds, and assume that the Lamé coefficients are uniformly bounded. Then there exist positive constants $\hat{\sigma}_\alpha$, $\hat{\sigma}_\beta$ such that if $\sigma_\alpha \geq \hat{\sigma}_\alpha$, $\sigma_\beta \geq \hat{\sigma}_\beta$, and*

$$\alpha_e = \frac{\sigma_\alpha}{h_e^3}, \quad \beta_e = \frac{\sigma_\beta}{h_e} \quad \textit{for } e \in \mathcal{E}_h, \tag{3.4}$$

then there exists a positive constant ζ depending on c_1, c_0 and the Lamé coefficients, such that

$$\mathcal{A}(\boldsymbol{\theta}, \omega; \boldsymbol{\theta}, \omega) \geq \zeta |||\boldsymbol{\theta}, \omega|||^2 \quad \textit{for all } (\boldsymbol{\theta}, \omega) \in \boldsymbol{\Theta}_h \times W_h. \tag{3.5}$$

Proof. We have to find ζ in such a way that the difference

$$\begin{aligned}\mathcal{A}(\boldsymbol{\theta}, \omega; \boldsymbol{\theta}, \omega) - \zeta|||\boldsymbol{\theta}, \omega|||^2 &= a_h(\boldsymbol{\theta}, \boldsymbol{\theta}) - \zeta\|e(\boldsymbol{\theta})\|^2_{0,h} + \epsilon^{-2}(1-\zeta)\|\boldsymbol{\theta} - \boldsymbol{\nabla}\omega\|^2_{0,h} \\ &+ (1-\zeta)\|\sqrt{\alpha}[\omega]\|^2_\Gamma + (1-\zeta)\|\sqrt{\beta}[\![\boldsymbol{\theta}]\!]\|^2_\Gamma + (1+\lambda_1)([\omega], \{\mathbf{div}\, \mathcal{C}\, e(\boldsymbol{\theta})\})_\Gamma \\ &- \zeta\left\|\frac{1}{\sqrt{\alpha}}\{\mathbf{div}\, \mathcal{C}\, e(\boldsymbol{\theta})\}\right\|^2_\Gamma - (1+\lambda_2)(\mathcal{C}\{e(\boldsymbol{\theta})\}, [\![\boldsymbol{\theta}]\!])_\Gamma - \zeta\left\|\frac{1}{\sqrt{\beta}}\{\mathcal{C}\, e(\boldsymbol{\theta})\}\right\|^2_\Gamma\end{aligned}$$

is positive. It follows from (1.3) that

$$a_h(\boldsymbol{\theta}, \boldsymbol{\theta}) - \zeta\|e(\boldsymbol{\theta})\|^2_{0,h} \geq (\Lambda_1 - \zeta)\|e(\boldsymbol{\theta})\|^2_{0,h}.$$

Using that $2ab \leq (\upsilon a^2 + b^2/\upsilon)$, for any real numbers a and b and for any $\upsilon > 0$, we have

$$([\omega], \{\mathbf{div}\, \mathcal{C}\, e(\boldsymbol{\theta})\})_\Gamma \geq -\frac{1}{2\upsilon_1}\|\sqrt{\alpha}[\omega]\|^2_\Gamma - \frac{\upsilon_1}{2}\left\|\frac{1}{\sqrt{\alpha}}\{\mathbf{div}\, \mathcal{C}\, e(\boldsymbol{\theta})\}\right\|^2_\Gamma,$$

$$-(\mathcal{C}\{e(\boldsymbol{\theta})\}, [\![\boldsymbol{\theta}]\!])_\Gamma \geq -\frac{1}{2\upsilon_2}\|\sqrt{\beta}[\![\boldsymbol{\theta}]\!]\|^2_\Gamma - \frac{\upsilon_2}{2}\left\|\frac{1}{\sqrt{\beta}}\mathcal{C}\{e(\boldsymbol{\theta})\}\right\|^2_\Gamma,$$

where υ_1 and υ_2 will be chosen below. Putting all these inequalities together we have

$$\begin{aligned}&\mathcal{A}(\boldsymbol{\theta}, \omega; \boldsymbol{\theta}, \omega) - \zeta|||\boldsymbol{\theta}, \omega|||^2 \\ &\quad \geq (\Lambda_1 - \zeta)\|e(\boldsymbol{\theta})\|^2_{0,h} + \epsilon^{-2}(1-\zeta)\|\boldsymbol{\theta} - \boldsymbol{\nabla}\omega\|^2_{0,h}\end{aligned}$$

$$+\left(1-\zeta-\frac{1}{\upsilon_1}\right)\|\sqrt{\alpha}[\omega]\|_\Gamma^2+\left(1-\zeta-\frac{1}{\upsilon_2}\right)\|\sqrt{\beta}[\![\boldsymbol{\theta}]\!]\|_\Gamma^2$$

$$-(\upsilon_1+\zeta)\left\|\frac{1}{\sqrt{\alpha}}\{\mathbf{div}\,\mathcal{C}\,e(\boldsymbol{\theta})\}\right\|_\Gamma^2-(\upsilon_2+\zeta)\left\|\frac{1}{\sqrt{\beta}}\{\mathcal{C}\,e(\boldsymbol{\theta})\}\right\|_\Gamma^2,$$

where we also used that λ_1 and λ_2 are bounded by one.

Using the penalization parameters (3.4), and the inverse inequalities (3.3), we gather that there exist constants $\hat{c}_0$, $\hat{c}_1$ such that

$$\left\|\frac{1}{\sqrt{\alpha}}\{\mathbf{div}\,\mathcal{C}\,e(\boldsymbol{\theta})\}\right\|_\Gamma^2\le\frac{\hat{c}_1}{\sigma_\alpha}\|e(\boldsymbol{\theta})\|_{0,h}^2,\quad\left\|\frac{1}{\sqrt{\beta}}\{\mathcal{C}\,e(\boldsymbol{\theta})\}\right\|_\Gamma^2\le\frac{\hat{c}_0}{\sigma_\beta}\|e(\boldsymbol{\theta})\|_{0,h}^2.$$

Thus

$$\begin{aligned}
&\mathcal{A}(\boldsymbol{\theta},\omega;\boldsymbol{\theta},\omega)-\zeta|\!|\!|\boldsymbol{\theta},\omega|\!|\!|^2\\
&\quad\ge\left(\Lambda_1-\zeta\left(1+\frac{\hat{c}_1}{\sigma_\alpha}+\frac{\hat{c}_0}{\sigma_\beta}\right)-\upsilon_1\frac{\hat{c}_1}{\sigma_\alpha}-\upsilon_2\frac{\hat{c}_0}{\sigma_\beta}\right)\|e(\boldsymbol{\theta})\|_{0,h}^2,\\
&\qquad+\epsilon^{-2}(1-\zeta)\|\boldsymbol{\theta}-\boldsymbol{\nabla}\omega\|_{0,h}^2+\left(1-\zeta-\frac{1}{\upsilon_1}\right)\|\sqrt{\alpha}[\omega]\|_\Gamma^2\\
&\qquad+\left(1-\zeta-\frac{1}{\upsilon_2}\right)\|\sqrt{\beta}[\![\boldsymbol{\theta}]\!]\|_\Gamma^2
\end{aligned}\tag{3.6}$$

If $\hat{\sigma}_\alpha$, $\hat{\sigma}_\beta$ satisfy

$$\frac{\hat{c}_1}{\hat{\sigma}_\alpha}+\frac{\hat{c}_0}{\hat{\sigma}_\beta}<\Lambda_1,$$

then there exists $\upsilon_1>0$, such that

$$1<\upsilon_1<\Lambda_1\left(\frac{\hat{c}_1}{\hat{\sigma}_\alpha}+\frac{\hat{c}_0}{\hat{\sigma}_\beta}\right)^{-1}.$$

Hence we get

$$1-\frac{1}{\upsilon_1}>0,\quad\Lambda_1-\upsilon_1\frac{c_1}{\hat{\sigma}_\alpha}-\upsilon_1\frac{c_0}{\hat{\sigma}_\beta}>0.\tag{3.7}$$

Choosing υ_2 such that $1<\upsilon_2<\upsilon_1$, we obtain

$$1-\frac{1}{\upsilon_2}>0,$$

and from the second inequality in (3.7) we have

$$\Lambda_1-\upsilon_1\frac{c_1}{\hat{\sigma}_\alpha}-\upsilon_2\frac{c_0}{\hat{\sigma}_\beta}>0.$$

Let $\zeta > 0$ be such that

$$\zeta < \min\left\{1 - \frac{1}{\upsilon_1}, 1 - \frac{1}{\upsilon_2}, 1, \frac{\Lambda_1 - \upsilon_1 \frac{c_1}{\hat{\sigma}_\alpha} - \upsilon_2 \frac{c_0}{\hat{\sigma}_\beta}}{1 + \frac{c_1}{\hat{\sigma}_\alpha} + \frac{c_0}{\hat{\sigma}_\beta}}\right\}.$$

Then for any $\sigma_\alpha \geq \hat{\sigma}_\alpha$, $\sigma_\beta \geq \hat{\sigma}_\beta$, the terms

$$1 - \zeta - \frac{1}{\upsilon_1}, \quad 1 - \zeta - \frac{1}{\upsilon_2}, \quad 1 - \zeta, \quad \Lambda_1 - \zeta\left(1 + \frac{\hat{c}_1}{\sigma_\alpha} + \frac{\hat{c}_0}{\sigma_\beta}\right) - \upsilon_1 \frac{\hat{c}_1}{\sigma_\alpha} - \upsilon_2 \frac{\hat{c}_0}{\sigma_\beta}$$

are all positive, and then the right-hand side of (3.6) is also positive, and our result follows. □

4. *A Priori* Error Analysis

Having continuity and coercivity of the bilinear form $\mathcal{A}$ in discrete spaces, we can proceed with error analysis of the method using standard techniques. Let us denote by $(\boldsymbol{\theta}, \omega)$ the exact solution of problem (1.2), and by $(\boldsymbol{\theta}_h, \omega_h)$ its approximation, the solution of (3.1). Next, let us denote by $(\boldsymbol{\theta}^i, \omega^i)$ some interpolant of $(\boldsymbol{\theta}, \omega)$ in $\boldsymbol{\Theta}_h \times W_h$ (we will fix this interpolants later).

We start by decomposing the approximation errors as follows:

$$\begin{aligned}\boldsymbol{\theta} - \boldsymbol{\theta}_h &= (\boldsymbol{\theta} - \boldsymbol{\theta}^i) + (\boldsymbol{\theta}^i - \boldsymbol{\theta}_h) \equiv e^i_{\boldsymbol{\theta}} - \xi_{\boldsymbol{\theta}},\\ \omega - \omega_h &= (\omega - \omega^i) + (\omega^i - \omega_h) \equiv e^i_\omega - \xi_\omega.\end{aligned}$$

Using the continuity and coercivity of the bilinear form, we readily get from the Galerkin orthogonality (3.2) that

$$\begin{aligned}|||\xi_{\boldsymbol{\theta}}, \xi_\omega|||^2 &\leq \zeta\mathcal{A}(\xi_{\boldsymbol{\theta}}, \xi_\omega; \xi_{\boldsymbol{\theta}}, \xi_\omega) = \zeta\mathcal{A}(e^i_{\boldsymbol{\theta}} - \boldsymbol{\theta} + \boldsymbol{\theta}_h, e^i_\omega - \omega + \omega_h; \xi_{\boldsymbol{\theta}}, \xi_\omega)\\ &= \zeta\mathcal{A}(e^i_{\boldsymbol{\theta}}, e^i_\omega; \xi_{\boldsymbol{\theta}}, \xi_\omega) - \zeta\mathcal{A}(\boldsymbol{\theta} - \boldsymbol{\theta}_h, \omega - \omega_h; \xi_{\boldsymbol{\theta}}, \xi_\omega)\\ &= \zeta\mathcal{A}(e^i_{\boldsymbol{\theta}}, e^i_\omega; \xi_{\boldsymbol{\theta}}, \xi_\omega) \leq c|||e^i_{\boldsymbol{\theta}}, e^i_\omega|||\, |||\xi_{\boldsymbol{\theta}}, \xi_\omega|||,\end{aligned}$$

and consequently,

$$|||\xi_{\boldsymbol{\theta}}, \xi_\omega||| \leq c|||e^i_{\boldsymbol{\theta}}, e^i_\omega|||.$$

This means that

$$|||\boldsymbol{\theta} - \boldsymbol{\theta}_h, \omega - \omega_h||| \leq c|||e^i_{\boldsymbol{\theta}}, e^i_\omega|||; \tag{4.1}$$

and to estimate the error of the method it is enough to estimate the interpolation error.

Proceeding as in Ref. 7 to choose appropriate interpolants, let us denote by π_W the natural projection onto $W_h \cap H^1(\Omega)$. For $\omega \in H^{p+1}(\Omega)$, let $\omega^i = \pi_W\omega$. It follows then from a well-known approximation estimate that for $0 \leq q \leq p+1$, there exists a constant c such that

$$\left\|\omega - \omega^i\right\|_{q,h} \leq ch^{p+1-q}\|\omega\|_{p+1,\Omega} \quad \text{for all } \omega \in H^{p+1}(\Omega). \tag{4.2}$$

Consider now the rotated Brezzi–Douglas–Marini space $\mathbf{BDM}_{p-1}^R$ of degree $p-1$, i.e. the space of all piecewise polynomial vector fields of degree at most $p-1$ subject to inter-element continuity of the tangential components; obviously $\mathbf{BDM}_{p-1}^R \subset \boldsymbol{\Theta}_h$. Let $\boldsymbol{\pi}_{\boldsymbol{\Theta}}$ denotes the natural projector of $\mathbf{H}^1(\Omega)$ into $\mathbf{BDM}_{p-1}^R$. Note that $\boldsymbol{\nabla} W_h \subseteq \boldsymbol{\Theta}_h$, and the following commutativity property of the projectors follows from integration by parts:

$$\boldsymbol{\pi}_{\boldsymbol{\Theta}} \boldsymbol{\nabla} \omega = \boldsymbol{\nabla} \pi_W \omega. \tag{4.3}$$

So, let $\boldsymbol{\theta}^i = \boldsymbol{\pi}_{\boldsymbol{\Theta}} \boldsymbol{\theta}$ be the interpolator of $\boldsymbol{\theta} \in \mathbf{H}^1(\Omega)$. Defining $\boldsymbol{\gamma} = \epsilon^{-2}(\boldsymbol{\theta} - \boldsymbol{\nabla}\omega)$ as the shear stress vector, and $\boldsymbol{\gamma}^i = \epsilon^{-2}(\boldsymbol{\theta}^i - \boldsymbol{\nabla}\omega^i)$, it follows that

$$\boldsymbol{\pi}_{\boldsymbol{\Theta}} \boldsymbol{\gamma} = \epsilon^{-2} \boldsymbol{\pi}_{\boldsymbol{\Theta}} (\boldsymbol{\theta} - \boldsymbol{\nabla}\omega) = \epsilon^{-2}(\boldsymbol{\pi}_{\boldsymbol{\Theta}} \boldsymbol{\theta} - \boldsymbol{\nabla}\pi_W \omega) = \epsilon^{-2}(\boldsymbol{\theta}^i - \boldsymbol{\nabla}\omega^i) = \boldsymbol{\gamma}^i.$$

Thus, $\boldsymbol{\gamma}^i$ interpolates $\boldsymbol{\gamma}$, and with this key condition, the next results for interpolation error estimates holds.[7] For $0 \le s \le l$, and $1 \le l \le p$

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^i\|_{s,h} \le c h^{l-s} \|\boldsymbol{\theta}\|_{l,\Omega} \quad \text{for all } \boldsymbol{\theta} \in \mathbf{H}^l(\Omega), \tag{4.4}$$

$$\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^i\|_{s,h} \le c h^{l-s} \|\boldsymbol{\gamma}\|_{l,\Omega} \quad \text{for all } \boldsymbol{\gamma} \in \mathbf{H}^l(\Omega). \tag{4.5}$$

The main result of this paper is the following.

Theorem 6. *Let $\Omega \subset \mathbb{R}^2$ be a polygonal convex domain and let $\mathcal{K}_h$ be a shape regular partition on Ω. Assume that the penalization parameters α and β are such that $\mathcal{A}(\cdot,\cdot;\cdot,\cdot)$ is coercive (according to Lemma 5). Assume also that the solution to (1.2) satisfy $(\boldsymbol{\theta}, \omega) \in \mathbf{H}^p(\Omega) \times H^{p+1}(\Omega)$ and that $p \ge 2$. Then $(\boldsymbol{\theta}_h, \omega_h) \in \boldsymbol{\Theta}_h \times W_h$, solution of discontinuous Galerkin finite element method (3.1), satisfy*

$$|||\boldsymbol{\theta} - \boldsymbol{\theta}_h, \omega - \omega_h||| \le c h^{p-1} \big(\|\boldsymbol{\theta}\|_p + \|\omega\|_{p+1} + \epsilon \|\boldsymbol{\gamma}\|_{p-1} \big), \tag{4.6}$$

where c does not depend on h or ϵ.

Proof. From (4.1) we get

$$\begin{aligned} |||\boldsymbol{\theta} - \boldsymbol{\theta}_h, \omega - \omega_h|||^2 \le c |||e_{\boldsymbol{\theta}}^i, e_\omega^i|||^2 = c \Bigg(& \|e(e_{\boldsymbol{\theta}}^i)\|_{0,h}^2 + \epsilon^{-2} \|e_{\boldsymbol{\theta}}^i - \boldsymbol{\nabla} e_\omega^i\|_{0,h}^2 \\ & + \|\sqrt{\alpha}[e_\omega^i]\|_\Gamma^2 + \|\sqrt{\beta}[\![e_{\boldsymbol{\theta}}^i]\!]\|_\Gamma^2 + \left\| \frac{1}{\sqrt{\alpha}} \{\mathbf{div}\, \mathcal{C}\, e(e_{\boldsymbol{\theta}}^i)\} \right\|_\Gamma^2 + \left\| \frac{1}{\sqrt{\beta}} \{\mathcal{C}\, e(e_{\boldsymbol{\theta}}^i)\} \right\|_\Gamma^2 \Bigg), \end{aligned}$$

so we have to estimate the terms on the right-hand side of the last inequality.

From (4.4) and (4.5) we obtain

$$\|e(e_{\boldsymbol{\theta}}^i)\|_{0,h}^2 \le c h^{2p-2} \|\boldsymbol{\theta}\|_{p,\Omega}^2,$$

$$\epsilon^{-2} \|e_{\boldsymbol{\theta}}^i - \boldsymbol{\nabla} e_\omega^i\|_{0,h}^2 = \epsilon^2 \|\boldsymbol{\gamma} - \boldsymbol{\gamma}^i\|_{0,h}^2 \le c \epsilon^2 h^{2p-2} \|\boldsymbol{\gamma}\|_{p-1,\Omega}^2.$$

Next, using trace inequality (2.3), the definitions of α and β, and the estimates (4.2), (4.4) we gather that

$$\begin{aligned}
\|\sqrt{\alpha}[e^i_\omega]\|^2_\Gamma &= \sum_{e\in\mathcal{E}_h} \alpha_e \|[e^i_\omega]\|^2_e \\
&\le c \sum_{K\in\mathcal{K}_h} h_K^{-3}\big(h_K^{-1}\|e^i_\omega\|^2_{0,K} + h_K|e^i_\omega|^2_{1,K}\big) \\
&\le ch^{2p-2}\|\omega\|^2_{p+1,\Omega}, \\
\|\sqrt{\beta}[\![e^i_{\boldsymbol\theta}]\!]\|^2_\Gamma &= \sum_{e\in\mathcal{E}_h} \beta_e \|[\![e^i_{\boldsymbol\theta}]\!]\|^2_e \\
&\le c \sum_{K\in\mathcal{K}_h} h_K^{-1}\big(h_K^{-1}\|e^i_{\boldsymbol\theta}\|^2_{0,K} + h_K|e^i_{\boldsymbol\theta}|^2_{1,K}\big) \\
&\le ch^{2p-2}\|\boldsymbol\theta\|^2_{p,\Omega}.
\end{aligned}$$

Similarly, using once again (2.3) and (4.2), we have

$$\begin{aligned}
\left\|\frac{1}{\sqrt{\alpha}}\{\mathbf{div}\,\mathcal{C}\,e(e^i_{\boldsymbol\theta})\}\right\|^2_\Gamma &= \sum_{e\in\mathcal{E}_h}\frac{1}{\alpha_e}\|\{\mathbf{div}\,\mathcal{C}\,e(e^i_{\boldsymbol\theta})\}\|^2_e \\
&\le c\sum_{K\in\mathcal{K}_h} h_K^3(h_K^{-1}\|e^i_{\boldsymbol\theta}\|^2_{2,K} + h_K|e^i_{\boldsymbol\theta}|^2_{3,K}) \\
&\le ch^{2p-2}\|\boldsymbol\theta\|^2_{p,\Omega}, \\
\left\|\frac{1}{\sqrt{\beta}}\{\mathcal{C}\,e(e^i_{\boldsymbol\theta})\}\right\|^2_\Gamma &= \sum_{e\in\mathcal{E}_h}\frac{1}{\beta_e}\|\{\mathcal{C}\,e(e^i_{\boldsymbol\theta})\}\|^2_e \\
&\le c\sum_{K\in\mathcal{K}_h} h_K(h_K^{-1}\|e^i_{\boldsymbol\theta}\|^2_{1,K} + h_K|e^i_{\boldsymbol\theta}|^2_{2,K}) \\
&\le ch^{2p-2}\|\boldsymbol\theta\|^2_{p,\Omega}.
\end{aligned}$$

Combining the inequalities above we have (4.6). □

Remark 7. Note that estimate (4.6) holds for any $\boldsymbol\Theta_h$ containing $\mathbf{BDM}^R_{p-1}$.[7] In fact, in the proof of Theorem 6, it was enough that the projection $\boldsymbol\pi_{\boldsymbol\Theta}$ is well-defined and that (4.3) holds. Particular choices include the case where W_h has only continuous functions, and the case of equal interpolation degree for all the unknowns, i.e. $\boldsymbol\Theta_h = \mathcal{S}^{p,h}(\mathcal{K}_h)\times\mathcal{S}^{p,h}(\mathcal{K}_h)$. This is particularly useful since using equal order interpolation for all spaces might make the computational implementation easier.

We proceed to consider an estimate in the norm

$$|\!|\!|\nu|\!|\!|^2_{2,h} = \|\nabla\boldsymbol\nabla\nu\|^2_{0,h} + \|\sqrt{\alpha}[\nu]\|^2_\Gamma + \|\sqrt{\beta}[\![\boldsymbol\nabla\nu]\!]\|^2_\Gamma.$$

The result follows from Theorem 6.

Theorem 8. *Under the assumptions of Theorem 6, it follows that there exists c such that*

$$|||\omega - \omega_h|||_{2,h} \le c(1 + \epsilon h^{-1}) h^{p-1} \big(\|\boldsymbol{\theta}\|_p + \|\omega\|_{p+1} + \epsilon \|\boldsymbol{\gamma}\|_{p-1} \big).$$

Proof. From the triangle inequality,

$$|||\omega - \omega_h|||_{2,h} \le |||\omega - \omega^i|||_{2,h} + |||\omega^i - \omega_h|||_{2,h},$$

and $|||\omega - \omega^i|||_{2,h} \le c h^{p-1} \|\omega\|_{p+1}$ from the approximation inequality (4.2). It suffices now to bound $|||\omega^i - \omega_h|||_{2,h}$. Adding and subtracting $\nabla \boldsymbol{\eta}$ and $\boldsymbol{\eta}$ for an arbitrary $\boldsymbol{\eta} \in \boldsymbol{\Theta}_h$, and using again the triangle inequality, we gather that

$$\begin{aligned} |||\nu|||_{2,h} \le \|\nabla(\boldsymbol{\nabla}\nu - \boldsymbol{\eta})\|_{0,h} + \|\nabla \boldsymbol{\eta}\|_{0,h} + \|\sqrt{\alpha}[\nu]\|_\Gamma^2 \\ + \|\sqrt{\beta}[\![\boldsymbol{\nabla}\nu - \boldsymbol{\eta}]\!]\|_\Gamma + \|\sqrt{\beta}[\![\boldsymbol{\eta}]\!]\|_\Gamma. \end{aligned}$$

Using the discrete Korn's inequality,[4,7]

$$\|\nabla \boldsymbol{\eta}\|_{0,h} \le c\Big(\|e(\boldsymbol{\eta})\|_{0,h} + \|\sqrt{\beta}[\![\boldsymbol{\eta}]\!]\|_\Gamma \Big) \le c|||\boldsymbol{\eta}, \nu|||,$$

the trace inequality (2.3), and the inverse inequality twice, we get

$$|||\nu|||_{2,h} \le c\left(\frac{1}{h} \|\boldsymbol{\nabla}\nu - \boldsymbol{\eta}\|_{0,h} + |||\boldsymbol{\eta}, \nu||| \right) \le c\Big(\frac{\epsilon}{h} + 1 \Big) |||\boldsymbol{\eta}, \nu|||.$$

The result follows by taking $\boldsymbol{\eta} = \boldsymbol{\theta}_i - \boldsymbol{\theta}_h$ and $\nu = \omega_i - \omega_h$, and from Theorem 6. ☐

5. Numerical Results

We consider now some numerical tests that display the performance of our method. We start by adapting the solution given in Ref. 16, and it follows that

$$\begin{aligned} \omega_1(x, y) &= \frac{1}{3} x^3 (x-1)^3 y^3 (y-1)^3, \\ \omega_2(x, y) &= y^3(y-1)^3 x(x-1)(5x^2 - 5x + 1) + x^3(x-1)^3 y(y-1)(5y^2 - 5y + 1), \\ \omega(x, y) &= \omega_1(x, y) - \epsilon^2 \frac{8(\mu + \lambda)}{3(2\mu + \lambda)} \omega_2(x, y), \\ \theta_1(x, y) &= y^3(y-1)^3 x^2 (x-1)^2 (2x - 1), \\ \theta_2(x, y) &= x^3(x-1)^3 y^2 (y-1)^2 (2y - 1), \end{aligned}$$

solves (1.2) in $\Omega = (0, 1) \times (0, 1)$ with

$$\begin{aligned} g = \frac{4(\mu + \lambda)\mu}{3(2\mu + \lambda)} \{ &12y(y-1)(5x^2 - 5x + 1)[2y^2(y-1)^2 + x(x-1)(5y^2 - 5y + 1)] \\ &+ 12x(x-1)(5y^2 - 5y + 1)[2x^2(x-1)^2 + y(y-1)(5x^2 - 5x + 1)] \}. \end{aligned}$$

In our numerical simulations we set the Lamé coefficients $\lambda = \mu = 1$.

Implementing the DG method described above in the PZ environment,[17] we proceed to check the convergence of the scheme (3.1) for the symmetric ($\lambda_1 = \lambda_2 = 1$), and non-symmetric ($\lambda_1 = \lambda_2 = -1$) cases. In both cases, we pick $\sigma_\alpha = \sigma_{\beta_e} = 10$. We used $\boldsymbol{\Theta}_h = \mathcal{S}^{p,h}(\mathcal{K}_h) \times \mathcal{S}^{p,h}(\mathcal{K}_h)$. As noted in Remark 7, the converge rates obtained in Theorem 6 are still valid under this choice.

We successively divide the domain using 2^{2L+1} triangles. Thus, if e_L denotes the error at the level of refinement L, the rate of convergence for such level is given by

$$r_L = \log\left(\frac{e_L}{e_{L-1}}\right)\Big/\log(0.5).$$

Figure 1 shows the error of the symmetric method for the vertical displacement at the top, and for the rotation at the bottom, as a function of the refinement level for $p = 2, 3$ and for different values of thicknesses ϵ. The errors were in the L^2 norm at the left column, and the H^1 norm at the right column. The nonsymmetric version of the method yields similar results. We observe that in the H^1 norm, the errors for all approximation orders exhibit similar behavior for ω and $\boldsymbol{\theta}$, confirming that in fact the

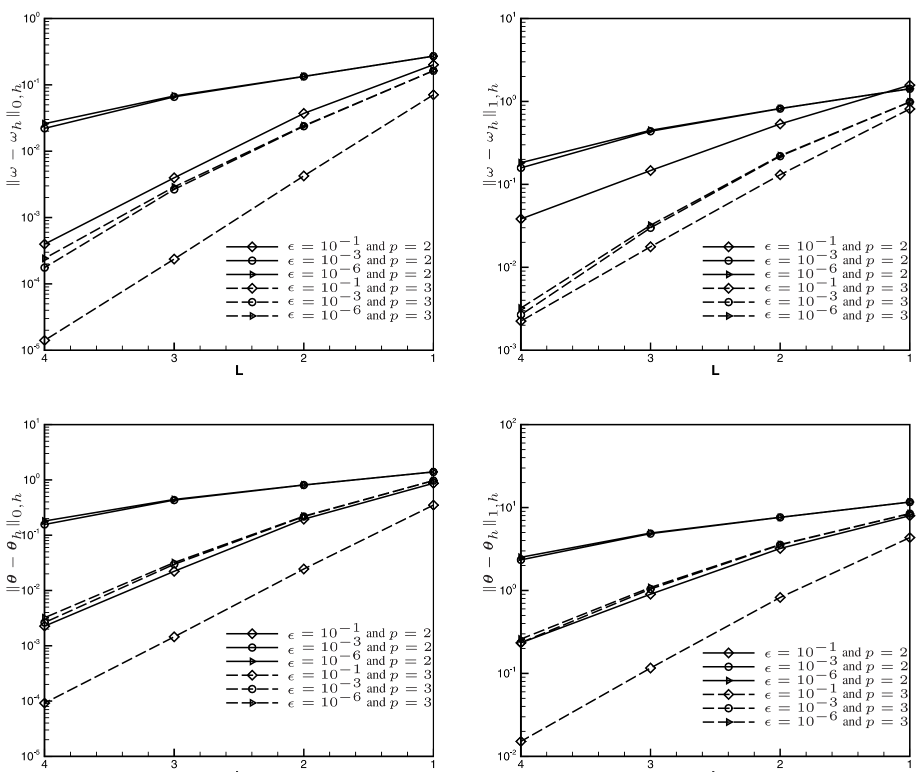

Fig. 1. Errors in ω (top), and $\boldsymbol{\theta}$ (bottom), with respect to the refinement level L. At the left we consider the L_2 norm, and at the right the $H^1(\mathcal{T}_h)$ norm. All the results are for the symmetric formulation. We considered $p = 2, 3$ and $\epsilon = 10^{-1}, 10^{-3}, 10^{-6}$.

constant in bound (4.6) does not depend on thickness ϵ. Note that, in the H^1 norm, the errors of approximation of vertical displacement is almost the same for all thickness, for a given approximation order, and the rotation error is less uniform in ϵ (since the error corresponding to thickness $\epsilon = 10^{-1}$ is at least one order better), indicating that the method approximates better the rotation for thicker plates. Indeed, for thick plates, far from the asymptotic limit, the Reissner−Mindlin equations behave as a "regular" second-order elliptic system. Since our numerical tests use order p for the rotation, the convergence rates in H^1 are of the same order. The errors in the L^2 norm are significantly better than in the H^1 norm and exhibit a similar behavior in respect to ϵ. We stress that the results are locking free.

We now investigate the convergence rates for both the vertical displacements and rotations. Table 1 contains the results for the symmetric formulation and in Table 2, we display the convergence rates for the nonsymmetric formulation. Since the norm $\|\cdot\|_{1,h}$ is bounded from above by a constant time $\|e(\cdot)\|_{0,h} + \|\beta \cdot\|_\Gamma$ (see Lemma 4.6 of Ref. 4, and also the more general results of Ref. 12), from the theoretically predicted rate of convergence for energy norm follows that the rate of convergence of the error

Table 1. Numerical convergence with the symmetric formulation and triangles.

		e_ω with $L^2(\mathcal{T}_h)$			e_ω with $H^1(\mathcal{T}_h)$			$e_{\boldsymbol\theta}$ with $L^2(\mathcal{T}_h)$			$e_{\boldsymbol\theta}$ with $H^1(\mathcal{T}_h)$		
p	$r_L\backslash\epsilon$	ϵ_1	ϵ_2	ϵ_3	ϵ_1	ϵ_2	ϵ_3	ϵ_1	ϵ_2	ϵ_3	ϵ_1	ϵ_2	ϵ_3
2	2	2.4	1.0	1.0	1.5	0.8	0.8	2.1	0.8	0.8	1.3	0.6	0.6
	3	3.2	1.0	1.0	1.9	0.9	0.9	3.1	0.9	0.9	1.8	0.6	0.6
	4	3.3	1.6	1.4	1.9	1.5	1.3	3.3	1.5	1.3	1.9	1.0	1.0
3	2	4.0	2.8	2.7	2.6	2.2	2.2	3.8	2.2	2.1	2.4	1.3	1.2
	3	4.2	3.2	3.1	2.9	2.9	2.8	4.1	2.9	2.8	2.8	1.8	1.7
	4	4.1	3.9	3.6	3.0	3.5	3.3	4.0	3.5	3.3	2.9	2.1	2.1
4	2	4.2	3.5	3.5	3.0	3.0	3.0	4.6	3.0	3.0	3.5	2.0	2.0
	3	4.8	5.0	4.8	3.8	4.0	3.9	4.8	4.1	3.9	3.8	2.9	2.8
	4	4.7	5.8	5.4	3.9	4.3	4.2	3.8	4.4	4.2	2.9	3.2	3.1

Note: $\epsilon_1 = 10^{-1}$, $\epsilon_2 = 10^{-3}$ and $\epsilon_3 = 10^{-6}$.

Table 2. Numerical convergence with the nonsymmetric formulation and triangles.

		e_ω with $L^2(\mathcal{T}_h)$			e_ω with $H^1(\mathcal{T}_h)$			$e_{\boldsymbol\theta}$ with $L^2(\mathcal{T}_h)$			$e_{\boldsymbol\theta}$ with $H^1(\mathcal{T}_h)$		
p	$r_L\backslash\epsilon$	ϵ_1	ϵ_2	ϵ_3	ϵ_1	ϵ_2	ϵ_3	ϵ_1	ϵ_2	ϵ_3	ϵ_1	ϵ_2	ϵ_3
2	2	2.4	1.0	1.0	1.6	0.8	0.8	2.2	0.8	0.8	1.4	0.6	0.6
	3	3.0	1.0	1.0	1.9	0.9	0.9	2.9	0.9	0.9	1.8	0.7	0.6
	4	2.8	1.6	1.4	1.9	1.5	1.3	2.8	1.5	1.3	2.0	1.1	1.0
3	2	4.0	2.7	2.7	2.6	2.1	2.1	3.8	2.1	2.1	2.4	1.3	1.2
	3	4.1	3.0	2.9	2.9	2.8	2.7	4.1	2.8	2.7	2.8	1.8	1.7
	4	4.0	3.2	3.0	3.0	3.3	3.1	4.0	3.3	3.1	2.9	2.1	2.1
4	2	4.2	3.5	3.4	3.0	3.0	3.0	4.7	3.0	3.0	3.5	2.0	2.0
	3	4.8	4.9	4.7	3.8	4.1	4.0	4.9	4.1	4.0	3.8	2.9	2.8
	4	4.6	4.8	4.5	3.9	4.4	4.2	3.9	4.4	4.2	3.0	3.2	3.1

Note: $\epsilon_1 = 10^{-1}$, $\epsilon_2 = 10^{-3}$ and $\epsilon_3 = 10^{-6}$.

of rotation in the H^1 norm should be $p-1$ in both formulations. This is clearly confirmed by our numerical experiments as can be seen in the last column of both Tables 1 and 2 that contain the results for $e_{\boldsymbol{\theta}}$ in $H^1(\mathcal{K}_h)$ norm. The convergence order in L^2 norm for the vertical displacement is approximately $p+1, p>2$ for symmetric version, that coincides with the similar results for biharmonic equation.[35] We remark also that for the nonsymmetric version these rates are reduced when compared to the symmetric case. For all the other cases, both formulations display similar results for all p. Finally, numerical tests for quadrilateral meshes yield convergence rates similar to the ones presented here.

6. Conclusion

Comparing our scheme with that of Ref. 7, we note that their choice of interpolation spaces are the same as ours, i.e. continuous or discontinuous polynomials of degree p for W_h, and discontinuous polynomials of degree $p-1$ for $\boldsymbol{\Theta}_h$, with $p \geq 2$. Also, they choose the space for the shear as being the same as the space for the rotation. We do not need such space in our formulation.

Splitting their analysis in two separate cases, depending on whether W_h is continuous or not, the authors of Ref. 7 obtain that the rotation error, in a norm slightly different from ours, plus the thickness times of the L^2 norm of the shear, is bounded by $ch^{p-1}(\|\boldsymbol{\theta}\|_{p,\Omega} + \epsilon\|\boldsymbol{\gamma}\|_{p-1,\Omega})$ if W_h is continuous, and bounded by $ch^{p-1}(\|\boldsymbol{\theta}\|_{p,\Omega} + \|\boldsymbol{\gamma}\|_{p-1,\Omega})$ if W_h is discontinuous. The undesirable term $\|\boldsymbol{\gamma}\|_{p-1,\Omega}$ can be replaced if a Helmholtz decomposition holds, and that is the case for $p=2$ and convex Ω. Thus, for $p=2$ the estimate does not blow up with ϵ.

On the other hand, our analysis is unified and does not require the Helmholtz decomposition. Our own estimate for the rotation error behaves like $ch^{p-1}(\|\boldsymbol{\theta}\|_{p,\Omega} + \|\omega\|_{p+1,\Omega} + \epsilon\|\boldsymbol{\gamma}\|_{p-1,\Omega})$ in general. For $p=2$, at least for smooth domains,[9] the term $\|\omega\|_{3,\Omega}$ can be uniformly bounded with respect to ϵ.

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