FINITE ELEMENT TECHNIQUES FOR LOCAL ACTIVE CONTROL OF NOISE IN ENCLOSURES

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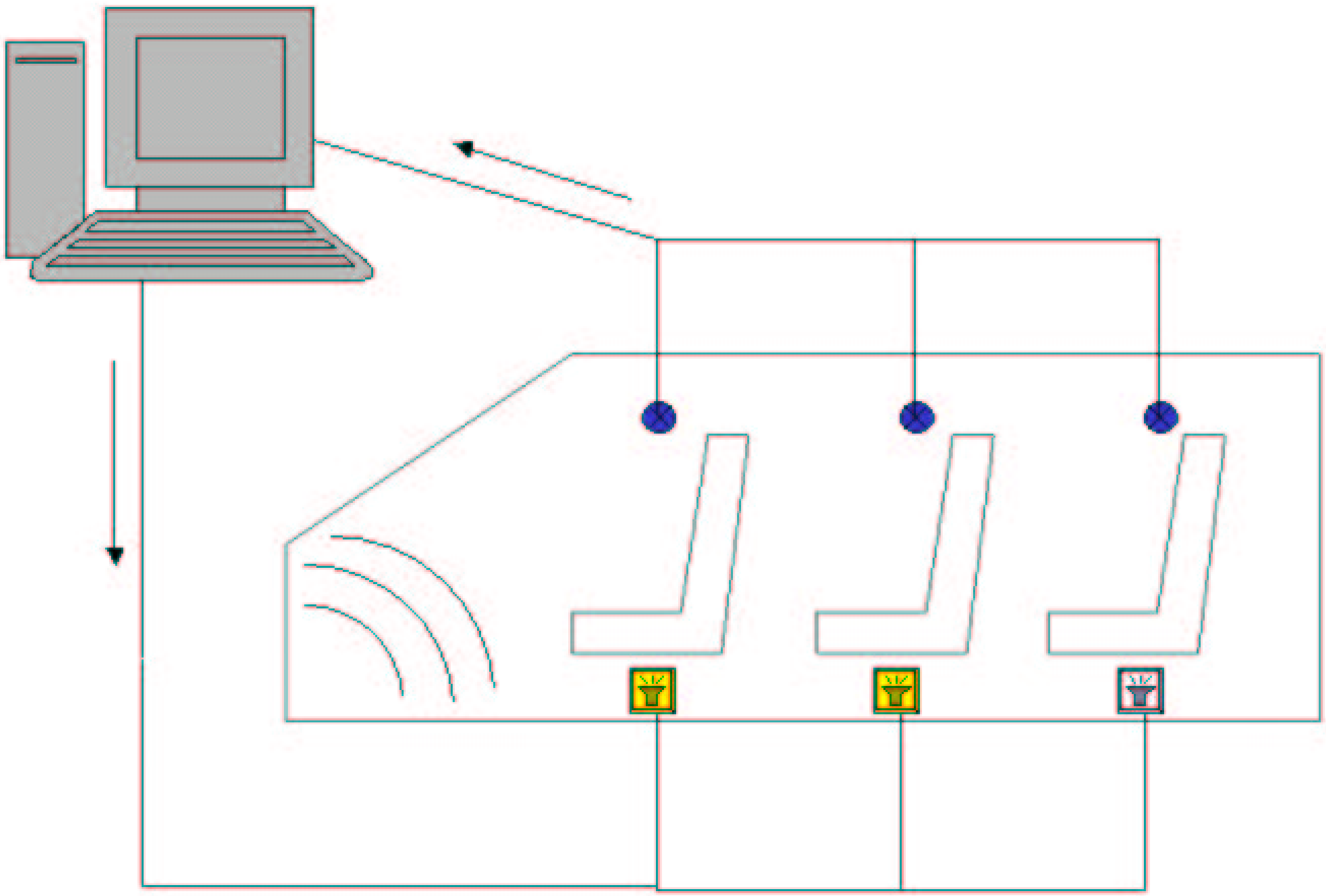

Passive-active control system of noise.

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1. THE ACOUSTICAL MODEL FOR ENCLOSURES

1) We consider the acoustic wave equation in a bounded domain Ω_{F} (enclosure):

$$\begin{cases} \dfrac{\partial^2 \hat{p}(\boldsymbol{x},t)}{\partial t^2} = c^2 \Delta \hat{p}(\boldsymbol{x},t) + \hat{f}(\boldsymbol{x},t) \qquad \text{in } \Omega_{\mathrm{F}}, \\ \text{boundary conditions} + \text{initial conditions}, \end{cases}$$

where $\hat{p}$ is the pressure, c the sound velocity and $\hat{f}$ is a sound source (which will be the control variable in our case).

2) We are going to study the steady state **harmonic problem** arising when the source is time-harmonic with angular frequency ω:

$$\hat{f}(\boldsymbol{x},t) := \mathrm{Re}\left[f(\mathbf{x})e^{-i\omega t}\right].$$

(We do not take account of the transient response.)

3) This leads to the following **Helmholtz problem**:

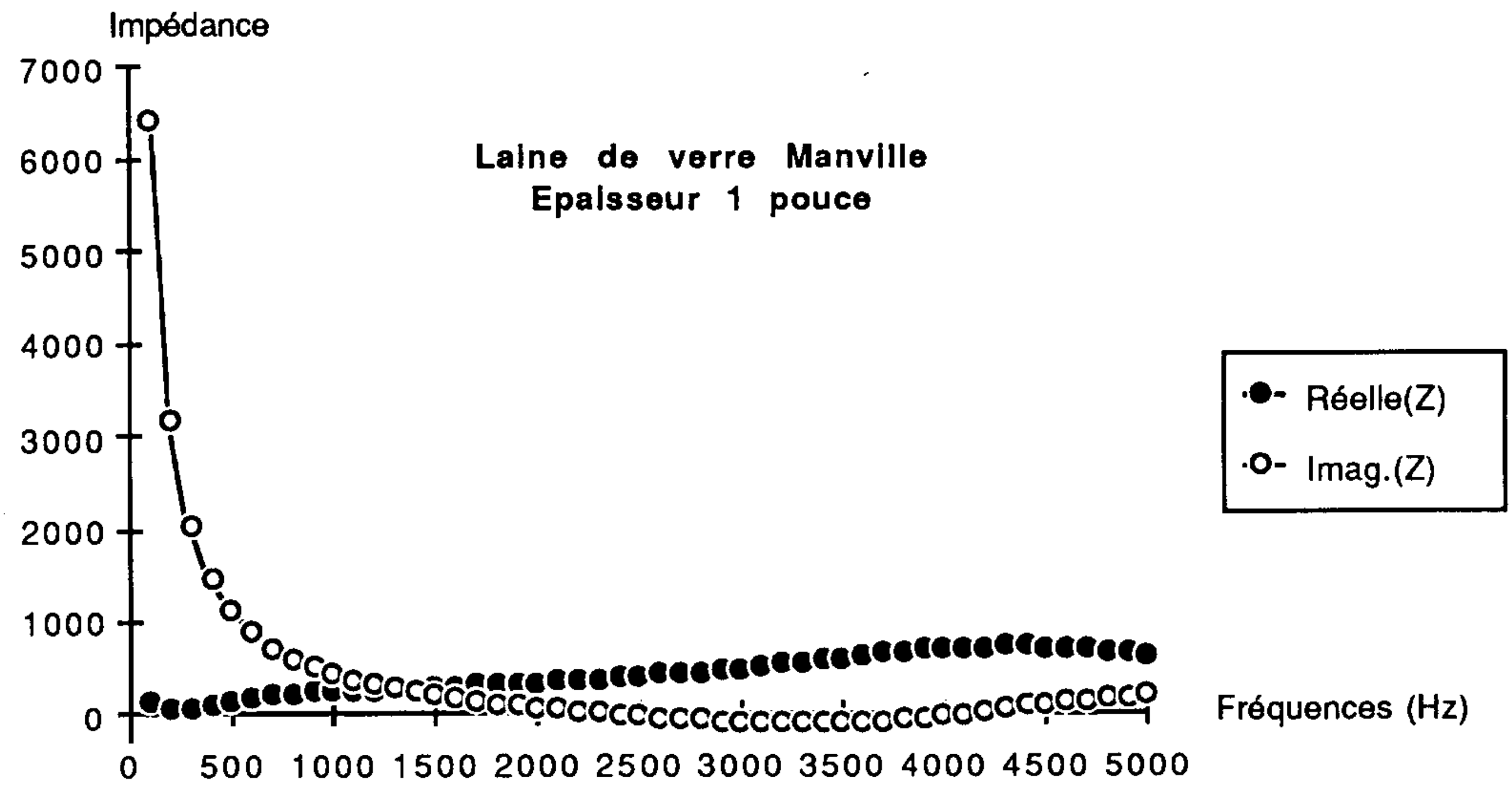
$$\begin{cases} -c^2\Delta p - \omega^2 p = f & \text{in} \quad \Omega_{\mathrm{F}}, \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = \dfrac{i\omega\rho}{Z(\omega)}\, p & \text{on} \quad \Gamma_{\mathrm{Z}} \subset \partial\Omega_{\mathrm{F}}, \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = g & \text{on} \quad \Gamma_{\mathrm{N}} \subset \partial\Omega_{\mathrm{F}}. \end{cases}$$

- $Z(\omega) \in \mathbb{C}$ is the wall impedance (Robin boundary condition). It models the behavior of thin layers of viscoelastic materials (for example, glass-wool),

$$Z(\omega) := \beta(\omega) + \frac{\alpha(\omega)}{\omega}\, i \qquad \text{(dashpot - spring system).}$$

- $\hat{g}(\boldsymbol{x}, t) := \operatorname{Re}\left[g(\boldsymbol{x})e^{-i\omega t}\right]$ is the time-harmonic boundary excitation, that is, the noise source (for example, the vibrations coming from an engine).

- In general $g(\boldsymbol{x})$ is a complex function, which allows delays between different points on the boundary Γ_{N} (Neumann boundary condition).



Acoustic impedance of a Manville glass-wool layer (1 inch).

2. RESONANCES

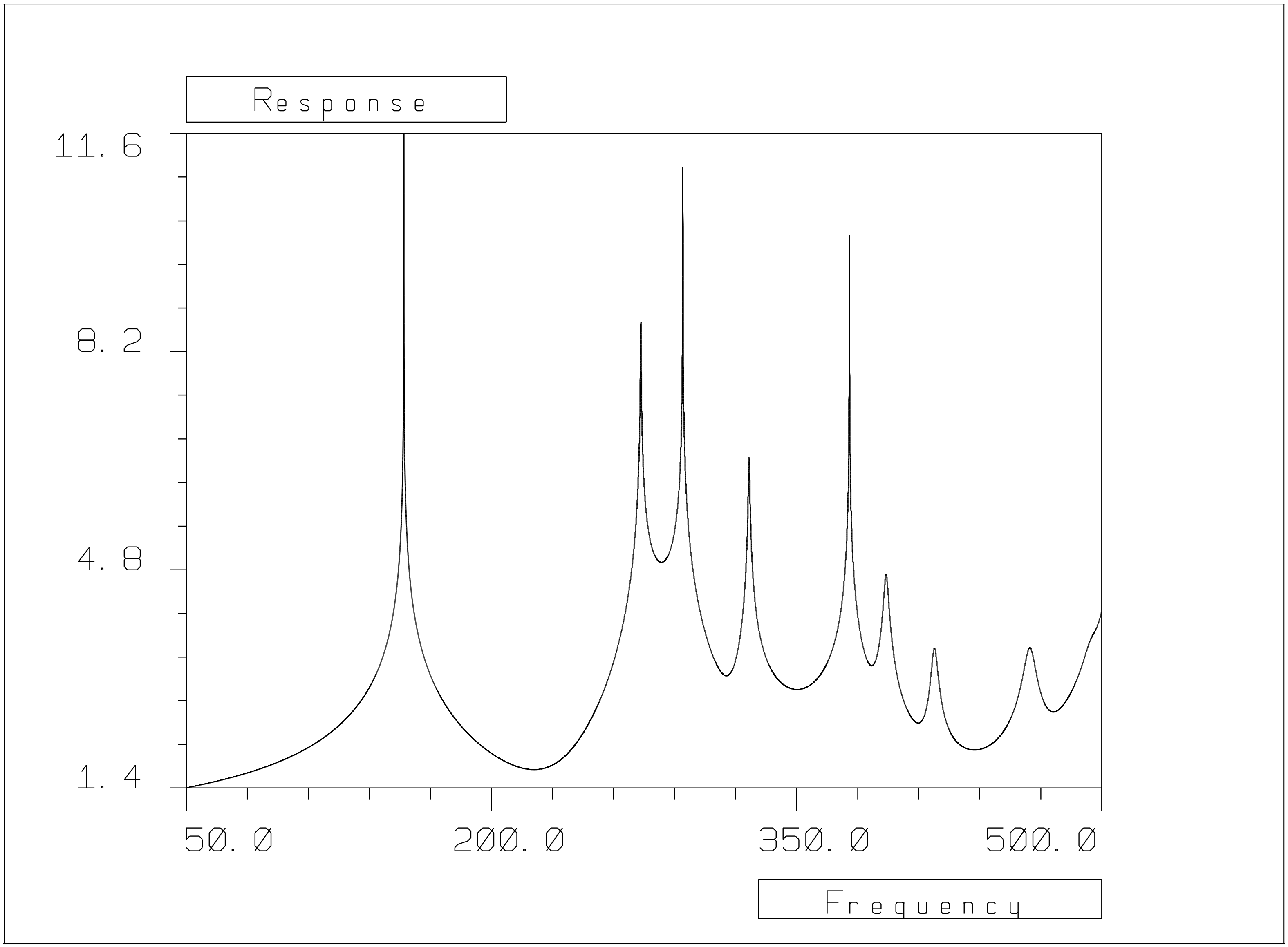
- For an enclosure Ω_{F} subject to a harmonic acoustic field pressure $p(\boldsymbol{x})$, the L^2 norm

$$\|p\|^2_{\mathrm{L}^2(\Omega_{\mathrm{F}})} := \int_{\Omega_{\mathrm{F}}} |p|^2 \, dx$$

  is a measure of the inside global noise level.

- The **frequency response curve** of the system is obtained by solving the source problem in a range of frequencies and plotting $\|p\|_{\mathrm{L}^2(\Omega_{\mathrm{F}})}$ versus ω.

- The "peaks" in the curve determine the so called **resonance frequencies**.



Frequency response curve.

- The active control is carried out for low frequencies (range where the passive systems are not efficient) in a band around the resonance frequencies, because the inside noise level is high in such bands.

- For the sake of simplicity, in our case, we consider that the source of noise is a pure tone.

- These resonance frequencies are close to the real part of the eigenvalues ω with a small imaginary part of the following non linear spectral problem:

Find $\omega \in \mathbb{C}$ *and* p *such that*

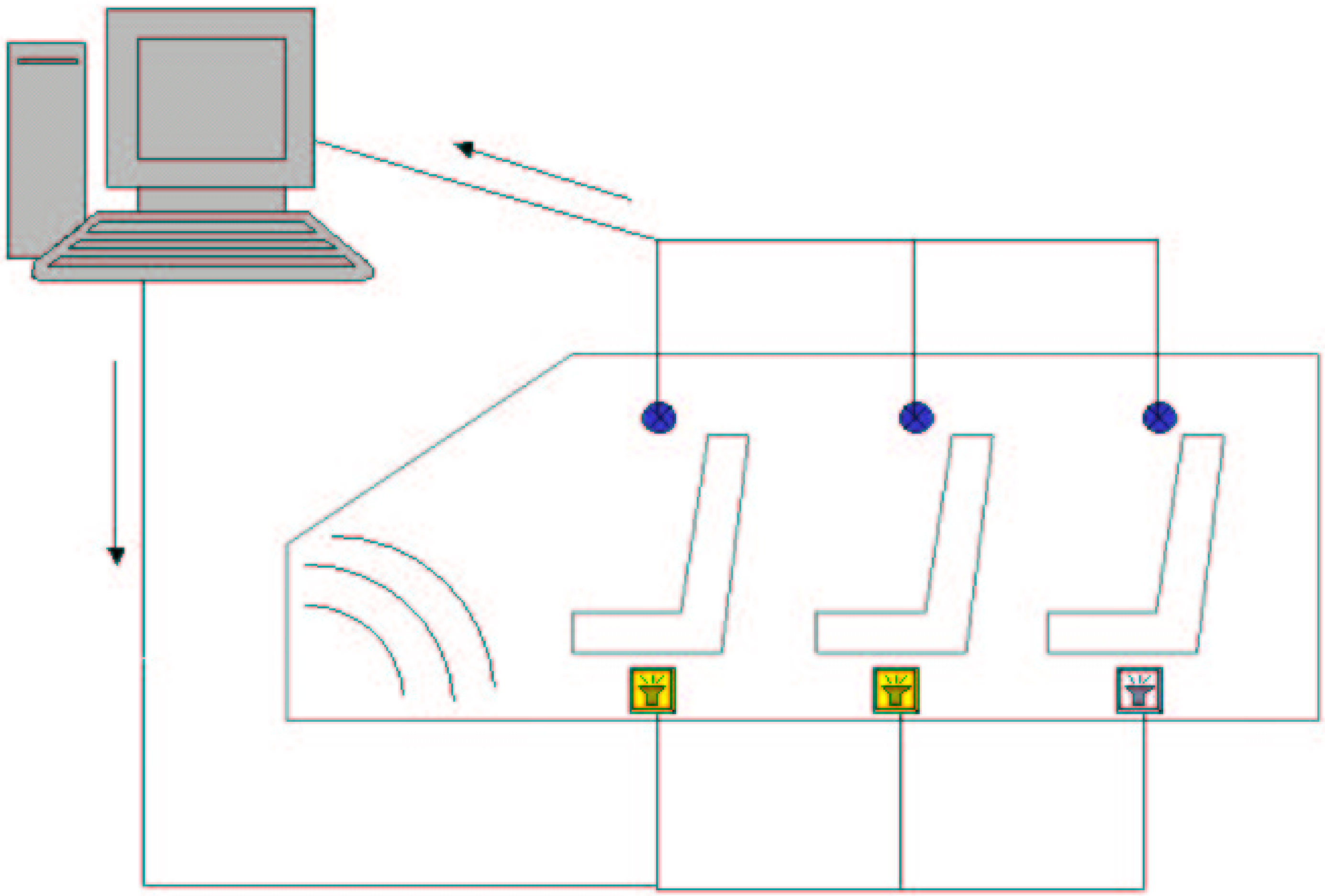
$$\begin{cases} -c^2 \Delta p = \omega^2 p & \text{in} \quad \Omega_{\mathrm{F}}, \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = \dfrac{i\omega\rho}{Z(\omega)} p & \text{on} \quad \Gamma_{\mathrm{Z}}, \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = 0 & \text{on} \quad \Gamma_{\mathrm{N}}. \end{cases}$$

- If $\beta(\omega) = \mathrm{Re}(Z) > 0$ (viscous part of the wall impedance), then the eigenvalues of the above spectral problem have non zero negative imaginary part.

- This means that the corresponding eigenfunctions show an exponential decay in time.

- On the other hand, since the eigenvalues are not real, the corresponding Helmholtz problem is well-posed (existence and uniqueness of solution) for all $\omega \in \mathbb{R}$.

3. ACTIVE CONTROL:
THE OPTIMAL CONTROL PROBLEM



3.1 Optimal source amplitudes

1) The Active Control of Sound in an enclosure can be posed in the framework of the Optimal Control Theory (J.L. LIONS).

2) In a first step, the **control variable** $\boldsymbol{u} \in \mathbb{C}^{N_a}$ are the **complex amplitudes** of the secondary sources (N_a loudspeakers), modeled as Dirac measures (monopoles) and placed at fixed points $\boldsymbol{x}_1^a, \ldots, \boldsymbol{x}_{N_a}^a$:

$$f(\boldsymbol{x}) = \sum_{i=1}^{N_a} u_i \, \delta(\boldsymbol{x} - \boldsymbol{x}_i^a), \qquad \boldsymbol{u} := (u_1, \ldots, u_{N_a}) \in \mathbb{C}^{N_a}.$$

The set of admissible controls will be a convex subset U of $\mathbb{C}^{N_a}$, v.g., $U = \left\{\boldsymbol{u} \in \mathbb{C}^{N_a} : \ |u_i| \leq u_{\max} \ , \ i = 1, \ldots, N_a\right\} \subset \mathbb{C}^{N_a}$.

3) Let $p(\boldsymbol{u})$ be the solution of the Helmholtz problem with Dirac measures sources of amplitudes $\boldsymbol{u} = (u_1, \ldots, u_{N_a})$:

$$\begin{cases} -c^2 \Delta p - \omega^2 p = \displaystyle\sum_{i=1}^{N_a} u_i \, \delta(\boldsymbol{x} - \boldsymbol{x}_i^a) & \text{in} \quad \Omega_{\mathrm{F}}, \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = \dfrac{i\omega\rho}{Z(\omega)} \, p & \text{on} \quad \Gamma_{\mathrm{Z}}, \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = g & \text{on} \quad \Gamma_{\mathrm{N}}. \end{cases}$$

The solution $p(\boldsymbol{u})$ belongs to $\mathrm{L}^2(\Omega_{\mathrm{F}})$ and, moreover, it is continuous in Ω_{F} except at the points $\boldsymbol{x}_1^a, \ldots, \boldsymbol{x}_{N_a}^a$ where the Dirac measures are supported (LIONS AND MAGENES).

4) The **observation** $\boldsymbol{z}$ is the vector of pressure values at certain points $\boldsymbol{x}_1^s, \dots, \boldsymbol{x}_{N_s}^s \in \Omega_{\mathrm{F}}$ (N_s number of sensors):

$$\boldsymbol{z} := \big(p(\boldsymbol{u})(\boldsymbol{x}_1^s), \dots, p(\boldsymbol{u})(\boldsymbol{x}_{N_s}^s)\big) \in \mathbb{C}^{N_s}.$$

5) In our case, the goal is to reduce the noise level only at the sensors location (local control) and, therefore, the **cost function** will be

$$J(\boldsymbol{u}) := \Phi(\boldsymbol{z}(\boldsymbol{u}), \boldsymbol{u}) = \frac{1}{2} \sum_{i=1}^{N_s} |p(\boldsymbol{u})(\boldsymbol{x}_i^s)|^2 + \frac{\nu}{2} \sum_{i=1}^{N_a} |u_i|^2,$$

where $\nu \geq 0$ is the so called control cost term.

Quadratic programming problem.

1) Since

- the set of admissible controls U belongs to a finite dimensional space,
- the application between the control $\boldsymbol{u}$ and the observation $\boldsymbol{z}$ is affine,
- and the function J is quadratic,

then, the optimal control problem turns out a **finite quadratic programming problem**.

2) To see this, we introduce the following definitions:

- The set of observations:

$$\boldsymbol{z}_0 := \left(p(\boldsymbol{e}_0)(\boldsymbol{x}_1^s), \ldots, p(\boldsymbol{e}_0)(\boldsymbol{x}_{N_s}^s)\right), \qquad \text{where } \boldsymbol{e}_0 := \boldsymbol{0},$$

$$\boldsymbol{z}_i := \left(p(\boldsymbol{e}_i)(\boldsymbol{x}_1^s), \ldots, p(\boldsymbol{e}_i)(\boldsymbol{x}_{N_s}^s)\right) - \boldsymbol{z}_0, \quad \text{where } (\boldsymbol{e}_i)_j := \delta_{ij}, \quad i, j = 1, \ldots, N_a.$$

Then, the affine map between control and observations reads:

$$\boldsymbol{z}(\boldsymbol{u}) = \boldsymbol{z}_0 + \sum_{i=1}^{N_a} u_i \boldsymbol{z}_i.$$

- The Hermitian matrix $\boldsymbol{Z}$ and the vector $\boldsymbol{b}_0$ of entries

$$(\boldsymbol{Z})_{ij} := \langle \boldsymbol{z}_j, \boldsymbol{z}_i \rangle, \qquad i, j = 1, \ldots, N_a,$$

$$(\boldsymbol{b}_0)_i := \langle \boldsymbol{z}_0, \boldsymbol{z}_i \rangle, \qquad i = 1, \ldots, N_a.$$

3) Then, the optimal control problem reads:

Find $\boldsymbol{u}_{\rm opt} \in U \subset \mathbb{C}^{N_a}$ *such that*

$$J(\boldsymbol{u}_{\rm opt}) = \min_{\boldsymbol{u}\in U} \left[\frac{1}{2} \Big(\left(\boldsymbol{Z} + \nu \boldsymbol{I} \right) \boldsymbol{u}, \boldsymbol{u} \Big) + \mathrm{Re} \left(\boldsymbol{b}_0, \boldsymbol{u} \right) \right].$$

4) The following variational inequality (optimality condition) determines the optimal control $\boldsymbol{u}_{\rm opt}$:

$$\mathrm{Re} \left(J'(\boldsymbol{u}_{\rm opt}) \, , \, \boldsymbol{v} - \boldsymbol{u}_{\rm opt} \right) = \mathrm{Re} \left((\boldsymbol{Z} + \nu \boldsymbol{I})\boldsymbol{u}_{\rm opt} + \boldsymbol{b}_0, \boldsymbol{v} - \boldsymbol{u}_{\rm opt} \right) \geq 0 \qquad \forall \boldsymbol{v} \in U.$$

5) It is easy to show that the optimal control problem attains a unique solution if:

- either $\nu > 0$
- or $\nu = 0$ and $\boldsymbol{Z}$ is non singular.

6) If U is a subspace of $\mathbb{C}^{N_a}$ (v.g., if no constraint like $|u_i| \leq u_{\max}$ must be satisfied), under any of the above assumptions, $\boldsymbol{u}_{\mathrm{opt}}$ is just the solution of the linear system of equations

$$(\boldsymbol{Z} + \nu \boldsymbol{I})\boldsymbol{u}_{\mathrm{opt}} = -\boldsymbol{b}_0.$$

If $\nu = 0$ and $\boldsymbol{Z}$ is singular, then $\boldsymbol{u}_{\mathrm{opt}}$ is not uniquely determined, but a minimum of the cost function can be obtained by solving a rank-deficient least-squares algebraic problem.

3.2. Optimal source positions

1) In this case, the control variable are the **amplitudes** and **positions** of the secondary sources (modeled as Dirac measures again):

$$(\boldsymbol{u}, \boldsymbol{X}^a) := \left(u_1, \ldots, u_{N_a}, \boldsymbol{x}_1^a, \ldots, \boldsymbol{x}_{N_a}^a\right).$$

2) The set of admissible controls is

$$U_{\mathrm{ad}} := U \times \bar{\Omega}_a^{N_a},$$

where $\bar{\Omega}_a$ is a convex closed subset of Ω_{F}.

3) The observation $\boldsymbol{z}$ is the pressure values at points $\boldsymbol{x}_1^s, \ldots, \boldsymbol{x}_{N_s}^s$ again (N_s number of sensors).

4) We consider the same cost function as in the above case,

$$\begin{aligned} J(\boldsymbol{u}, \boldsymbol{X}^a) &:= \Phi\big(\boldsymbol{z}(\boldsymbol{u}, \boldsymbol{X}^a), \boldsymbol{u}\big) \\ &= \frac{1}{2} \sum_{i=1}^{N_s} \left| p(\boldsymbol{u}, \boldsymbol{X}^a)(\boldsymbol{x}_i^s) \right|^2 + \frac{\nu}{2} \sum_{i=1}^{N_a} \left| u_i \right|^2 , \end{aligned}$$

where, now, the state p depends on the source amplitudes and positions.

5) The relation between the state p and the source positions is non linear.

6) Then, the optimal control problem reads:

Find $(\boldsymbol{u}_{\rm opt}, \boldsymbol{X}^a_{\rm opt}) \in U_{\rm ad} = U \times \bar{\Omega}_a^{N_a}$ *such that*

$$J(\boldsymbol{u}_{\rm opt}, \boldsymbol{X}^a_{\rm opt}) = \min_{(\boldsymbol{u}, \boldsymbol{X}^a) \in U_{\rm ad}} \left[\frac{1}{2} \sum_{i=1}^{N_s} \left| p(\boldsymbol{u}, \boldsymbol{X}^a)(\boldsymbol{x}_i^s) \right|^2 + \frac{\nu}{2} \sum_{i=1}^{N_a} |u_i|^2 \right].$$

7) In general, the **solution is not unique** and there could be **local minima**.

8) Therefore, the gradient like methods are not suitable in this case.

Optimization on a finite set of "available" positions.

- From the practical point of view, we pose the optimization problem on a **finite set** which consists of N_p "available" positions for the N_a secondary sources ($N_a \leq N_p$).

- The number of possible **configurations** will be $\binom{N_p}{N_a}$.

- For each configuration (fixed positions for the secondary sources) we minimize J with respect to the amplitudes as described above.

- Then, the **optimal configuration** will be the configuration for which the value of J is minimum.

- Simulated annealing and/or genetic algorithms methods are used to look for the optimal configuration instead of an exhaustive search, when the number of configurations is large (Baek and Elliot).

4. NUMERICAL APPROXIMATION OF THE STATE EQUATION

1) Before studying the optimal control approximation we must analyze the approximation of the state equation.

Helmholtz problem:

$$\left\{\begin{array}{rcll} -c^2\Delta p - \omega^2 p &=& f & \text{in} \quad \Omega_{\mathrm{F}}, \\ \dfrac{\partial p}{\partial \boldsymbol{n}} &=& \dfrac{i\omega\rho}{Z(\omega)}\, p & \text{on} \quad \Gamma_{\mathrm{Z}}, \\ \dfrac{\partial p}{\partial \boldsymbol{n}} &=& g & \text{on} \quad \Gamma_{\mathrm{N}}. \end{array}\right.$$

2) First, we write a variational formulation of the Helmholtz problem:

Find $p \in \mathrm{H}^1(\Omega_\mathrm{F})$ *such that*

$$\int_{\Omega_\mathrm{F}} \nabla p \cdot \nabla \bar{q}\, dx - \frac{i\omega\rho}{Z(\omega)} \int_{\Gamma_\mathrm{Z}} p\bar{q}\, d\Gamma - \frac{\omega^2}{c^2} \int_{\Omega_\mathrm{F}} p\bar{q}\, dx$$
$$= \int_{\Omega_\mathrm{F}} f\,\bar{q}\, dx + \int_{\Gamma_\mathrm{N}} g\bar{q}\, d\Gamma \qquad \forall q \in \mathrm{H}^1(\Omega_\mathrm{F}).$$

3) Then, as usual, we consider a regular family of triangulations $\{\mathcal{T}_h\}_{h>0}$ in Ω_{F} to build a sequence of discrete spaces $V_h \subset \mathrm{H}^1(\Omega_{\mathrm{F}})$ (finite element spaces). We denote by h the mesh-size.

4) The finite element spaces V_h are made up of global continuous functions in Ω_{F}, linear in each element of the mesh (a triangle if $\Omega_{\mathrm{F}} \subset \mathbb{R}^2$ or a tetrahedron if $\Omega_{\mathrm{F}} \subset \mathbb{R}^3$).

5) For each finite element space $V_h \subset \mathrm{H}^1(\Omega_{\mathrm{F}})$, the following variational discrete problem is obtained:

Find $p_h \in V_h$ *such that*

$$\int_{\Omega_{\mathrm{F}}} \nabla p_h \cdot \nabla \bar{q}_h \, dx - \frac{i\omega\rho}{Z(\omega)} \int_{\Gamma_{\mathrm{Z}}} p_h \bar{q}_h \, d\Gamma - \frac{\omega^2}{c^2} \int_{\Omega_{\mathrm{F}}} p_h \bar{q}_h \, dx$$
$$= \int_{\Omega_{\mathrm{F}}} f \bar{q}_h \, dx + \int_{\Gamma_{\mathrm{N}}} g \bar{q}_h \, d\Gamma \qquad \forall q_h \in V_h \subset \mathrm{H}^1(\Omega_{\mathrm{F}}).$$

The solution p_h of this discrete problem is an approximation of the exact solution p.

Convergence results.

The following results have been proved for a polygonal (polyhedral) convex domain Ω_{F} and h small enough:

- If $f(\boldsymbol{x}) \in \mathrm{L}^2(\Omega_{\mathrm{F}})$,

$$\|p - p_h\|_{\mathrm{L}^2(\Omega_{\mathrm{F}})} \leq C\,h^2, \qquad \text{for } \Omega_{\mathrm{F}} \subset \mathbb{R}^2 \text{ or } \Omega_{\mathrm{F}} \subset \mathbb{R}^3.$$

(Mikhlin, discrete LBB condition).

- If $f(\boldsymbol{x})$ is a Dirac measure supported at $\boldsymbol{x}_0$,

$$\|p - p_h\|_{\mathrm{L}^2(\Omega_{\mathrm{F}})} \leq \begin{cases} C(\boldsymbol{x}_0)\,h, & \text{for } \Omega_{\mathrm{F}} \subset \mathbb{R}^2, \\ C(\boldsymbol{x}_0)\,h^{\frac{1}{2}}, & \text{for } \Omega_{\mathrm{F}} \subset \mathbb{R}^3, \end{cases}$$

where $C(\boldsymbol{x}_0) \to \infty$ as $\boldsymbol{x}_0$ gets close to $\partial\Omega_{\mathrm{F}}$ (Casas, Scott).

5. NUMERICAL APPROXIMATION OF THE OPTIMAL SOURCE AMPLITUDES

Let us now consider the optimization problem with respect to the source amplitudes (their positions being fixed):

- From the FEM approximation of the state equation, we obtain the **approximated observations** $\boldsymbol{z}_{0h}, \dots, \boldsymbol{z}_{N_a h}$:

$$\boldsymbol{z}_{0h} := \left(p_h(\boldsymbol{e}_0)(\boldsymbol{x}_1^s), \dots, p_h(\boldsymbol{e}_0)(\boldsymbol{x}_{N_s}^s)\right),$$

$$\boldsymbol{z}_{ih} := \left(p_h(\boldsymbol{e}_i)(\boldsymbol{x}_1^s), \dots, p_h(\boldsymbol{e}_i)(\boldsymbol{x}_{N_s}^s)\right), \qquad i = 1, \dots, N_a.$$

where $p_h(\boldsymbol{u})$ is the solution of the discrete Helmholtz problem with Dirac measures sources of amplitudes $\boldsymbol{u} = (u_1, \dots, u_{N_a})$.

- We define $\boldsymbol{Z}_h$ and $\boldsymbol{b}_{0h}$ analogously to their continuous counterparts:

$$(\boldsymbol{Z}_h)_{ij} := \langle \boldsymbol{z}_j h, \boldsymbol{z}_i h \rangle \,, \qquad i,j = 1,\dots,N_a;$$
$$(\boldsymbol{b}_0)_i := \langle \boldsymbol{z}_0 h, \boldsymbol{z}_i h \rangle \,, \qquad i = 1,\dots,N_a;$$

- This leads to an **approximated quadratic programming problem**:

Find $\boldsymbol{u}^h_{\text{opt}} \in U \subset \mathbb{C}^{N_a}$ *such that*

$$J_h(\boldsymbol{u}^h_{\text{opt}}) = \min_{\boldsymbol{u}\in U} \left[\frac{1}{2} \Big(\left(\boldsymbol{Z}_h + \nu \boldsymbol{I} \right) \boldsymbol{u}, \boldsymbol{u} \Big) + \operatorname{Re} \left(\boldsymbol{b}_{0h}, \boldsymbol{u} \right) \right].$$

Its solution $\boldsymbol{u}^h_{\text{opt}}$ is an **approximation of the optimal control $\boldsymbol{u}_{\text{opt}}$**.

- To obtain an error bound for the optimal control, we need pointwise error estimates, since the observations consist of point values of the pressure.

- Hypothesis. *We suppose that the following pointwise estimate holds for the Helmholtz problem with Dirac measures:*

$$|p(\boldsymbol{x}) - p_h(\boldsymbol{x})| \leq C h^2 \frac{\log\left(|\boldsymbol{x} - \boldsymbol{x}_0|/h\right)}{|\boldsymbol{x} - \boldsymbol{x}_0|^n},$$

for $\boldsymbol{x}, \boldsymbol{x}_0 \in \tilde{\Omega}_{\mathrm{F}} \subset\subset \Omega_{\mathrm{F}} \subset \mathbb{R}^n$ *(*$n = 2$ *or* 3*), with* $|\boldsymbol{x} - \boldsymbol{x}_0| \geq Ch$.

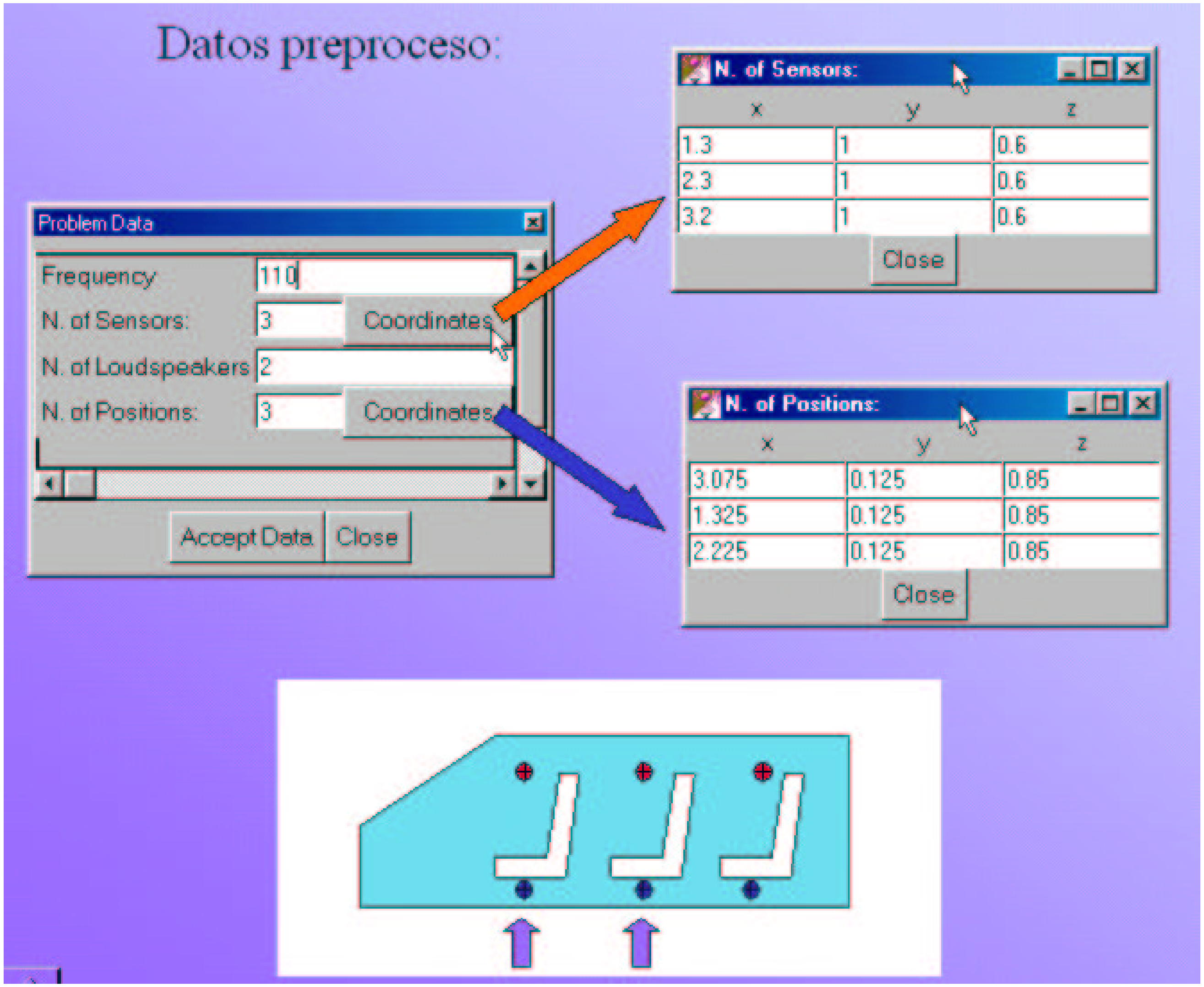
(A similar estimate is proved by Wahlbin for the Poisson problem.)

- Under the above assumption, the following **error estimate** holds:

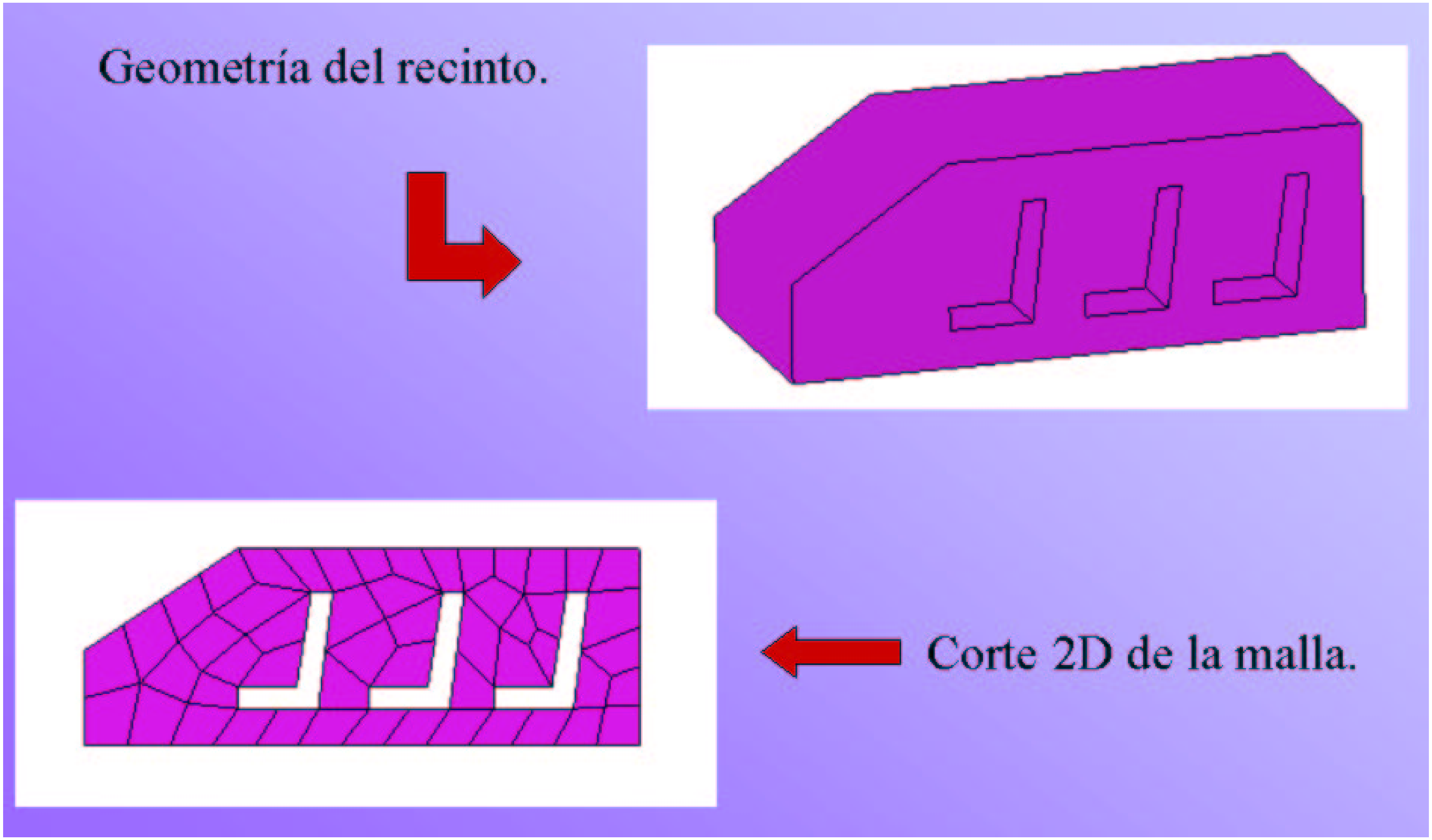
There exists $h_0 > 0$ *such that,* $\forall h < h_0$,

$$\frac{\|\boldsymbol{u}_{\mathrm{opt}} - \boldsymbol{u}^h_{\mathrm{opt}}\|_\infty}{\|\boldsymbol{u}_{\mathrm{opt}}\|_\infty} \leq C \operatorname{cond}_\infty(\boldsymbol{Z} + \nu \boldsymbol{I}) \left(\frac{\|\boldsymbol{Z} - \boldsymbol{Z}_h\|_\infty}{\|\boldsymbol{Z} + \nu \boldsymbol{I}\|_\infty} + \frac{\|\boldsymbol{b}_0 - \boldsymbol{b}_{0h}\|_\infty}{\|\boldsymbol{b}_0\|_\infty} \right) \leq C \log\left(\frac{1}{h}\right) h^2.$$

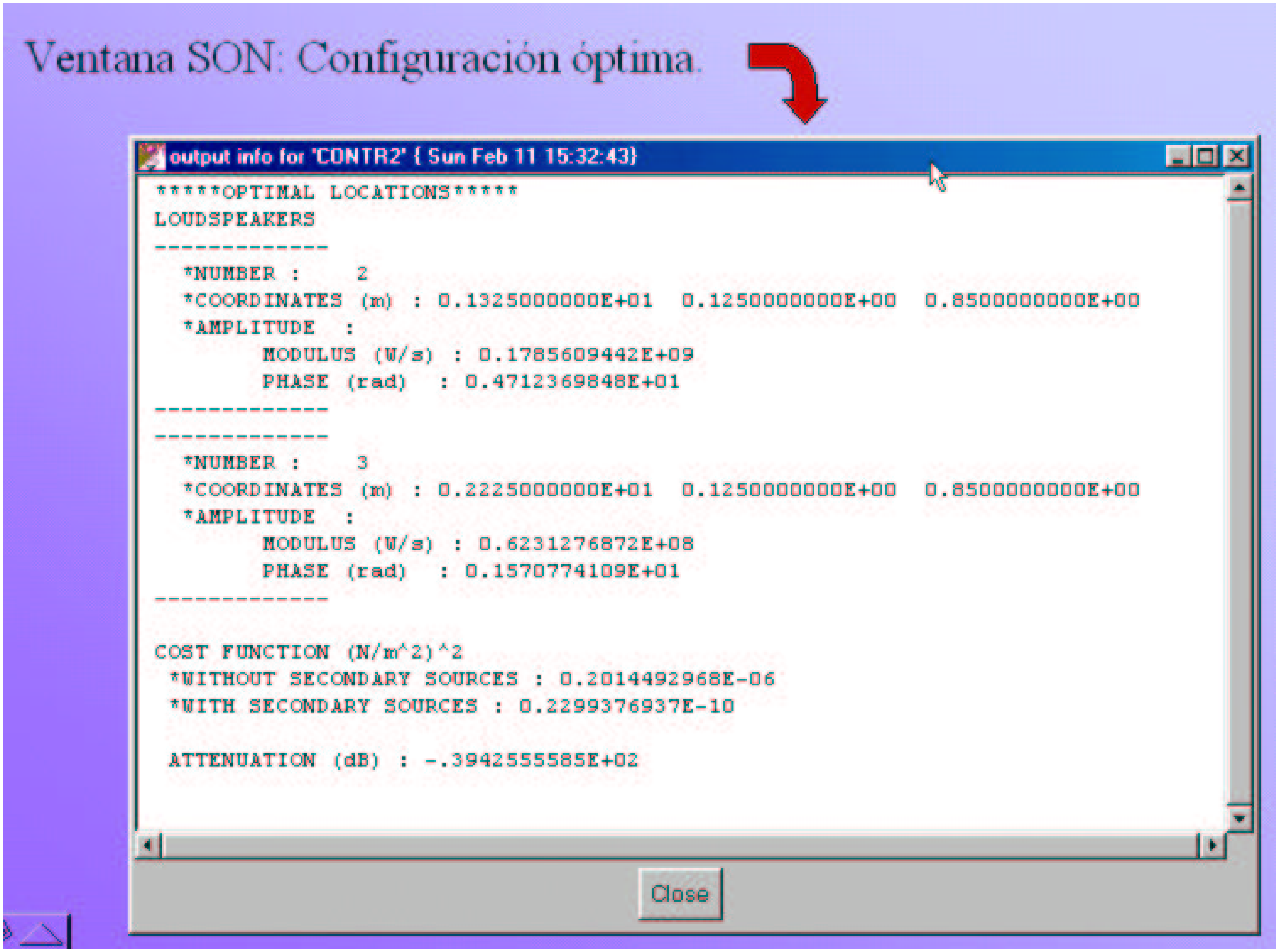
6. SOME NUMERICAL RESULTS



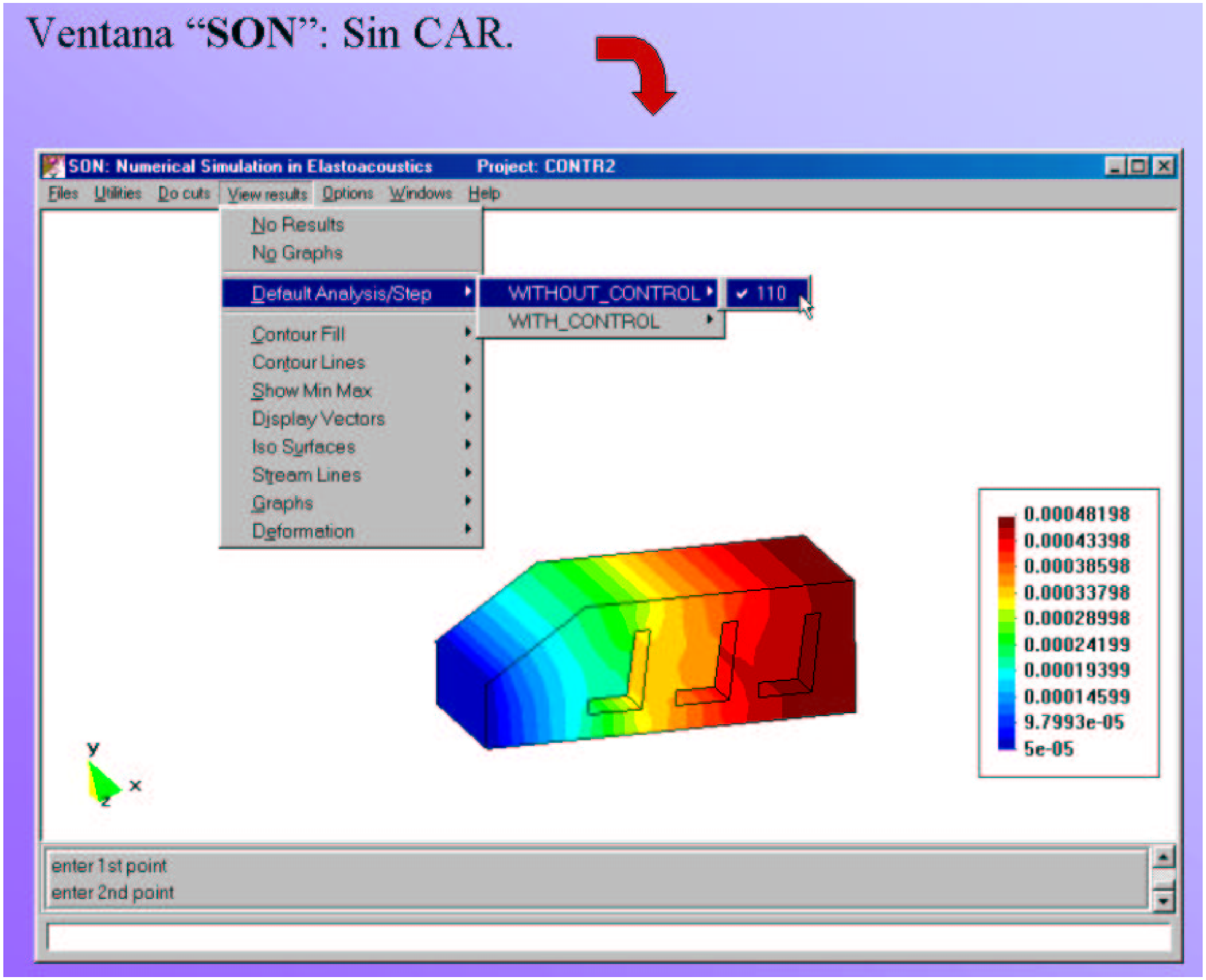
Problem data.



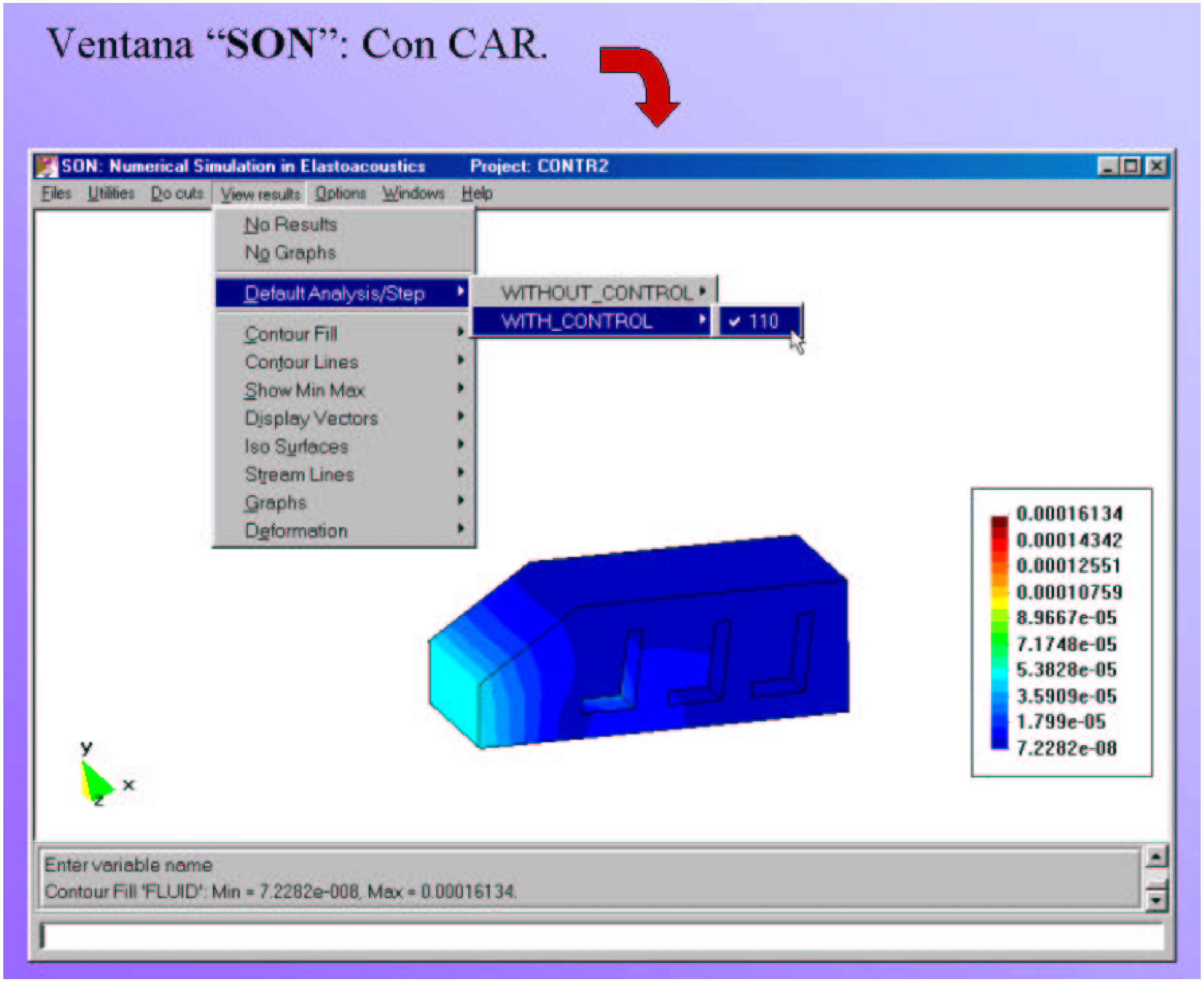
Geometry and vertical section of the mesh.



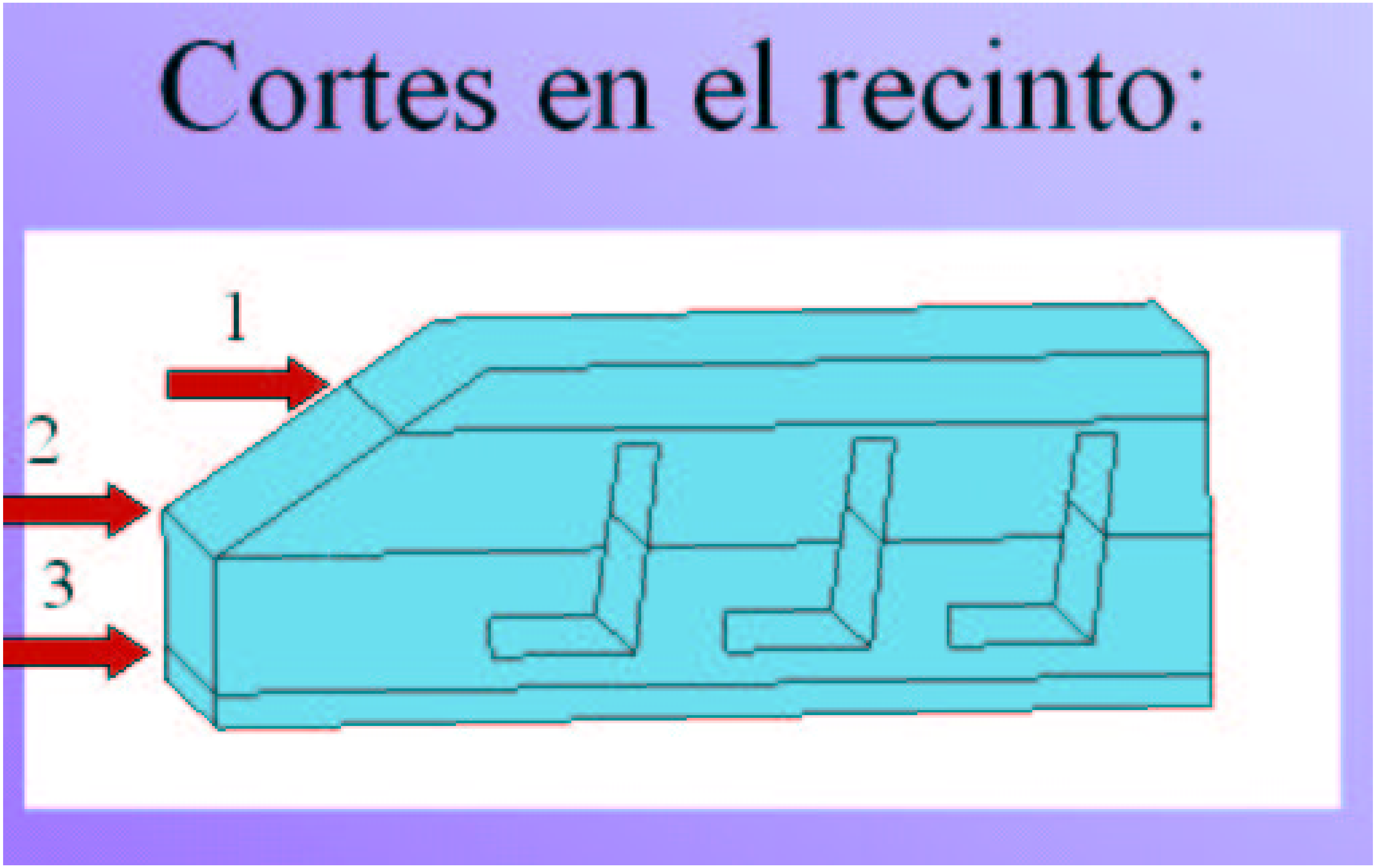
Results: optimal configuration and source amplitudes.



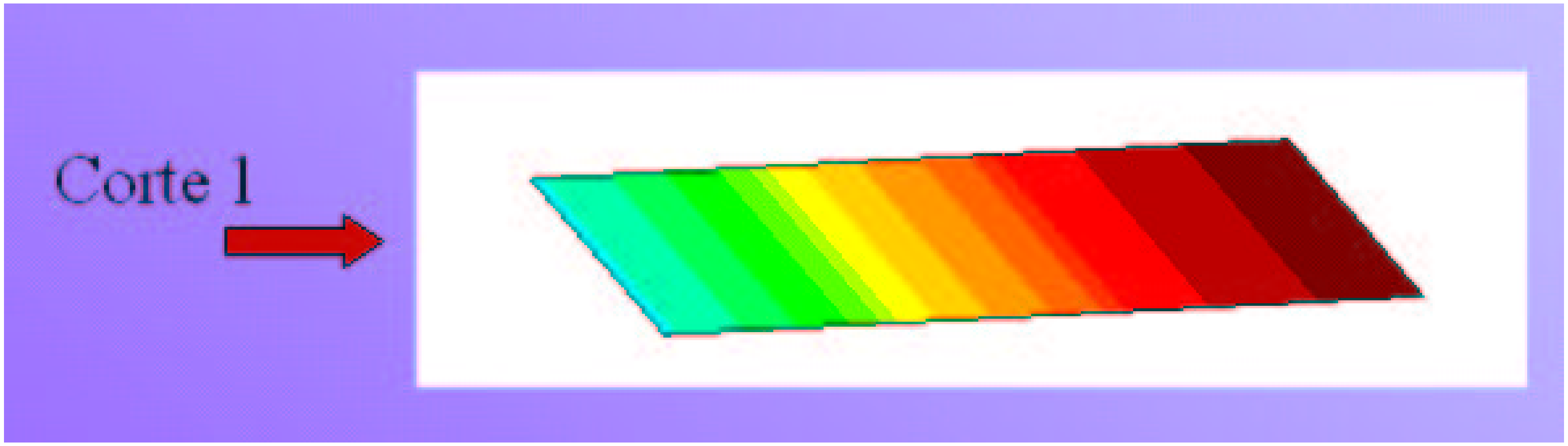
Pressure field without control.



Pressure field with control.

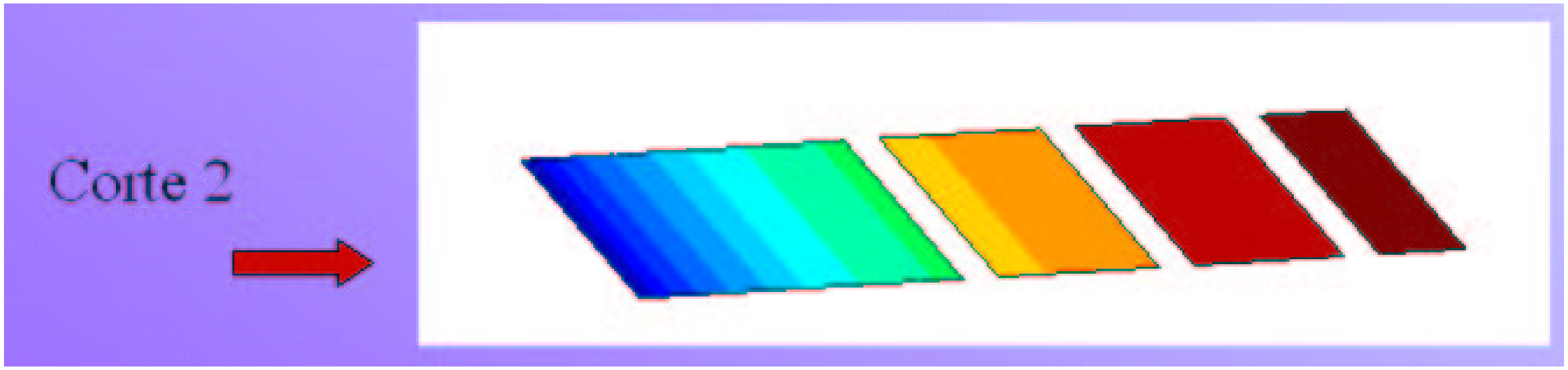


Horizontal sections of the enclosure.

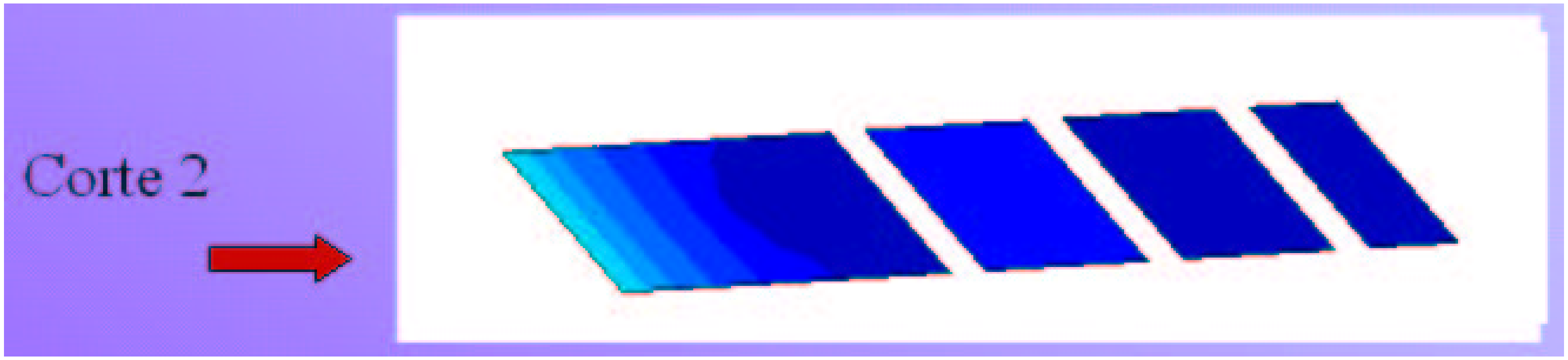


without control

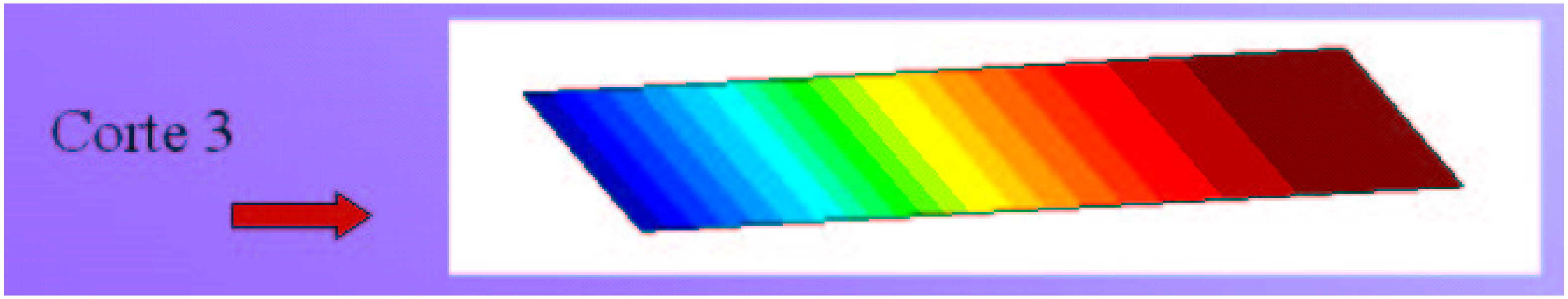
with control



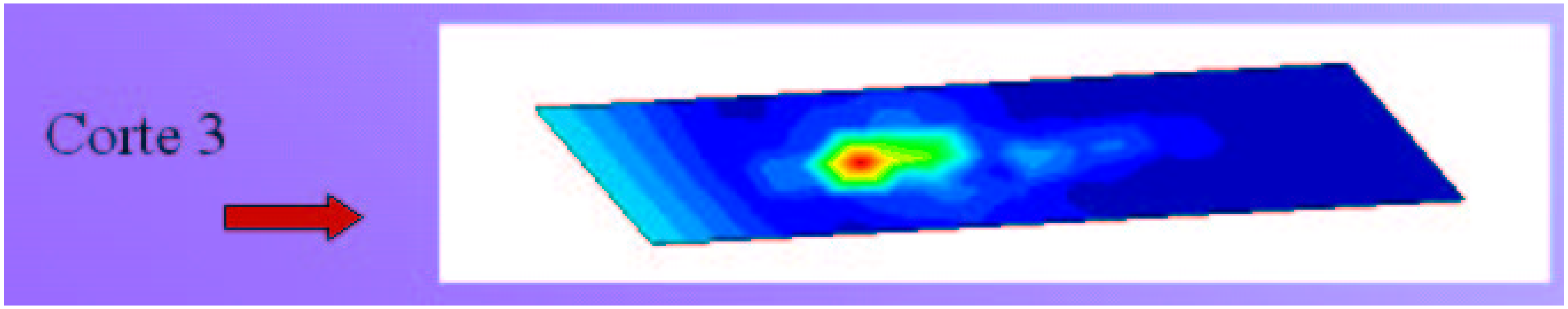
without control



with control



without control



with control