Computing vibration modes of a fluid-plate coupled system

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Acoustic fluid in a rigid cavity

Navier Stokes equations for a compressible fluid

$$\rho_{\mathrm{F}} \frac{Dv}{Dt} = -\nabla p + (\lambda + \mu)\,\nabla(\operatorname{div} v) + \mu \Delta v$$

- v: fluid velocity field,
- p: fluid pressure,
- ρ_{F}; fluid density,
- μ: first coefficient of viscosity,
- $\zeta := \lambda + \frac{2}{3}\mu$: second coefficient of viscosity,

$$+ \begin{cases} \text{state equation,} \\ \text{boundary conditions,} \\ \text{initial conditions.} \end{cases}$$

Acoustic fluid equations

Hypotheses:

- **Dissipative viscosity terms** very small with respect to pressure gradients:

$$|(\lambda+\mu)\,\nabla(\operatorname{div} v)| + |\mu\Delta v| \ll |\nabla\, p|\,.$$

- **Small displacements** from an equilibrium position:

$$v \approx \frac{\partial u}{\partial t} \qquad \text{and} \qquad \frac{Dv}{Dt} \approx \frac{\partial v}{\partial t}.$$

- **Isentropic state equation**:

$$\frac{\partial p}{\partial t} = -\rho_{\mathrm{F}}\, c^2 \operatorname{div} v,$$

  c: **acoustic speed** (speed of sound in the fluid).

Neglecting the viscosity terms and integrating in time the state equation:

$$\begin{cases} \rho_{\mathrm{F}}\,\ddot{u} = -\,\nabla\, p, \\ p = -\rho_{\mathrm{F}}\, c^2 \operatorname{div} u. \end{cases}$$

Note: $\ddot{u}$, $\dot{p}$, etc. denote time derivatives.

Dynamic equations in a rigid cavity Ω

Boundary slipping conditions:

- null normal displacements: $u \cdot n = 0$ on $\partial\Omega$,
- no constraint on the tangential displacements.

Pressure formulation:

$$\begin{cases} \ddot{p} = \Delta p \quad \text{in } \Omega, \\ \dfrac{\partial p}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Numerical solution: standard finite elements.

Displacement formulation:

$$\begin{cases} \rho_{\text{F}}\, \ddot{u} = \nabla(\rho_{\text{F}}\, c^2 \operatorname{div} u) \quad \text{in } \Omega, \\ u \cdot n = 0 \quad \text{on } \partial\Omega, \end{cases}$$

+ irrotational condition: $\quad u = \nabla \phi \quad \text{in } \Omega.$

Free vibration problem

Natural vibration modes: harmonic in time solutions.

For instance, for the displacement formulation, we look for

$$u(x,y,z,t) = u(x,y,z)\cos\omega t, \qquad (x,y,z) \in \Omega, \quad t \in \mathbb{R},$$

solution of

$$\begin{cases} \rho_{\mathrm{F}}\ddot{u} = \nabla(\rho_{\mathrm{F}} c^2 \operatorname{div} u) \quad \text{in } \Omega, \\ u \cdot n = 0 \quad \text{on } \partial\Omega, \\ u = \nabla\phi \quad \text{in } \Omega. \end{cases}$$

Spectral problem:

$$\boxed{\begin{array}{l} \textit{Find } \omega \in \mathbb{C} \textit{ and } u \neq 0 \textit{ satisfying} \\ \qquad \begin{cases} -\omega^2 \rho_{\mathrm{F}} u = \nabla(\rho_{\mathrm{F}} c^2 \operatorname{div} u) \quad \text{in } \Omega, \\ u \cdot n = 0 \quad \text{on } \partial\Omega. \end{cases} \end{array}}$$

Remark: If $\omega \neq 0$, then $u = \nabla\phi \quad \text{in } \Omega$, with $\phi = -\dfrac{c^2}{\omega^2}\operatorname{div} u$.

Variational formulation

Sobolev spaces:

$$
\begin{aligned}
\mathrm{H}^1(\Omega) &:= \left\{\phi \in \mathrm{L}^2(\Omega) : \ \nabla \phi \in \mathrm{L}^2(\Omega)^3\right\}, \\
\mathrm{H}(\mathrm{div}, \Omega) &:= \left\{v \in \mathrm{L}^2(\Omega)^3 : \ \mathrm{div}\, v \in \mathrm{L}^2(\Omega)\right\}, \\
\mathcal{V} := \mathrm{H}_0(\mathrm{div}, \Omega) &:= \left\{v \in \mathrm{H}(\mathrm{div}, \Omega) : \ v \cdot n = 0 \text{ on } \partial\Omega\right\}.
\end{aligned}
$$

Variational spectral problem:

$$
\boxed{\begin{array}{l}
\textit{Find } \omega \in \mathbb{C} \textit{ and } 0 \neq u \in \mathcal{V} \textit{ satisfying} \\
\displaystyle \qquad \int_\Omega \rho_{\mathrm{F}}\, c^2 \,\mathrm{div}\, u \,\mathrm{div}\, v = \omega^2 \int_\Omega \rho_{\mathrm{F}}\, u \cdot v \qquad \forall v \in \mathcal{V}.
\end{array}}
$$

It is simple to show that $\omega \geq 0$, however...

Drawback: since the irrotational constraint is not imposed on the variational formulation, $\omega = 0$ is an eigenvalue of the spectral problem with eigenspace

$$\mathcal{K} := \{v \in \mathcal{V} : \ \mathrm{div}\, v = 0 \text{ in } \Omega\},$$

which corresponds to the set of purely **rotational fluid motions**.

Helmholtz decomposition

$$\boxed{\mathcal{V} = \mathcal{K} \oplus \mathcal{G}}$$

with

$$\mathcal{K} := \{v \in \mathcal{V} : \ \operatorname{div} v = 0 \text{ in } \Omega\}$$

and

$$\mathcal{G} := \left\{v \in \mathcal{V} : \ \exists \phi \in \mathrm{H}^1(\Omega) \text{ with } v = \nabla\, \phi \text{ in } \Omega\right\}.$$

PROOF: Given $v \in \mathcal{V} = \mathrm{H}_0(\operatorname{div}, \Omega)$, let ϕ be the solution of

$$\begin{cases} -\Delta\phi = \operatorname{div} v \quad \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

This Neumann problem is well posed, since $\displaystyle\int_\Omega \operatorname{div} v = \int_{\partial\Omega} v \cdot n = 0.$

Then, it attains a solution $\phi \in \mathrm{H}^1(\Omega)$, unique up to an additive constant.

Let $\chi := v - \nabla\, \phi$. Then $\chi \in \mathcal{K}$.

Hence, $v = \chi + \nabla\, \phi$, with $\chi \in \mathcal{K}$ and $\nabla\, \phi \in \mathcal{G}$. $\square$

Spectral characterization

Find $\omega \in \mathbb{C}$ *and* $0 \neq u \in \mathcal{V}$ *satisfying*

$$\int_\Omega \rho_{\mathrm{F}} c^2 \operatorname{div} u \operatorname{div} v = \omega^2 \int_\Omega \rho_{\mathrm{F}}\, u \cdot v \qquad \forall v \in \mathcal{V}.$$

THEOREM. The solutions of the spectral problem above are:

1. The eigenvalue $\omega = 0$ with eigenspace $\mathcal{K}$.
2. A sequence of strictly positive vibration frequencies ω_n, such that $\omega_n \xrightarrow{n} \infty$. Each vibration frequency $\omega_n > 0$ is of finite multiplicity and the corresponding eigenfunctions satisfy $u_n \in \mathcal{G}$ (irrotational).

Remarks:

- The irrotational eigenfunctions $u_n \in \mathcal{G}$ and their associated vibration frequencies ω_n correspond to the natural vibration modes of the fluid. These are the physically relevant magnitudes that should be computed.
- All the purely rotational motions in $\mathcal{K}$ are spurious solutions, which arise because no irrotational constrain has been imposed.

Finite element numerical solution

$\mathcal{V}_h \subset \mathcal{V}$ finite dimensional subspace ($\dim \mathcal{V}_h = N$).

Discrete spectral problem:

Find $\omega_h \geq 0$ *and* $0 \neq u_h \in \mathcal{V}_h$ *satisfying*

$$\int_\Omega \rho_{\mathrm{F}} c^2 \operatorname{div} u_h \operatorname{div} v_h = \omega_h^2 \int_\Omega \rho_{\mathrm{F}}\, u_h \cdot v_h \qquad \forall v_h \in \mathcal{V}_h.$$

Let $\{v_1, \ldots, v_N\}$ basis of $\mathcal{V}_h$, $u_h = \sum_{i=1}^{N} \alpha_i v_i$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\lambda = \omega_h^2$.

Matrix generalized eigenvalue problem:

Find $\lambda \geq 0$ *and* $0 \neq \boldsymbol{\alpha} \in \mathbb{R}^N$ *satisfying*

$$\mathbf{K}\boldsymbol{\alpha} = \lambda\, \mathbf{M}\boldsymbol{\alpha}.$$

- $\mathbf{K} \in \mathbb{R}^{N\times N}$: fluid **stiffness matrix** of entries $K_{ij} = \int_\Omega \rho_{\mathrm{F}} c^2 \operatorname{div} v_i \operatorname{div} v_j$;
- $\mathbf{M} \in \mathbb{R}^{N\times N}$: fluid **mass matrix** of entries $M_{ij} = \int_\Omega \rho_{\mathrm{F}}\, v_i \cdot v_j$.

Spurious modes

Spurious modes arise when standard finite elements are used.[a]

If $\mathcal{V}_h$ is the space of four-node standard tetrahedral elements (i.e., piecewise linear and continuous), then 0 is not an eigenvalue of the algebraic spectral problem

$$\mathbf{K}\boldsymbol{\alpha} = \lambda\,\mathbf{M}\boldsymbol{\alpha}.$$

Hence, the infinite-multiplicity eigenvalue 0 of the variational spectral problem

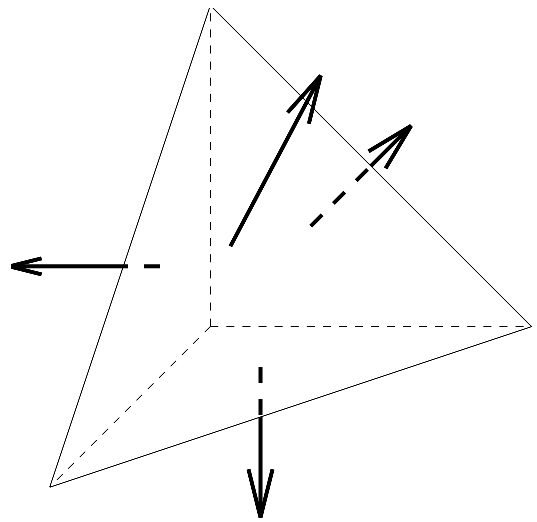
$$\int_\Omega \rho_{\mathrm{F}}\, c^2 \operatorname{div} u \operatorname{div} v = \omega^2 \int_\Omega \rho_{\mathrm{F}}\, u \cdot v \qquad \forall v \in \mathcal{V}$$

is approximated by a lot of strictly positive spurious eigenvalues λ, with eigenfunctions corresponding to rotational motions.

The non-zero frequencies $\omega_h = \sqrt{\lambda}$ of these spurious modes appear interspersed among the physically relevant vibration frequencies.

[a] L. Kiefling, G.C. Feng, *AIAA J.* **14** (1976) 199-203.

Raviart-Thomas finite elements



$v_h|_T$: incomplete linear polynomial vector field of the form

$$v_h(x,y,z) = (a + dx, b + dy, c + dz) \qquad (x,y,z) \in T.$$

These polynomial vector fields have constant normal components on any plane.

In fact, consider a general plane of equation

$$\alpha x + \beta y + \gamma z = \delta, \qquad \text{with } \alpha^2 + \beta^2 + \gamma^2 = 1.$$

Then $n = (\alpha, \beta, \gamma)$ is its unit normal vector and, for any point (x, y, z) in the plane,

$$\begin{aligned} v_h(x,y,z) \cdot n &= \alpha(a + dx) + \beta(b + dy) + \gamma(c + dz) \\ &= \alpha a + \beta b + \gamma c + (\alpha x + \beta y + \gamma z)d \\ &= \alpha a + \beta b + \gamma c + \delta d \qquad \text{(constant).} \end{aligned}$$

Raviart-Thomas finite elements (cont.)

In particular, these fields have constant normal components on each of the four faces of the tetrahedron.

The values of these constants define a unique polynomial vector field of this type.

In fact, let the equations of the four faces of the tetrahedron be

$$\alpha_i\, x + \beta_i\, y + \gamma_i\, z = \delta_i, \qquad \text{with } \alpha_i^2 + \beta_i^2 + \gamma_i^2 = 1, \qquad\qquad i = 1, \ldots, 4.$$

The system of linear equations

$$\begin{cases} \alpha_1\, a + \beta_1\, b + \gamma_1\, c + \delta_1\, d = V_1, \\ \alpha_2\, a + \beta_2\, b + \gamma_2\, c + \delta_2\, d = V_2, \\ \alpha_3\, a + \beta_3\, b + \gamma_3\, c + \delta_3\, d = V_3, \\ \alpha_4\, a + \beta_4\, b + \gamma_4\, c + \delta_4\, d = V_4, \end{cases}$$

yields the coefficients of the unique vector field $v_h(x, y, z) = (a + dx, b + dy, c + dz)$ with normal components on each face $v_h \cdot n_i = V_i, \quad i = 1, \ldots, 4.$

Raviart-Thomas finite elements (cont.)

The global discrete displacement field v_h is allowed to have discontinuous tangential components on the faces of the tetrahedra of the triangulation but its (constant) normal components must be continuous through these faces, these constant values being its **degrees of freedom**.

Because of this, $\operatorname{div} v_h$ is globally well defined on Ω.

$$\mathcal{V}_h := \{v_h \in \mathrm{H}_0(\operatorname{div}, \Omega) : \ v_h|_T(x, y, z) = (a + dx, b + dy, c + dz)\} \subset \mathcal{V}.$$

Discrete spectral problem:

Find $\omega_h \geq 0$ and $0 \neq u_h \in \mathcal{V}_h$ satisfying

$$\boxed{\int_\Omega \rho_{\mathrm{F}} c^2 \operatorname{div} u_h \operatorname{div} v_h = \omega_h^2 \int_\Omega \rho_{\mathrm{F}}\, u_h \cdot v_h \qquad \forall v_h \in \mathcal{V}_h.}$$

Spectral approximation

THEOREM.[a] The solutions of the discrete spectral problem above are:

1. The eigenvalue $\omega_h = 0$ with eigenspace

$$\mathcal{K}_h := \{v_h \in \mathcal{V}_h : \operatorname{div} v_h = 0 \text{ in } \Omega\} \xrightarrow{h \to 0} \mathcal{K}.$$

2. A finite sequence of strictly positive vibration frequencies ω_{hn} satisfying

$$\omega_{hn} \xrightarrow{h \to 0} \omega_n.$$

   Each vibration frequency ω_{hn} is of finite multiplicity and the corresponding eigenfunctions u_{hn} satisfy

$$u_{hn} \xrightarrow{h \to 0} u_n.$$

   Optimal order error estimates are valid.

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[a] A. Bermúdez, R.R., *Comp. Methods Appl. Mech. Eng.*, **119** (1994) 355-370.

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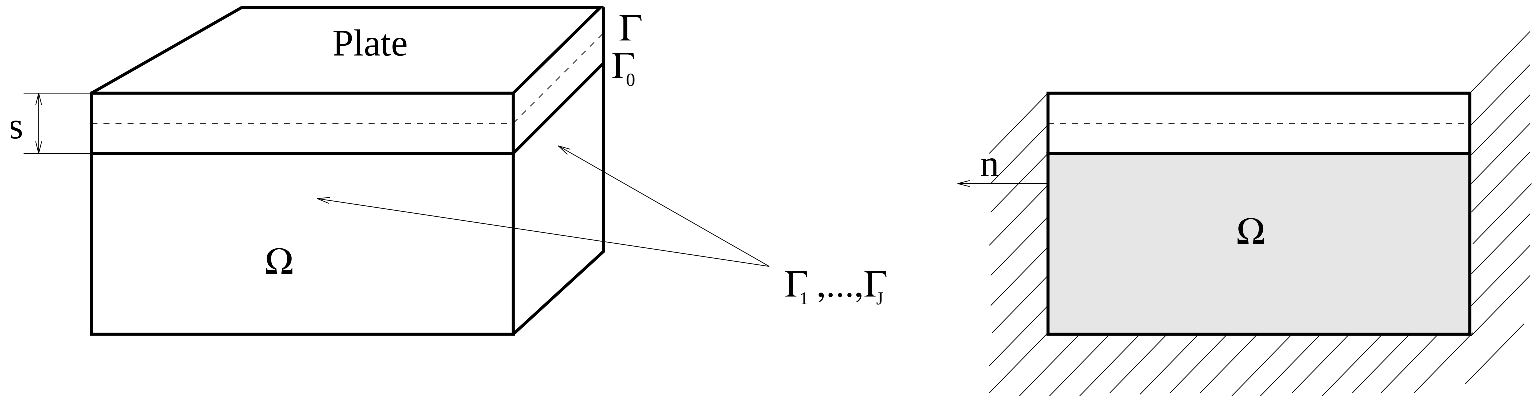
Fluid-plate coupled system

Reissner-Mindlin plates



- s: plate thickness.
- $\partial\Omega = \Gamma_{\mathrm{R}} \cup \Gamma$: $\qquad \Gamma_{\mathrm{R}} = \Gamma_1 \cup \cdots \cup \Gamma_J$: rigid walls; $\qquad \Gamma \sim \Gamma_0$: plate.

Reissner-Mindlin model for the deflection of the plate:

$$v(x, y, z) = \big(-z\,\theta_1(x, y), -z\,\theta_2(x, y),\, w(x, y) \big)\,.$$

- w: transversal displacement;
- $\theta = (\theta_1, \theta_2)$: rotations vector.

Elastodynamic equations for a fluid-loaded clamped plate

$$w,\, \theta_1,\, \theta_2 \in \mathrm{H}^1_0(\Gamma) := \left\{ r \in \mathrm{H}^1(\Gamma) :\; r = 0 \text{ on } \partial\Gamma \right\},$$

$$s^3 a\,(\theta, \eta) + \kappa s \int_\Gamma (\nabla w - \theta) \cdot (\nabla r - \eta) + s \int_\Gamma \rho_{\mathrm{P}}\, \ddot{w} r + \frac{s^3}{12} \int_\Gamma \rho_{\mathrm{P}}\, \ddot{\theta} \cdot \eta = \int_\Gamma pr$$
$$\forall (r, \eta) \in \mathrm{H}^1_0(\Gamma) \times \mathrm{H}^1_0(\Gamma)^2.$$

- $a\,(\theta, \eta) := \dfrac{E}{12(1+\nu)} \displaystyle\int_\Gamma \left[\sum_{i,j=1}^{2} \varepsilon_{ij}(\theta)\, \varepsilon_{ij}(\eta) + \frac{\nu}{1-\nu} \operatorname{div} \theta \, \operatorname{div} \eta \right],$
- $\varepsilon_{ij}(\theta) := \dfrac{1}{2} \left(\dfrac{\partial \theta_j}{\partial x_i} + \dfrac{\partial \theta_i}{\partial x_j} \right)$ linear strain tensor,
- E: Young modulus,
- ν: Poisson ratio,
- $\kappa := Ek/[2(1+\nu)]$,
- k: correction factor (for clamped plates, $k = \frac{5}{6}$),
- ρ_{P}: plate density,
- p: pressure on the fluid-solid interface.

Dynamic equations for an acoustic fluid in contact with a plate

$$\begin{cases} \rho_{\mathrm{F}} \ddot{u} = -\nabla p & \text{in } \Omega, \\ p = -\rho_{\mathrm{F}} c^2 \operatorname{div} u & \text{in } \Omega, \\ u \cdot n = 0 & \text{on } \Gamma_{\mathrm{R}}, \\ u \cdot n = w & \text{on } \Gamma. \end{cases}$$

Variational formulation:

$$u \in \mathrm{H}_{\Gamma_{\mathrm{R}}}(\operatorname{div}, \Omega) := \left\{ v \in \mathrm{H}(\operatorname{div}, \Omega) : \ v \cdot n = 0 \text{ on } \Gamma_{\mathrm{R}} \right\},$$

$$\int_{\Omega} \rho_{\mathrm{F}} \ddot{u} \cdot v + \int_{\Omega} \rho_{\mathrm{F}} c^2 \operatorname{div} u \operatorname{div} v = -\int_{\Gamma} p \, v \cdot n \qquad \forall v \in \mathrm{H}_{\Gamma_{\mathrm{R}}}(\operatorname{div}, \Omega).$$

Dynamic equations for the fluid-plate coupled system

$$\left\{\begin{array}{l} s^3 a\left(\theta,\eta\right) + \kappa s \displaystyle\int_\Gamma (\nabla w - \theta)\cdot(\nabla r - \eta) + s\int_\Gamma \rho_{\mathrm{P}}\ddot{w} r + \frac{s^3}{12}\int_\Gamma \rho_{\mathrm{P}}\ddot{\theta}\cdot\eta = \int_\Gamma p r \\ \hfill \forall (r,\eta) \in \mathrm{H}_0^1(\Gamma)\times \mathrm{H}_0^1(\Gamma)^2. \\ \\ \displaystyle\int_\Omega \rho_{\mathrm{F}}\ddot{u}\cdot v + \int_\Omega \rho_{\mathrm{F}} c^2 \operatorname{div} u \operatorname{div} v = -\int_\Gamma p\, v\cdot n \qquad \forall v \in \mathrm{H}_{\Gamma_{\mathrm{R}}}(\operatorname{div},\Omega). \end{array}\right.$$

Coupled dynamic problem:

$$(w,\theta,u) \in \mathcal{W} := \left\{(r,\eta,v) \in \mathrm{H}_0^1(\Gamma)\times \mathrm{H}_0^1(\Gamma)^2 \times \mathrm{H}_{\Gamma_{\mathrm{R}}}(\operatorname{div},\Omega): \quad v\cdot n = r \text{ on } \Gamma\right\},$$

$$\begin{aligned} s^3 a(\theta,\eta) + \kappa s \int_\Gamma (\nabla w - \theta)\cdot(\nabla r - \eta) + \int_\Omega \rho_{\mathrm{F}} c^2 \operatorname{div} u \operatorname{div} v \\ + s\int_\Gamma \rho_{\mathrm{P}}\ddot{w} v + \frac{s^3}{12}\int_\Gamma \rho_{\mathrm{P}}\ddot{\theta}\cdot\eta + \int_\Omega \rho_{\mathrm{F}}\ddot{u}\cdot v = 0 \qquad \forall (r,\eta,v)\in\mathcal{W}. \end{aligned}$$

Free vibration problem for a fluid-plate coupled system

Natural vibration modes: harmonic in time solutions

$$
\begin{aligned}
w(x,y,t) &= w(x,y)\cos\omega t, && (x,y)\in\Gamma, \quad t\in\mathbb{R},\\
\theta(x,y,t) &= \theta(x,y)\cos\omega t, && (x,y)\in\Gamma, \quad t\in\mathbb{R},\\
u(x,y,z,t) &= u(x,y,z)\cos\omega t, && (x,y,z)\in\Omega, \quad t\in\mathbb{R}.
\end{aligned}
$$

Variational coupled spectral problem:

Find $\omega\in\mathbb{C}$ *and* $0\neq(w,\theta,u)\in\mathcal{W}$ *satisfying*

$$
s^3 a(\theta,\eta)+\kappa s\int_\Gamma(\nabla w-\theta)\cdot(\nabla r-\eta)+\int_\Omega \rho_{\mathrm{F}} c^2\,\mathrm{div}\,u\,\mathrm{div}\,v
$$

$$
=\omega^2\left(s\int_\Gamma \rho_{\mathrm{P}} wv+\frac{s^3}{12}\int_\Gamma \rho_{\mathrm{P}}\theta\cdot\eta+\int_\Omega \rho_{\mathrm{F}} u\cdot v\right) \qquad \forall (r,\eta,v)\in\mathcal{W}.
$$

Spectral characterization

THEOREM. The solutions of the spectral problem above are:

1. The eigenvalue $\omega = 0$ with eigenspace $\{0\} \times \{0\} \times \mathcal{K}$.
2. A sequence of strictly positive vibration frequencies ω_n, such that $\omega_n \xrightarrow{n} \infty$.

   Each vibration frequency $\omega_n > 0$ is of finite multiplicity and the corresponding eigenfunctions (w_n, θ_n, u_n) satisfy $u_n = \nabla \phi$, $\phi \in \mathrm{H}^1(\Omega)$ (irrotational).

Remarks:

- The eigenspace $\{0\} \times \{0\} \times \mathcal{K}$ consists of spurious circulation modes inducing neither plate vibrations nor variations of the fluid pressure.

  They are mathematical solutions of the spectral problem with no physical entity, which arise because no irrotational constraint is imposed to the fluid displacements.

- The rest of the spectrum corresponds to the natural vibration modes of the coupled fluid-plate system.

  Each of these vibration modes can be seen as a perturbation of those of corresponding uncoupled problems: either the fluid contained in a perfectly rigid cavity or the plate in vacuo.

Finite element numerical solution

- For the plate, **MITC3**[a] (also named **DL3**) locking-free finite elements:

$$(w_h, \theta_h) \in \mathcal{DL}(\Gamma),$$

combined with reduced integration for the shear energy

$$\int_\Gamma (\nabla w_h - \theta_h) \cdot (\nabla r_h - \eta_h) \approx \int_\Gamma \Pi(\nabla w_h - \theta_h) \cdot \Pi(\nabla r_h - \eta_h),$$

where Π denotes the interpolation of the shear terms on the rotated Raviart-Thomas elements.

- For the fluid, **Raviart-Thomas** finite elements (with vanishing normal components on Γ_{R}):

$$u_h \in \mathcal{RT}(\Omega).$$

- **Non-conforming coupling** on the fluid-solid interface:

$$\int_T u_h \cdot n = \int_T w_h \qquad \forall \text{ triangle of the mesh } T \subset \Gamma.$$

Remark: If we had imposed $u_h \cdot n \neq w_h$ on Γ, then $u_h \cdot n = w_h \equiv 0$ on Γ.

[a] R. Durán, E. Liberman, *Math. Comp.*, **58** (1992) 561–573.

Finite element numerical solution (cont.)

$$\mathcal{W}_h := \left\{ (w_h, \theta_h, u_h) \in \mathcal{DL}(\Gamma) \times \mathcal{RT}(\Omega) : \int_T u_h \cdot n = \int_T w_h \ \forall T \subset \Gamma \right\} \not\subset \mathcal{W}.$$

Discrete spectral coupled problem:

Find $\omega_h \in \mathbb{C}$ *and* $0 \neq (w_h, \theta_h, u_h) \in \mathcal{W}_h$ *satisfying*

$$s^3 a(\theta_h, \eta_h) + \kappa s \int_\Gamma (\nabla w_h - \Pi\theta_h) \cdot (\nabla r_h - \Pi\eta_h) + \int_\Omega \rho_{\mathrm{F}} c^2 \operatorname{div} u_h \operatorname{div} v_h$$
$$= \omega^2 \left(s \int_\Gamma \rho_{\mathrm{P}} w_h v_h + \frac{s^3}{12} \int_\Gamma \rho_{\mathrm{P}} \theta_h \cdot \eta_h + \int_\Omega \rho_{\mathrm{F}} u_h \cdot v_h \right) \qquad \forall (r_h, \eta_h, v_h) \in \mathcal{W}_h.$$

Spectral approximation

THEOREM.[a] The solutions of the discrete spectral problem above are:

1. The eigenvalue $\omega_h = 0$ with eigenspace

$$\{0\} \times \{0\} \times \mathcal{K}_h, \qquad \text{with } \mathcal{K}_h := \{v_h \in \mathcal{V}_h : \ \operatorname{div} v_h = 0 \text{ in } \Omega\} \xrightarrow{h \to 0} \mathcal{K}.$$

2. A finite sequence of strictly positive vibration frequencies ω_{hn} satisfying

$$\omega_{hn} \xrightarrow{h \to 0} \omega_n.$$

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[a] R. DURÁN, L. HERVELLA, E. LIBERMAN, R.R., J. SOLOMIN. *Math. Comp.*, 68 (1999) 1447-1463.

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Spectral approximation (cont.)

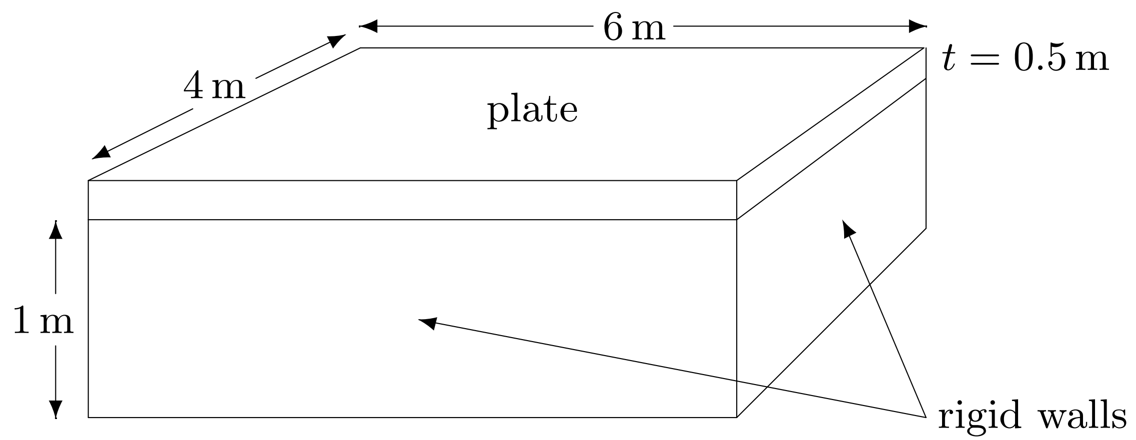
Each vibration frequency ω_{hn} is of finite multiplicity and the corresponding eigenfunctions $(w_{hn}, \theta_{hn}, u_{hn})$ satisfy

$$
\begin{aligned}
w_{hn} &\xrightarrow{h \to 0} w_n, \\
\theta_{hn} &\xrightarrow{h \to 0} \theta_n, \\
u_{hn} &\xrightarrow{h \to 0} u_n.
\end{aligned}
$$

Optimal order locking-free error estimates are valid.

Numerical example

Geometrical data:



Physical parameters:

Plate material: steel.

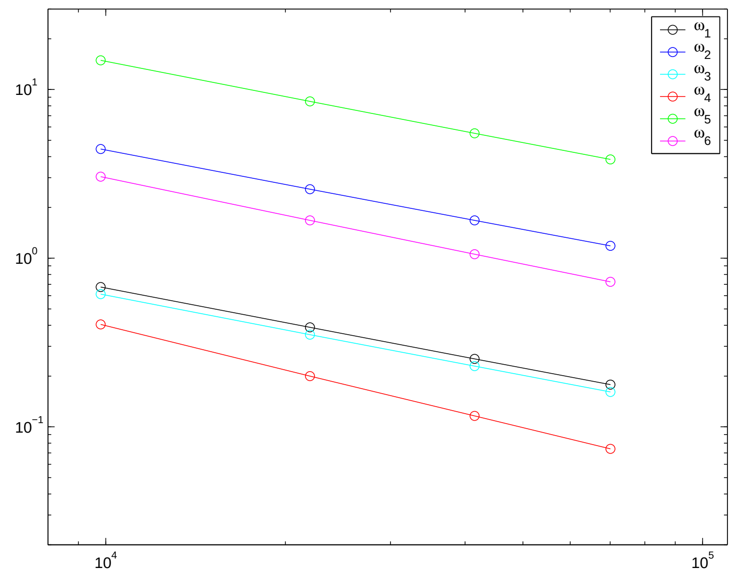
- Density: $\rho_{\mathrm{P}} = 7700\,\mathrm{kg/m^3}$
- Young modulus: $E = 1.44 \times 10^{11}\,\mathrm{Pa}$
- Poisson coefficient: $\nu = 0.35$

Fluid: water.

- density: $\rho_{\mathrm{F}} = 1000\,\mathrm{kg/m^3}$
- sound speed: $c = 1430\,\mathrm{m/s}$

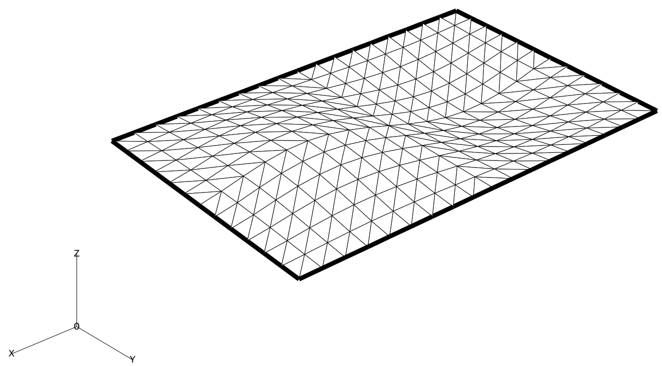
Numerical example (cont.)

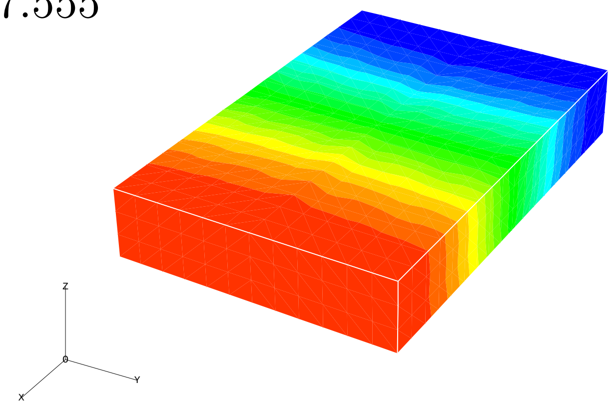
Mode	$N = 3$	$N = 4$	$N = 5$	$N = 6$	order	ω_n
ω_1^h	696.880	697.166	697.302	697.377	1.92	697.555
ω_2^h	1019.075	1017.201	1016.310	1015.819	1.91	1014.635
ω_3^h	1081.299	1081.559	1081.682	1081.750	1.93	1081.911
ω_4^h	1317.326	1317.121	1317.037	1316.995	2.44	1316.921
ω_5^h	1470.968	1464.565	1461.561	1459.916	1.95	1456.063
ω_6^h	1504.253	1505.621	1506.242	1506.574	2.07	1507.298



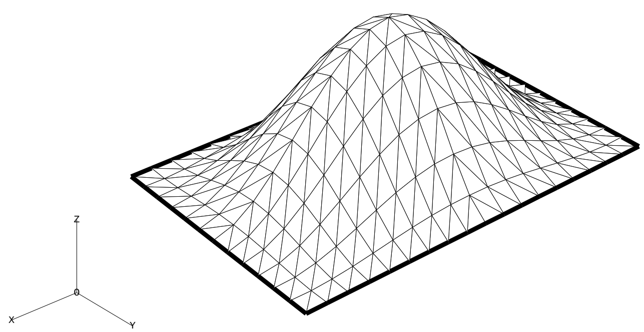
Numerical example (cont.)

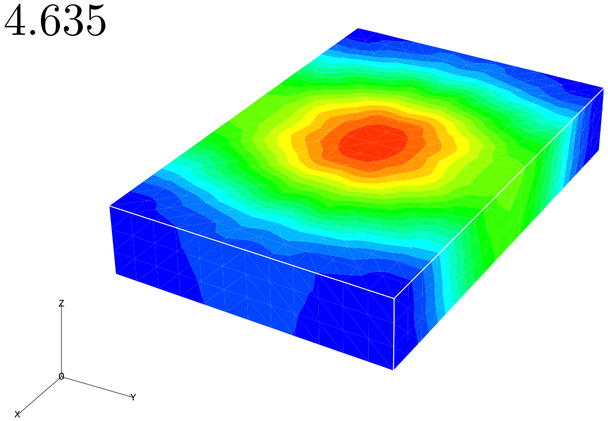
$$\omega_1 = 697.555$$





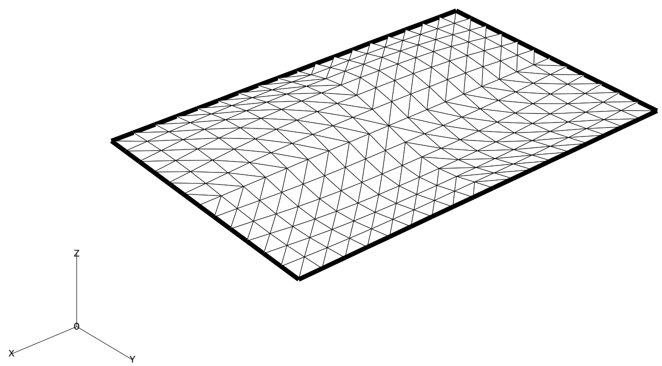
$$\omega_2 = 1014.635$$

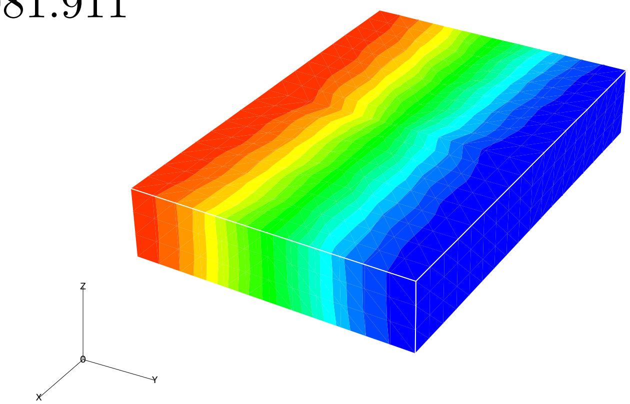




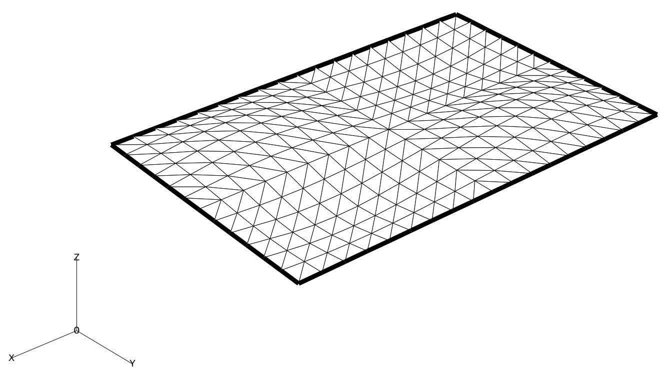
Numerical example (cont.)

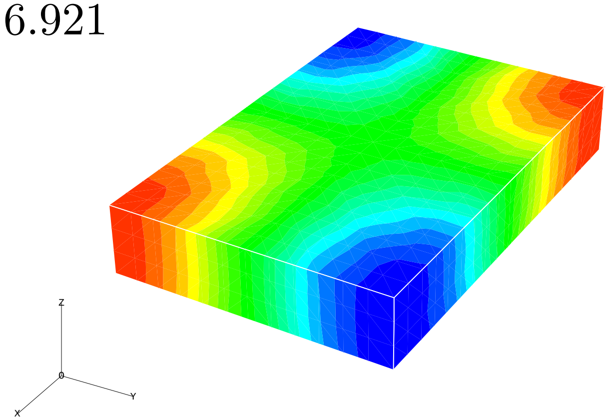
$\omega_3 = 1081.911$





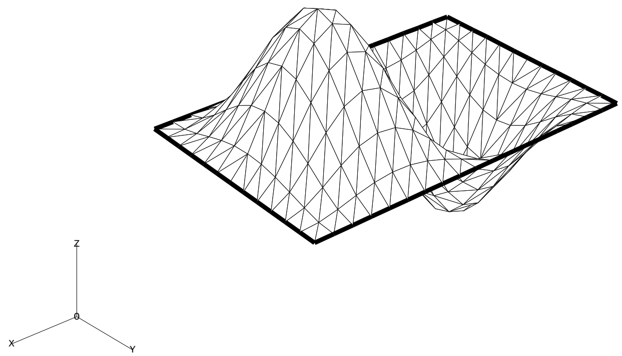
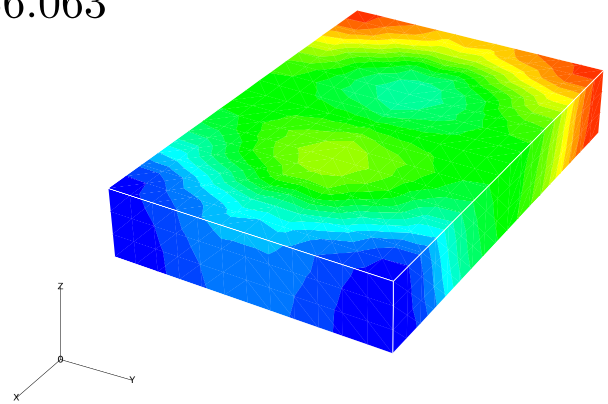
$\omega_4 = 1316.921$





Numerical example (cont.)

$$\omega_5 = 1456.063$$

$$\omega_6 = 1507.298$$