Some Remarks on the Reissner–Mindlin Plate Model

Alexandre L. Madureira

LNCC – Brazil

Talk at the Workshop of Numerical Analysis of PDEs – LNCC

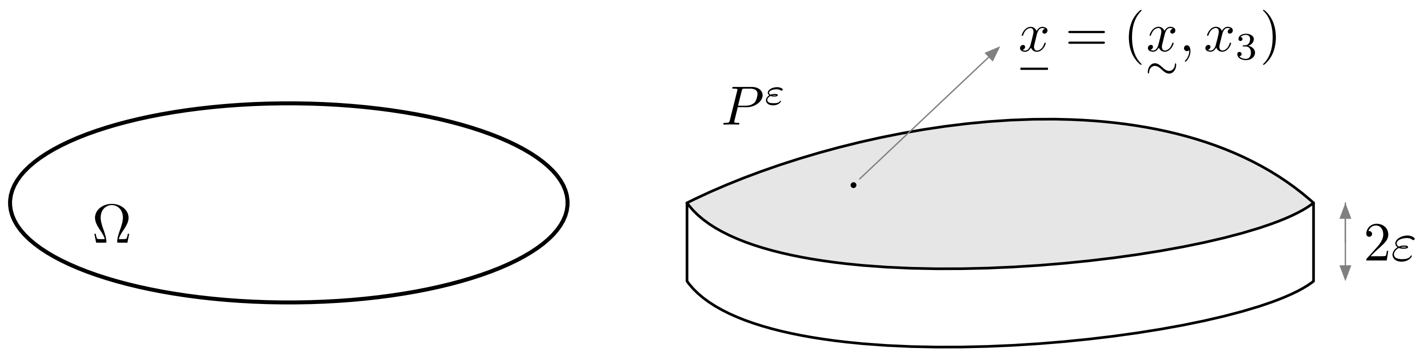
February 10, 2003

Outline

- The 3D Problem and its Modeling
- Full Discretization–Playing Around with the Timoshenko Beam
- Conclusions

- The 3D Problem and its Modeling
  - 3D linearly elastic plates
  - A Hellinger–Reissner Principle
  - Reissner–Mindlin Model Derivation
  - Convergence Estimate
- Full Discretization–Playing Around with the Timoshenko Beam
- Conclusions

The Domain



Middle surface: $\Omega \subset \mathbb{R}^2$,

3D Domain: $P^\varepsilon = \Omega \times (-\varepsilon, \varepsilon)$,

Lateral Boundary: $\partial P^\varepsilon_L = \partial\Omega \times (-\varepsilon, \varepsilon)$,

Top and bottom boundaries: $\partial P^\varepsilon_\pm = \Omega \times \{-\varepsilon, \varepsilon\}$.

Bending of Linearly Elastic Plate

Find $\underline{u}^\varepsilon : P^\varepsilon \to \mathbb{R}^3$ and $\underline{\sigma}^\varepsilon : P^\varepsilon \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ such that

$$-\underline{\operatorname{div}}\,\underline{\sigma}^\varepsilon = 0, \qquad \underline{\sigma}^\varepsilon = \underline{\underline{C}}\,\underline{e}(\underline{u}^\varepsilon), \quad \text{in } P^\varepsilon,$$

$$\underline{\sigma}^\varepsilon \underline{n} = \underline{g}^\varepsilon \quad \text{on } \partial P^\varepsilon_\pm, \qquad \underline{u}^\varepsilon = 0 \quad \text{on } \partial P^\varepsilon_L,$$

where $\underline{g}^\varepsilon = (0, 0, g_3) : \partial P^\varepsilon_\pm \to \mathbb{R}^3$ is the traction load.

Also, $\underline{e}(\underline{u}^\varepsilon) = \dfrac{1}{2}(\underline{\nabla}\,\underline{u}^\varepsilon + \underline{\nabla}^T\,\underline{u}^\varepsilon)$ and

$$\underline{\underline{C}}\,\underline{\tau} = 2\nu\underline{\tau} + \lambda\,\mathrm{tr}(\underline{\tau})\underline{\delta},$$

where μ and λ are the Lamé coeff., and $\underline{\delta}$ is the 3×3 ident. matrix.

Assume $g_3(\underset{\sim}{x}, \varepsilon) = g_3(\underset{\sim}{x}, -\varepsilon)$.

Dimension Reduction

The goal of dimension reduction is to pose a system of equations in the middle surface Ω that is "easy" to solve and which solution approximates the original 3D solution.

The most popular models are

- variants of Reissner–Mindlin
- Kirchhoff–Love (also known as biharmonic model)
- formal higher order models

Notation:

$$\underline{u} = \begin{pmatrix} \underset{\sim}{u} \\ u_3 \end{pmatrix} \in \begin{pmatrix} \mathbb{R}^2 \\ \mathbb{R} \end{pmatrix}, \qquad \underline{\underline{\sigma}} = \begin{pmatrix} \underset{\approx}{\sigma} & \underset{\sim}{\sigma} \\ \underset{\sim}{\sigma}^t & \sigma_{33} \end{pmatrix} \in \begin{pmatrix} \mathbb{R}^{2\times 2} & \mathbb{R}^2 \\ {\mathbb{R}^2}^t & \mathbb{R} \end{pmatrix},$$

Hellinger–Reissner Principle

Let $\underline{\underline{S}}_{\underline{g}}(P^\varepsilon) = \{\, \underline{\underline{\tau}} \in \underline{\underline{H}}(\mathrm{div}, P^\varepsilon) \,:\, \underline{\underline{\tau}}\,\underline{n} = \underline{g} \text{ on } \partial P^\varepsilon_\pm \,\}$.

Hellinger-Reissner Principle: $(\underline{\underline{\sigma}}^\varepsilon, \underline{u}^\varepsilon)$ is critical point of

$$L(\underline{\underline{\tau}}, \underline{v}) = \frac{1}{2}\int_{P^\varepsilon} \underline{\underline{\underline{\underline{C}}}}^{-1}\underline{\underline{\tau}} : \underline{\underline{\tau}}\, d\underline{x} + \int_{P^\varepsilon} \underline{\mathrm{div}}\, \underline{\underline{\tau}} \cdot \underline{v}\, d\underline{x}$$

on $\underline{\underline{S}}_{\underline{g}}(P^\varepsilon) \times \underline{L}^2(P^\varepsilon)$.

So $\underline{\underline{\sigma}}^\varepsilon \in \underline{\underline{S}}_{\underline{g}}(P^\varepsilon)$ and $\underline{u}^\varepsilon \in \underline{L}^2(P^\varepsilon)$ satisfy

$$\int_{P^\varepsilon} \underline{\underline{\underline{\underline{C}}}}^{-1}\underline{\underline{\sigma}}^\varepsilon : \underline{\underline{\tau}}\, d\underline{x} + \int_{P^\varepsilon} \underline{u}^\varepsilon \cdot \underline{\mathrm{div}}\, \underline{\underline{\tau}}\, d\underline{x} = 0 \quad \text{for all } \underline{\underline{\tau}} \in \underline{\underline{S}}_0(P^\varepsilon),$$

$$\int_{P^\varepsilon} \underline{\mathrm{div}}\, \underline{\underline{\sigma}}^\varepsilon \cdot \underline{v}\, d\underline{x} = 0 \quad \text{for all } \underline{v} \in \underline{L}^2(P^\varepsilon).$$

We now derive a model of Reissner–Mindlin type by looking for critical points of L within subspaces of $\underline{S}_{\underline{g}}(P^\varepsilon) \times \underline{L}^2(P^\varepsilon)$.

Consider subspaces composed of functions which are polynomial in the transverse direction, with the following degrees:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}.$$

Then our solution will be of the form

$$\underline{u}_R = \begin{pmatrix} -\underset{\sim}{\theta}(\underset{\sim}{x})x_3 \\ \omega(\underset{\sim}{x}) + \omega_2(\underset{\sim}{x})r(x_3) \end{pmatrix}, \quad \underline{\sigma}_R = \begin{pmatrix} \underset{\approx}{\sigma}(\underset{\sim}{x})x_3 & \underset{\sim}{\sigma}(\underset{\sim}{x})q(x_3) \\ \times & g_3 x_3/\varepsilon + \sigma_{33}s(x_3) \end{pmatrix},$$

where $r(z) = 3(z^2/\varepsilon^2 - 1/5)/2$, $q(z) = 3(1 - z^2/\varepsilon^2)/2$, and $s(z) = 5(z/\varepsilon - z^3/\varepsilon^3)/4$.

Consider the eqtn $\int_{P^\varepsilon} \underline{\mathrm{div}}\, \underline{\underline{\sigma}} \cdot \underline{v}\, d\underline{x} = 0$, where v is a test function.

Choose $\underline{v} = (\underset{\sim}{\psi}(\underset{\sim}{x})x_3, 0)$, with $\underset{\sim}{\psi} \in \underset{\sim}{L}^2(\Omega)$ arbitrary, to obtain

$$-\frac{\varepsilon^2}{3}\int_\Omega \underset{\sim}{\mathrm{div}}\, \underset{\approx}{\sigma} \cdot \underset{\sim}{\psi}\, d\underset{\sim}{x} + 2\varepsilon \int_\Omega \underset{\sim}{\sigma} \cdot \underset{\sim}{\psi}\, d\underset{\sim}{x} = 0 \quad \text{for all } \underset{\sim}{\psi} \in \underset{\sim}{L}^2(\Omega).$$

Choosing $\underline{v} = (0, 0, w(\underset{\sim}{x}))$, with $w \in L^2(\Omega)$ arbitrary, it follows that

$$\varepsilon \int_\Omega \mathrm{div}\, \underset{\sim}{\sigma}\, w\, d\underset{\sim}{x} = \int_\Omega g_3 w\, d\underset{\sim}{x}, \quad \text{for all } w \in L^2(\Omega)$$

Finally, with $\underline{v} = (0, 0, w_2(\underset{\sim}{x})r(x_3))$, and $w_2 \in L^2(\Omega)$ arbitrary,

$$\int_\Omega \sigma_{33} w_2\, d\underset{\sim}{x} = \frac{2}{5}\int_\Omega g_3 w_2\, d\underset{\sim}{x}, \quad \text{for all } w_2 \in L^2(\Omega)$$

Hence

$$-\frac{\varepsilon^2}{3}\int_\Omega \operatorname{div}\underset{\sim}{\underset{\sim}{\sigma}} \cdot \underset{\sim}{\psi}\, d\underset{\sim}{x} + 2\varepsilon \int_\Omega \underset{\sim}{\sigma} \cdot \underset{\sim}{\psi}\, d\underset{\sim}{x} = 0 \quad \text{for all } \underset{\sim}{\psi} \in L^2(\Omega)$$

$$\varepsilon \int_\Omega \operatorname{div}\underset{\sim}{\sigma}\, w\, d\underset{\sim}{x} = \int_\Omega g_3 w\, d\underset{\sim}{x}, \quad \text{for all } w \in L^2(\Omega)$$

$$\int_\Omega \sigma_{33} w_2\, d\underset{\sim}{x} = \frac{2}{5}\int_\Omega g_3 w_2\, d\underset{\sim}{x}, \quad \text{for all } w_2 \in L^2(\Omega)$$

Similarly for the stresses, choosing arbitrary test functions with polynomial profile, we gather that

$$\int_\Omega \underset{\approx}{A}\underset{\sim}{\sigma} : \underset{\sim}{\tau} - 2\varepsilon\underset{\sim}{\theta} \cdot \underset{\sim}{\operatorname{div}}\, \underset{\sim}{\tau}\, d\underset{\sim}{x} = \frac{\nu}{E}\int_\Omega (\sigma_{33} + 2g_3)\operatorname{tr}(\underset{\sim}{\tau})\, d\underset{\sim}{x}$$

$$\text{for all } \underset{\sim}{\tau} \in H^1(\Omega),$$

$$\int_\Omega \left(\frac{1}{2\mu}\frac{6}{5}\underset{\sim}{\sigma} + \underset{\sim}{\theta}\right) \cdot \underset{\sim}{\tau} + \omega \operatorname{div} \underset{\sim}{\tau}\, d\underset{\sim}{x} = 0 \quad \text{for all } \underset{\sim}{\tau} \in H^1(\Omega),$$

$$\int_\Omega \omega_2 \tau_{33}\, d\underset{\sim}{x} = \frac{\varepsilon}{E}\int_\Omega \left(\frac{1}{3}g_3 + \frac{5}{21}\sigma_{33} - \frac{\nu}{6}\operatorname{tr}(\underset{\sim}{\sigma})\right)\tau_{33}\, d\underset{\sim}{x}$$

$$\text{for all } \tau_{33} \in H^1(\Omega).$$

Integrating by parts, we can write the above equations in terms of the displacement unknowns only.

Reissner–Mindlin Model: Find $\underset{\sim}{\theta}$ and ω such that

$$-\frac{2\varepsilon^3}{3\alpha}\,\underset{\sim}{\operatorname{div}}\,\underset{\approx}{e}(\underset{\sim}{\theta}) + 4\varepsilon\mu\frac{5}{6}(\underset{\sim}{\theta} - \underset{\sim}{\nabla}\,\omega) = -\varepsilon^2\frac{4}{5}\frac{\beta}{\alpha}\,\underset{\sim}{\nabla}\,g_3 \text{ in } \Omega,$$

$$4\varepsilon\mu\frac{5}{6}\operatorname{div}(\underset{\sim}{\theta} - \underset{\sim}{\nabla}\,\omega) = 2g_3 \text{ in } \Omega,$$

$$\underset{\sim}{\theta} = 0, \quad \omega = 0 \text{ on } \Omega.$$

After that we can perform the following computations.

$$\underset{\approx}{\sigma} = -2\varepsilon\underset{\approx}{A}^{-1}\underset{\approx}{e}(\underset{\sim}{\theta}) + \frac{12}{5}\frac{\lambda}{2\mu+\lambda}g_3\underset{\approx}{\delta}, \qquad \sigma_{33} = \frac{2}{5}g_3,$$

$$\underset{\sim}{\sigma} = 2\mu\frac{5}{6}(-\underset{\sim}{\theta} + \underset{\sim}{\nabla}\,\omega),$$

$$\omega_2 = \frac{\varepsilon}{E}\left(\frac{1}{3}g_3 + \frac{5}{21}\sigma_{33} - \frac{\nu}{6}\operatorname{tr}(\underset{\approx}{\sigma})\right).$$

Convergence of Reissner–Mindlin

Theorem 1 *Assume* $g_3 = \varepsilon^\alpha \hat{g}_3$*, where* $\hat{g}_3$ *is* ε*-independent and smooth, and* α *is a nonnegative number. Then*

$$\frac{\|\underline{u}^\varepsilon - \underline{u}_R\|_{E(P^\varepsilon)}}{\|\underline{u}^\varepsilon\|_{E(P^\varepsilon)}} + \frac{\|\underline{\underline{\sigma}}^\varepsilon - \underline{\underline{\sigma}}_R\|_{L^2(P^\varepsilon)}}{\|\underline{\underline{\sigma}}^\varepsilon\|_{L^2(P^\varepsilon)}} \leq C\varepsilon^{1/2}.$$

The constant C *depends on* Ω*, and* $\hat{g}_3$ *only.*

The proof of this theorem relies on the two-energies principle. This principle says that if, given a stress that is statically admissible, and a displacement that is kinematically admissible, then the sum of the squared energy and complementary energy norms is *equal* to the complementary energy norm of the constitutive residual.

The complementary energy norm is simply

$$\|\underline{\sigma}\|_{C^{\varepsilon}} = \left[\int_{P^{\varepsilon}} (\underline{\underline{C}}^{-1}\underline{\sigma}) : \underline{\sigma}\, d\underline{x}\right]^{1/2}$$

Theorem 2 *(The two energies principle.) Suppose that* $\underline{\underline{\sigma}} \in \underline{\underline{H}}(\mathrm{d\underline{i}v}, P^\varepsilon)$*, is statically admissible, i.e.*

$$\mathrm{d\underline{i}v}\underline{\underline{\sigma}} = 0 \quad in \quad P^\varepsilon, \qquad \underline{\underline{\sigma}}\,\underline{n} = \underline{g}^\varepsilon \quad on \quad \partial P^\varepsilon_\pm,$$

and suppose $\underline{u} \in \underline{H}^1(P^\varepsilon)$ *is kinematically admissible, i.e.*

$$\underline{u} = 0 \quad on \quad \partial P^\varepsilon_L.$$

Then

$$\|\underline{u}^\varepsilon - \underline{u}\|^2_{E^\varepsilon} + \|\underline{\underline{\sigma}}^\varepsilon - \underline{\underline{\sigma}}\|^2_{C^\varepsilon} = \|\underline{\underline{\sigma}} - \underline{\underline{\underline{\underline{C}}}}\,\underline{\underline{e}}(\underline{u})\|^2_{C^\varepsilon}.$$

Our derivation of the Reissner–Mindlin system yields an statically admissible stress field. Also, we add a boundary layer to make the displacement field kinematically admissible.

The constitutive residual $\underline{\rho} = \underline{\underline{C}}^{-1}\underline{\sigma}_R - \underline{e}(\underline{u}_R)$ is given by

$$\underset{\sim}{\varrho} = 0, \quad \underset{\sim}{\rho} = \frac{5}{8\mu}\left(\frac{x_3^2}{\varepsilon^2} - \frac{1}{5}\right)[\mu(\underset{\sim}{\theta} - \underset{\sim}{\nabla} w)],$$
$$\rho_{33} = \frac{x_3}{2\mu + \lambda}(\lambda \operatorname{div} \underset{\sim}{\theta} + g_3).$$

We shall need the following *a priori* estimates:

Lemma 3 *Let* $\underset{\sim}{\gamma} = \mu(\underset{\sim}{\theta} - \underset{\sim}{\nabla} w)$*. Then there exists a constant* C *only dependent on* Ω *such that*

$$\|\underset{\sim}{\theta}\|_{\underset{\sim}{H}^1} + \|w\|_{H^1} + \varepsilon^{-1}\|\underset{\sim}{\gamma}\|_{\underset{\sim}{L}^2} \leq C\|g_3\|_{L^2}.$$

- The 3D Problem and its Modeling
- Full Discretization–Playing Around with the Timoshenko Beam
  - Equations
  - Full Discretization
  - Results for Constant Traction
- Conclusions

Consider the beam (2D elasticity) problem of finding

$$\underset{\sim}{u}^{\varepsilon} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{and } \underset{\approx}{\sigma}^{\varepsilon} = \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{1,2} & \sigma_{2,2} \end{pmatrix}$$

such that

$$-\underset{\sim}{\operatorname{div}}\, \underset{\approx}{\sigma}^{\varepsilon} = 0, \qquad \underset{\approx}{\sigma}^{\varepsilon} = \underset{\approx\approx}{C}\, \underset{\approx}{e}(\underset{\sim}{u}^{\varepsilon}), \quad \text{in } (-1,1) \times (-\varepsilon, \varepsilon),$$

$$\underset{\approx}{\sigma}^{\varepsilon} \underset{\sim}{n} = \underset{\sim}{g}^{\varepsilon} \quad \text{on } (-1,1) \times \{-\varepsilon, \varepsilon\}, \qquad \underset{\sim}{u}^{\varepsilon} = 0 \quad \text{on } \{-1,1\} \times (-\varepsilon, \varepsilon),$$

where $\underset{\sim}{g}^{\varepsilon} = (0, g_2)$ is the traction , and $\underset{\approx}{e}(\underset{\sim}{u}^{\varepsilon}) = \frac{1}{2}(\underset{\approx}{\nabla}\, \underset{\sim}{u}^{\varepsilon} + \underset{\approx}{\nabla}^{T}\, \underset{\sim}{u}^{\varepsilon})$.

Hellinger-Reissner Principle: $(\underset{\approx}{\sigma}^\varepsilon, \underset{\sim}{u}^\varepsilon)$ is critical point of

$$L(\underset{\approx}{\tau}, \underset{\sim}{v}) = \frac{1}{2}\int_{P^\varepsilon} \underset{\approx\approx}{C}^{-1}\underset{\approx}{\tau} : \underset{\approx}{\tau}\, d\underset{\sim}{x} + \int_{P^\varepsilon} \underset{\sim}{\operatorname{div}}\, \underset{\approx}{\tau} \cdot \underset{\sim}{v}\, d\underset{\sim}{x}$$

on $\underset{\approx}{S}_{\underset{\sim}{g}}(P^\varepsilon) \times \underset{\sim}{L}^2(P^\varepsilon)$.

Full discretization by choosing

$$\underset{\approx}{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \in \begin{pmatrix} P_0^2(-1,1) \otimes x_2 & P_0^1(-1,1) \otimes q(x_2) \\ \times & P_{-1}^0(-1,1) \otimes s(x_2) \end{pmatrix}$$

$$\underset{\sim}{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \begin{pmatrix} P_{-1}^1(-1,1) \otimes x_2 \\ P_{-1}^0(-1,1) \otimes \{1, r(x_2)\} \end{pmatrix}$$

$P_0^k(-1,1)$: continuous piecewise polynomials of degree k in $(-1,1)$

$P_{-1}^k(-1,1)$: discontinuous polynomials of degree k in $(-1,1)$

Our (Timoshenko Beam Model) solution will be of the form

$$\underline{\underline{\sigma}}_T(x_1, x_2) = \begin{pmatrix} \sigma_{11}(x_1)x_2 & \sigma_{12}(x_1)q(x_2) \\ \times & g_3x_3/\varepsilon + \sigma_{22}s(x_2) \end{pmatrix},$$

$$\underline{u}_T(x_1, x_2) = \begin{pmatrix} -\theta(x_1)x_2 \\ \omega(x_1) + \omega_2(x_1)r(x_2) \end{pmatrix},$$

where $\sigma_{11} \in P_0^2(-1,1)$, $\sigma_{12} \in P_0^1(-1,1)$, $\sigma_{22} \in P_{-1}^0(-1,1)$,

$\theta \in P_{-1}^1(-1,1)$, $\omega \in P_{-1}^0(-1,1)$

$$-\frac{\varepsilon^2}{3}\int_{-1}^{1}\frac{d\sigma_{11}}{dx_1}\psi\,dx_1+2\varepsilon\int_{-1}^{1}\sigma_{12}\psi\,dx_1=0 \qquad \forall\psi\in P^1_{-1}(-1,1),$$

$$\varepsilon\int_{-1}^{1}\frac{d\sigma_{12}}{dx_1}w\,dx_1=\int_{-1}^{1}g_2w\,dx_1 \qquad \forall w\in P^0_{-1}(-1,1),$$

$$\int_{-1}^{1}\sigma_{22}w_2\,dx_1=\frac{2}{5}\int_{-1}^{1}g_2w_2\,dx_1 \qquad \forall w_2\in P^0_{-1}(-1,1),$$

$$\int_{-1}^{1}\alpha\sigma_{11}\tau_{11}-\beta\sigma_{22}\tau_{11}-2\varepsilon\theta\frac{d\tau_{11}}{dx_1}\,dx_1=2\beta\int_{-1}^{1}g_2\tau_{11}\,dx_1,$$

$$\int_{-1}^{1}\left(\frac{1}{2\mu}\frac{6}{5}\sigma_{12}+\theta\right)\tau_{12}+\omega\frac{d\tau_{12}}{dx_1}\,dx_1=0,$$

$$\int_{-1}^{1}\omega_2\tau_{22}\,dx_1=\int_{-1}^{1}\varepsilon\left(\frac{1}{2\mu}-\beta\right)\left[\frac{1}{3}g_2+\frac{5}{21}\sigma_{22}\right]\tau_{22}-\beta\frac{\varepsilon}{6}\sigma_{11}\tau_{22}\,dx_1,$$

for all $\tau_{11}\in P^2_0(-1,1)$, $\tau_{12}\in P^1_0(-1,1)$, $\tau_{22}\in P^0_{-1}(-1,1)$.

Assume g_2 constant by parts. It is possible to solve for the stresses using the equations above:

$$\sigma_{11}^h = \frac{6}{\varepsilon^2} P_2 + \frac{6}{\varepsilon} c_0 x_1 + c_1, \qquad \sigma_{12}^h = \frac{P}{\varepsilon} + c_0, \qquad \sigma_{22}^h = \frac{2}{5} g_2,$$

where $P(x) = \int_{-1}^{x} g_2(\xi)\, d\xi$, $P_2(x) = \int_{-1}^{x} P(\xi)\, d\xi$, and we compute c_0 and c_1 from

$$(1+\frac{5\mu\alpha}{3\varepsilon^2})c_0 = -\frac{1}{2\varepsilon}\int_{-1}^{1} P\, dx_1 - \frac{5\mu\alpha}{2\varepsilon^3}\int_{-1}^{1} P_2 x_1\, dx_1 - \beta\mu\int_{-1}^{1} g_2 x_1\, dx_1.$$

$$2\alpha c_1 = \beta\frac{6}{5}\int_{-1}^{1} g_2\, dx_1 - \frac{6\alpha}{\varepsilon^2}\int_{-1}^{1} P_2\, dx_1.$$

Hence, we have to solve for $\theta \in P^1_{-1}(-1,1)$ and $\omega \in P^0_{-1}(-1,1)$, where

$$-\int_{-1}^{1} 2\varepsilon\theta \frac{d\tau_{11}}{dx_1}\,dx_1 = \int_{-1}^{1} \Big(\beta\sigma_{22} - \alpha\sigma_{11} + 2\beta g_2\Big)\tau_{11}\,dx_1,$$

$$\int_{-1}^{1} \omega \frac{d\tau_{12}}{dx_1}\,dx_1 = -\int_{-1}^{1} \Big(\frac{1}{2\mu}\frac{6}{5}\sigma_{12} + \theta\Big)\tau_{12}\,dx_1,$$

for all $\tau_{11} \in P^2_0(-1,1)$, $\tau_{12} \in P^1_0(-1,1)$.

- For the plate problem, things are not that easy. Indeed, the planar stress is no longer a scalar, but a 2×2 symmetric matrix with rows in $H(\mathrm{div}, P^{\varepsilon})$. It then becomes nontrivial to find stable elements for stresses and displacements.
- An option is to solve for the planar stress in terms of the displacement and shear stress. This is the same as start with Reissner–Mindlin, and include shear stress, and it's traditionally considered to develop stable elements.

- The 3D Problem and its Modeling
- Full Discretization–Playing Around with the Timoshenko Beam
- Conclusions

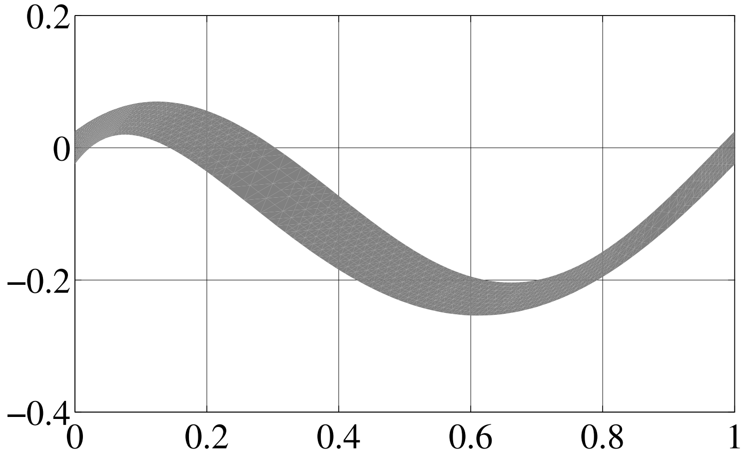
Conclusions

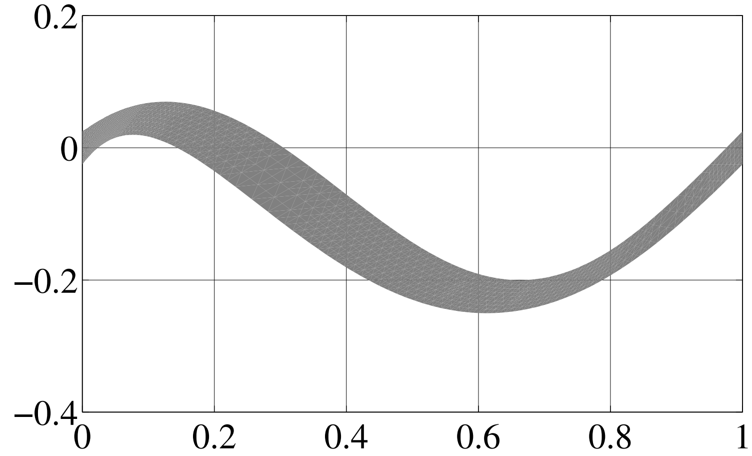
- The derivation of the Reissner–Mindlin Model using Hellinger–Reissner Principle is “mathematically” consistent. The model obtained is actually the first in a hierarchical sequence.
- It also makes the derivation of the modeling error through the two energies principle easier, since it yields statically admissible stress field.
- Reissner–Mindlin is better than the biharmonic: if there is shear, the biharmonic fails to deliver a meaningful result.

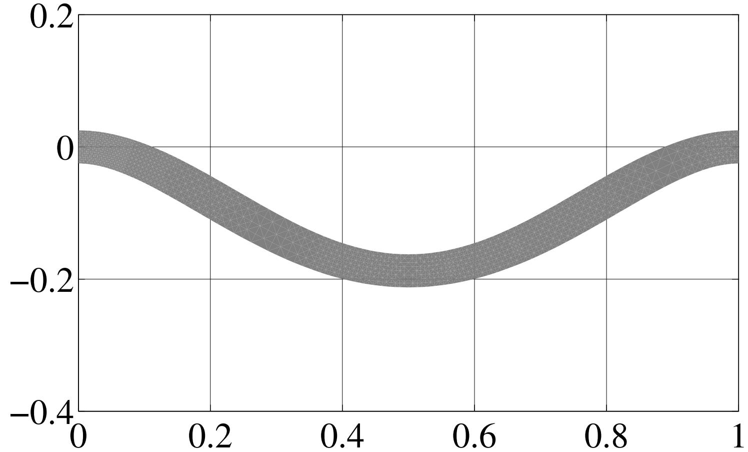
Example: General load

- Domain: $(0,1) \times (-\varepsilon, \varepsilon)$, with $\varepsilon = 1/40$
- Traction: $g^\varepsilon = (1, 10^{-3})$ on the top and $g^\varepsilon = (-1, 10^{-3})$ on the bottom
- Clamped lateral boundary conditions

Planar elasticity, Reissner–Mindlin and Kirchhoff–Love solutions:







- Reissner–Mindlin is the same as the biharmonic: for purely transverse load, both models converge at the same rate. I now of no proof indicating that one is better than the other in this case.

- Both suffer from a quite low convergence rate: $O(\varepsilon^{1/2})$ in relative energy norm. Is there a way to improve this rate?

Thank You!