MIXED FINITE ELEMENTS FOR PLATES

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- Necessity of $2D$ models.
- Reissner-Mindlin Equations.
- Finite Element Approximations.
- Locking.
- Mixed interpolation or reduced integration.
- General Error Analysis.
- L^2 Error estimates
- Examples.

Why do we need to use 2D models if 3D elasticity can be solved by FE?

3D ELASTICITY EQUATIONS

$D \subset \mathbb{R}^3$ Initial configuration of elastic solid.

$u = (u_1, u_2, u_3)$ Displacement.

$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ Strain tensor.

$$2\mu \int_D \varepsilon(u) : \varepsilon(v) + \lambda \int_D \operatorname{div} u \operatorname{div} v = \int_D f \cdot v + \int_\Gamma g \cdot v$$
$$\forall v \in V \subset H^1(D)^3$$

Coercivity of this bilinear form follows from "Korn's inequality":

$$\|u\|_1 \leq K \|\varepsilon(u)\|_0$$

Under appropriate boundary conditions.

Remark 1. *: The constant K depends on the geometry of D.*

RECALL CEA'S LEMMA:

If u is the exact solution and u_h the FE approximation then,

$$\|u - u_h\|_1 \leq \frac{M}{\alpha}\|u - v_h\|_1 \quad \forall v_h \in V_h$$

where M and α are the continuity and coercivity constants of the bilinear form.

In our problem α depends on the Korn's constant K. If K is too large then α is too small and the constant in Cea's Lemma is large.

Consider a PLATE of thickness t:

$$D = \Omega \times (-t/2\,,\, t/2)$$

where $\Omega \subset \mathbb{R}^2$ and $t > 0$, with t small in comparison with the dimensions of Ω.

In this case:

$$K = O(t^{-1})$$

CONSEQUENCE:

THE METHOD IS NOT EFFICIENT: VERY SMALL MESH SIZE WILL BE NEEDED!

This is a serious computational drawback specially in 3D.

SOLUTION: USE 2D MODELS!

REISSNER-MINDLIN EQUATIONS

$$\Omega \subset \mathbb{R}^2 \qquad D = \Omega \times (-t/2\,,\, t/2)$$

Displacements are approximated by

$$\begin{aligned} u_1(x,y,z) &\sim -z\theta_1(x,y) \\ u_2(x,y,z) &\sim -z\theta_2(x,y) \\ u_3(x,y,z) &\sim w(x,y) \end{aligned}$$

θ_1, θ_2 "rotations" ,
w "transverse displacement".

Assuming a transverse load of the form

$$t^3 f(x,y)$$

and a that the plate is clamped,

$$\theta = (\theta_1, \theta_2) \quad \text{and} \quad w$$

satisfy the system of equations:

$$a(\theta,\eta)+\kappa t^{-2}(\nabla w-\theta,\nabla v-\eta)=(f,v)$$
$$\forall \eta\in H^1_0(\Omega)^2, v\in H^1_0(\Omega).$$

$$a(\theta,\eta):=\frac{E}{12(1-\nu^2)}\int_\Omega[(1-\nu)\varepsilon(\theta)\varepsilon(\eta)$$
$$+\nu\operatorname{div}\theta\operatorname{div}\eta],$$

E Young modulus,
ν Poisson ratio,
$\kappa := Ek/2(1+\nu)$ shear modulus,
k correction factor usually taken as $5/6$.

We change notation and keep only the parameter t. So, our equations are

$$a(\theta,\eta)+t^{-2}(\nabla w-\theta,\nabla v-\eta)=(f,v)$$

For our purposes, the only important fact about a is that it is coercive in H^1_0 (which follows from the 2D Korn inequality). We will not make other use of the explicit form o a.

The deformation energy is given by

$$\frac{1}{2}a(\theta,\theta) + \frac{t^{-2}}{2}\int_\Omega |\nabla w - \theta|^2 - \int_\Omega fw$$

It can be shown that the second term remains bounded when $t \to 0$. In particular,

$$t \to 0 \quad \Rightarrow \quad |\nabla w - \theta| \to 0$$

For the limit problem:

$$\nabla w = \theta \quad \text{“Kirchkoff constraint”}$$

THIS IS A PROBLEM FOR THE NUMERICAL SOLUTION!

If t is small the problem is close to a constrained minimization problem.

FOR EXAMPLE: If we use standard $\mathcal{P}_1$ finite elements for θ and w, the restriction of the limit problem is too strong.

Indeed, if

$$\nabla w_h = \theta_h$$

then, ∇w_h piecewise constant and continuous

$$\Rightarrow \quad \nabla w_h \quad \text{constant}$$

But,

$$\theta_h \in H_0^1(\Omega)^2 \quad \Rightarrow \nabla w_h = \theta_h = 0$$

CONSEQUENCE: For t small $\theta_h, w_h \sim 0$.

This is called "LOCKING"

Remark 2. *: Indeed, now the continuity constant of the bilinear form is too large. It seems that we have a problem similar to the original 3D problem!*

AND SO, WHAT IS THE ADVANTAGE OF USING THE 2D MODEL?

SOLUTION: Mixed Interpolation or Reduced Integration.

IDEA: Relax the restriction of the limit problem.

$$\nabla w - \theta = 0 \quad \text{replaced by} \quad \Pi(\nabla w - \theta) = 0$$

Π is some interpolation or projection onto some space Γ_h.

So, in the discrete problem, the restriction is verified only at some points or in some average sense.

FINITE ELEMENT APPROXIMATION:

$$\theta_h \in H_h \subset H_0^1(\Omega)^2, \quad w_h \in W_h \subset H_0^1(\Omega)$$

are such that

$$a(\theta_h, \eta) + t^{-2}(\Pi(\nabla w_h - \theta_h), \Pi(\nabla v - \eta)) = (f, v) \quad \forall \eta \in H_h, v \in W_h$$

In the usual methods

$$\nabla W_h \subset \Gamma_h$$

So,

$$a(\theta_h, \eta) + t^{-2}(\nabla w_h - \Pi\theta_h, \nabla v - \Pi\eta) = (f, v)$$
$$\forall \eta \in H_h, v \in W_h$$

MIXED FORM:

Introducing the shear stress

$$\gamma = t^{-2}(\nabla w - \theta)$$

$$\begin{cases} a(\theta_h, \eta) + (\gamma_h, \nabla v - \Pi\eta) = (f, v) \\ \gamma_h = t^{-2}(\nabla w_h - \Pi\theta_h) \end{cases}$$
$$\forall \eta \in H_h, v \in W_h$$

MAIN PROBLEM: How to choose the spaces H_h, W_h, Γ_h and the operator Π?

EXAMPLE: The Bathe-Dvorkin MITC4 rectangular elements (Mixed Interpolation Tensorial Components).

H_h and W_h are the standard $\mathcal{Q}_1$ elements and Γ_h is locally defined by $(a + by, c + dx)$ (is a rotated Raviart-Thomas space).

The operator Π is defined by

$$\int_\ell \Pi\eta \cdot t_\ell = \int_\ell \eta \cdot t_\ell$$

for all side ℓ of an element, where t_ℓ is the unit tangent vector on ℓ.

GENERAL ERROR ANALYSIS

Continuous problem:

$$\begin{cases} a(\theta, \eta) + (\gamma, \nabla v - \Pi\eta) = (f, v) + (\gamma, \eta - \Pi\eta) \\ \gamma = t^{-2}(\nabla w - \theta) \end{cases}$$

$$\forall \eta \in H_0^1(\Omega)^2, v \in H_0^1(\Omega)$$

Discrete Problem:

$$\begin{cases} a(\theta_h, \eta) + (\gamma_h, \nabla v - \Pi\eta) = (f, v) \\ \gamma_h = t^{-2}(\nabla w_h - \Pi\theta_h) \end{cases}$$

$$\forall \eta \in H_h, v \in W_h$$

Error equation:

$$a(\theta - \theta_h, \eta) + (\gamma - \gamma_h, \nabla v - \Pi\eta)$$
$$= (\gamma, \eta - \Pi\eta) \qquad \forall \eta \in H_h, v \in W_h$$

Lemma 1. *Let* $\theta_I \in H_h$*,* $w_I \in W_h$ *and* $\gamma_I = t^{-2}(\nabla w_I - \Pi\theta_I) \in \Gamma_h$.
Suppose

$$\|\gamma - \Pi\gamma\|_0 \leq Ch\|\gamma\|_1$$

and

$$(\gamma - \Pi\gamma, \eta) = 0 \quad \forall \eta \in \mathcal{P}^2_{k-2}$$

Let P *be the* L^2 *projection into* $\mathcal{P}^2_{k-2}$*. Then,*

$$\|\theta - \theta_h\|_1 + t\|\gamma - \gamma_h\|_0$$
$$\leq C(\|\theta_I - \theta\|_1 + t\|\gamma_I - \gamma\|_0 + h\|\gamma - P\gamma\|_0$$

Proof.

$$a(\theta_I - \theta_h, \eta) + (\gamma_I - \gamma_h, \nabla v - \Pi\eta)$$
$$= a(\theta_I - \theta, \eta) + (\gamma_I - \gamma, \nabla v - \Pi\eta) + (\gamma, \eta - \Pi\eta)$$
$$\forall \eta \in H_h, v \in W_h$$

Take $\eta = \theta_I - \theta_h$ and $v = w_I - w_h$. So,

$$\gamma_I - \gamma_h = t^{-2}(\nabla v - \Pi\eta)$$

Using the coercivity of a we obtain

$$\|\theta_I - \theta_h\|_1^2 + t^2\|\gamma_I - \gamma_h\|_0^2$$

$$= a(\theta_I - \theta, \theta_I - \theta_h) + t^2(\gamma_I - \gamma, \gamma_I - \gamma_h)$$
$$+(\gamma - P\gamma, \theta_I - \theta_h - \Pi(\theta_I - \theta_h))$$

and the lemma follows. □

To apply the Lemma we need to find approximations θ_I and w_I such that the associated γ_I be also a good approximation. This will follow from the existence of approximations satisfying the following property

$$\boxed{\nabla w_I - \Pi\theta_I = \Pi(\nabla w - \theta)}$$

and

$$\|\theta - \theta_I\|_1 \leq Ch^k \|\theta\|_{k+1}$$
$$\|\gamma - \Pi\gamma\|_0 \leq Ch^k \|\gamma\|_k$$

This property can be seen as a generalization of the known Fortin property basic in the analysis of mixed methods. In fact, introducing the operators

$$I(\theta, w) = (\theta_I, w_I)$$
$$B(\theta, w) = \nabla w - \theta$$

and

$$B_h(\theta_h, w_h) = \nabla w_h - \Pi\theta$$

the property can be summarized by the following commutative diagram:

$$\begin{array}{ccc} (H^1_0)^2 \times H^1_0 & \xrightarrow{B} & (L^2)^2 \\ I \big\downarrow & & \big\downarrow \Pi \\ H_h \times W_h & \xrightarrow{B_h} & \Gamma_h \end{array}$$

When Π is an L^2 projection, this is exactly the Fortin property, which is known to be equivalent to the inf-sup condition. This will be the case in one of our examples (the Arnold-Falk elements). In that case error estimates for the shear stress γ can also be obtained.

Remark 3. *Indeed it is enough to have the commutative diagram with* Π *replaced by some operator with good approximation properties.*

If approximations satisfying the properties above (i.e, commutative diagram + approximation properties) exist, then it follows from the Lemma:

ERROR ESTIMATES

$$\|\theta - \theta_h\|_1 + t\|\gamma - \gamma_h\|_0 + \|w - w_h\|_1$$
$$\leq Ch^k(\|\theta\|_{k+1} + t\|\gamma\|_k + \|\gamma\|_{k-1})$$

With constant C independent of h and t.

EXAMPLES

Example 1: MITC3 or DL element:

$H_h : \mathcal{P}_1^2 \oplus \{\lambda_2\lambda_3 t_1, \lambda_3\lambda_1 t_2, \lambda_1\lambda_2 t_3\}, \mathcal{C}^0$

Degrees of freedom: $\theta(V_i)$, $\int_{\ell_i} \theta \cdot t_i$

$W_h : \quad \mathcal{P}_1 \quad \mathcal{C}^0$

$\Gamma_h : \quad (a - by, c + bx) \quad \mathcal{C}^0$ t. c.

Degrees of freedom: $\int_{\ell_i} \gamma \cdot t_i$

Π defined by

$$\int_\ell \Pi\gamma \cdot t = \int_\ell \gamma \cdot t$$

Example 2: Arnold-Falk element:

H_h : $\mathcal{P}_1^2 \oplus \{\lambda_1\lambda_2\lambda_3\}$, $\mathcal{C}^0$

Degrees of freedom: $\theta(V_i)$, $\int_T \theta$

W_h : $\mathcal{P}_1$ Non Conforming

Degrees of freedom: $\int_\ell w$

Γ_h : $\mathcal{P}_0^2$

Π defined by

$$\int_T \Pi\gamma = \int_T \gamma$$

L^2-projection!

Error estimates for Examples 1 and 2:

$$\|\theta - \theta_h\|_1 + \|w - w_h\|_1 \le Ch$$

Example 3: Bathe-Brezzi second order rectangular elements:

$H_h: \mathcal{Q}_2, \mathcal{C}^0$

W_h: $\mathcal{Q}_2^r : \{1, x, y, xy, x^2, y^2, x^2y, xy^2\}, \mathcal{C}^0$

Γ_h: $\{1, x, y, xy, y^2\} \times \{1, x, y, xy, x^2\}$

Π defined by

$$\begin{cases} \int_\ell \Pi\gamma \cdot tp_1 = \int_\ell \gamma \cdot tp_1 \\ \int_R \Pi\gamma = \int_R \gamma \end{cases}$$

Error estimates:

$$\|\theta - \theta_h\|_1 + \|w - w_h\|_1 \leq Ch^2$$

Remark 4. *Here the constant C is NOT independent of t because the right hand side involves higher order norms of the solution which depend on t.*

L^2- ERROR ESTIMATES

$$\|\theta - \theta_h\|_0 + \|w - w_h\|_0 \le Ch^2$$

Recall the error equation

$$a(\theta - \theta_h, \eta) + (\gamma - \gamma_h, \nabla v - R\eta) = (\gamma, \eta - R\eta)$$
$$\forall (\eta, v) \in H_h \times W_h,$$

Duality argument: $(\varphi, u) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$ solution of

$$\begin{cases} a(\eta, \varphi) + (\nabla v - \eta, \delta) \\ = (v, w - w_h) + (\eta, \theta - \theta_h) \\ \delta = t^{-2}(\nabla u - \varphi). \end{cases}$$
$$\forall (\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega),$$

A priori estimate (we assume Ω is convex)

$$\|\varphi\|_2 + \|u\|_2 + \|\delta\|_0 + t\,\|\delta\|_1$$
$$\le C(\|\theta - \theta_h\|_0 + \|w - w_h\|_0\Big),$$

Take $v = w - w_h$ and $\eta = \theta - \theta_h$ in the dual problem and use the error equation with $(\eta, v) = (\varphi_I, u_I)$:

$$\begin{gathered}\|w - w_h\|_0^2 + \|\theta - \theta_h\|_0^2 \\ = a(\theta - \theta_h, \varphi - \varphi_I) + t^2(\gamma - \gamma_h, \delta - \Pi\delta) \\ + (\theta_h - R\theta_h, \delta) + (\gamma, \varphi_I - \Pi\varphi_I),\end{gathered}$$

where we have used the commutative diagram property $\Pi\delta = t^{-2}(\nabla u_I - \Pi\varphi_I)$.

PROBLEM: The last two terms.

For the MITC3 elements we have:

Lemma 2. *If $\gamma \in H(div, \Omega)$, $\psi \in H_0^1(\Omega)$ and ψ_A is a piecewise-linear average interpolant:*

$$|(\gamma, \psi_A - \Pi\psi_A)| \leq Ch^2 \|div\gamma\|_0 \|\psi\|_1$$

ISOPARAMETRIC ELEMENTS

$$F_K : \hat{K} \longrightarrow K$$

$\hat{K} = (0,1) \times (0,1)$: reference element

K: shape regular quadrilateral

F_K: $\mathcal{Q}_1^2$ transformation

W_h : Standard isoparametric $\mathcal{Q}_1$ elements:

$$v \circ F_K = (a + b\hat{x} + c\hat{y} + d\hat{x}\hat{y})$$

Γ_h : Rotated Raviart-Thomas elements:

$$\eta \circ F_K = DF_K^{-T}(a + b\hat{y}, c + d\hat{x})$$

The operator Π is defined by

$$\int_\ell \Pi\eta \cdot t_\ell = \int_\ell \eta \cdot t_\ell$$

for all side ℓ of an element, where t_ℓ is the unit tangent vector on ℓ.

For H_h we consider two cases:

MITC4: Standard $\mathcal{Q}_1^2$ isoparametric elements

DL4: $\mathcal{Q}_1^2 \oplus \langle p_1, p_2, p_3, p_4 \rangle$

$p_i = (\hat{p}_i \circ F_K^{-1}) t_i$

$\hat{p}_i$ is a cubic polynomial vanishing on $\hat{\ell}_j$, $j \neq i$.

ERROR ESTIMATES

$$\|\theta - \theta_h\|_1 + t\|\gamma - \gamma_h\|_0 + \|w - w_h\|_1 \\ \leq Ch(\|\theta\|_2 + t\|\gamma\|_1 + \|\gamma\|_0)$$

ASSUMPTIONS: For DL4 general shape regular meshes. For MITC4 the mesh $\mathcal{T}_h$ is a refinement of a coarser $\mathcal{T}_{2h}$ obtained by joining the edge midpoints and $\mathcal{T}_{2h}$ is obtained in the same way from $\mathcal{T}_{4h}$.

L^2- ERROR ESTIMATES

$$\|\theta - \theta_h\|_0 + \|w - w_h\|_0 \leq Ch^2$$

ASSUMPTION: Asymptotically parallelogram meshes.