Optimal error estimates of the SDFEM on layer adapted meshes

...joint work with Martin Stynes, National University College Cork

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Problem and solution decomposition

$$-\varepsilon\Delta u + b\cdot\nabla u + c\,u = f \quad \text{in } \Omega = (0,1)^2, \quad u = 0 \quad \text{on } \Gamma := \partial\Omega$$

$$0 < \varepsilon \ll 1, \quad c - \frac{1}{2}\text{div}\, b \geq c_0 > 0, \quad b_i(x,y) \geq \beta_i > 0 \quad i = 1,2,$$

Assumption

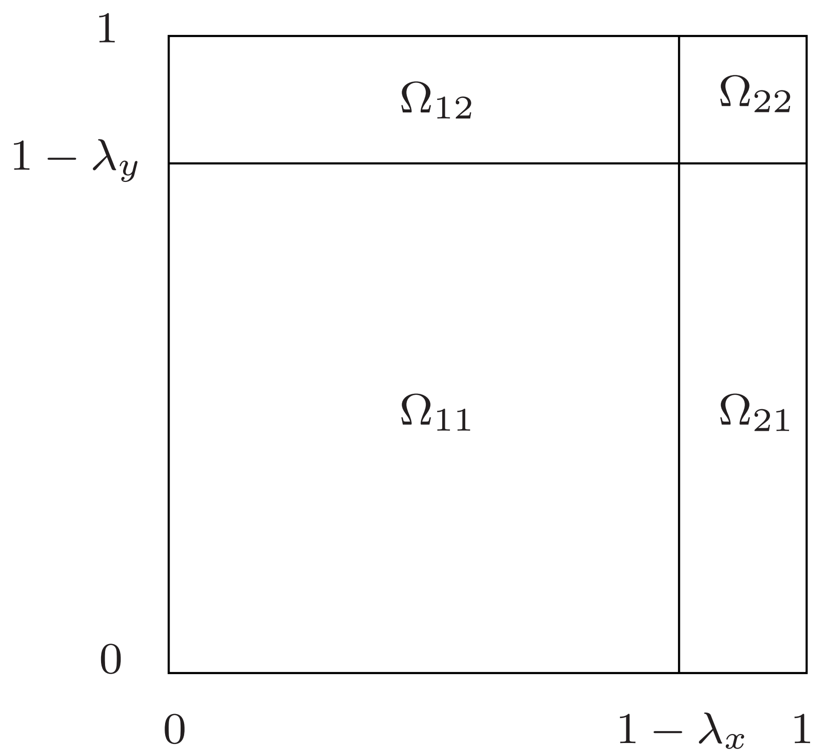
$$u = S + E_{21} + E_{12} + E_{22},$$

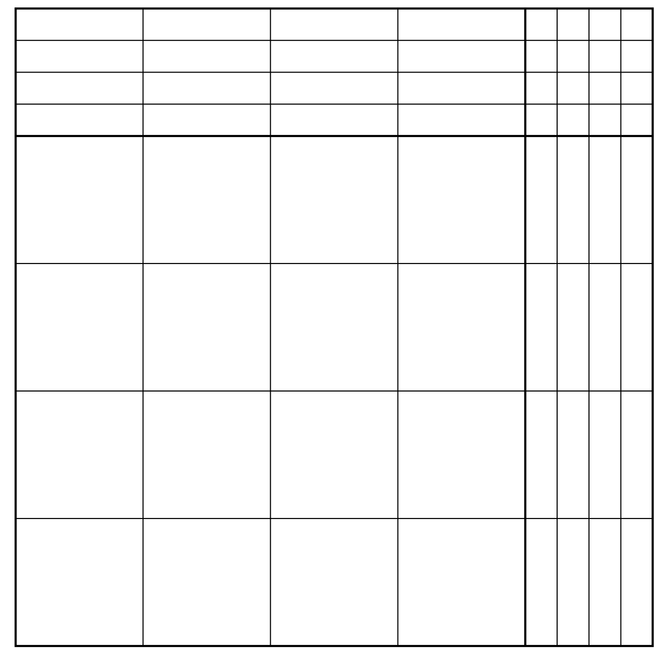
where there exists a constant C such that for all $(x,y) \in \Omega$

$$\begin{aligned}
\left|\frac{\partial^{i+j} S}{\partial x^i \partial y^j}(x,y)\right| &\leq C \qquad \text{for } 0 \leq i+j \leq 2,\\
\left|\frac{\partial^{i+j} E_{21}}{\partial x^i \partial y^j}(x,y)\right| &\leq C\,\varepsilon^{-i} e^{-\beta_1(1-x)/\varepsilon} \quad \text{for } 0 \leq i,j \leq 2,\\
\left|\frac{\partial^{i+j} E_{12}}{\partial x^i \partial y^j}(x,y)\right| &\leq C\,\varepsilon^{-j} e^{-\beta_2(1-y)/\varepsilon} \quad \text{for } 0 \leq i,j \leq 2,\\
\left|\frac{\partial^{i+j} E_{22}}{\partial x^i \partial y^j}(x,y)\right| &\leq C\,\varepsilon^{-(i+j)} e^{-(\beta_1(1-x)+\beta_2(1-y))/\varepsilon} \quad \text{for } 0 \leq i,j \leq 2.\\
\|S\|_3 &\leq C, \qquad S \in H^3(\Omega)
\end{aligned}$$

Linß /Stynes: Asymptotic analysis . . . J. Math. Anal. Appl. 261(2001), 604-632

Shishkin mesh





$$\lambda_x = \min\left(\frac{1}{2}, \frac{5}{2}\frac{\varepsilon}{\beta_1}\ln N\right) \qquad \lambda_y = \min\left(\frac{1}{2}, \frac{5}{2}\frac{\varepsilon}{\beta_2}\ln N\right)$$

We have for $(x,y) \in \Omega_{11} \cup \Omega_{12}$

$$|E_{21}(x,y)| \le C\ \exp(-\frac{5}{2}\ln N) \le C\ N^{-\frac{5}{2}}$$

Galerkin FEM on layer adapted meshes

- Stynes/O'Riordan: A uniform convergent Galerkin method on a Shishkin mesh for a convection–diffusion problem. J. Math. Anal. Appl. 214(1997), 36-54
  On a piecewise uniform mesh (Shishkin-type mesh)

$$\|u^I - u^N_{Gal}\|_{1,\varepsilon} = \mathcal{O}(N^{-1} \ln N)$$

- Linß: Uniform superconvergence of a Galerkin finite element method on Shishkin-type meshes. Numer. Methods Partial Diff. Eqs. 16(2000), 426-440
  On a graded Bachvalov-type mesh

$$\|u^I - u^N_{Gal}\|_{1,\varepsilon} = \mathcal{O}(N^{-2})$$

- Zhang: Finite element superconvergence on Shishkin mesh for 2-d convection–diffusion problems. Math. Comp. 72(2003), 1147-1177

$$\|u^I - u^N_{Gal}\|_{1,\varepsilon,d} = \mathcal{O}(N^{-2} \ln^2 N)$$

Drawback: Stability of the discrete problem with respect to $\|\cdot\|_{1,\varepsilon} = \left(\varepsilon|\cdot|_1^2 + \|\cdot\|_0^2\right)^{1/2}$

SDFEM on uniform and layer adapted meshes

- Nävert: A finite element method for convection–diffusion problems.
  PhD thesis, Chalmers University, Göteborg 1982
  On a uniform mesh for linear and bilinear elements, respectively

$$\|u - u^N\|_{SD} = \mathcal{O}(N^{-3/2}|u|_2)$$

- Stynes/Tobiska: Analysis of the streamline-diffusion finite element method for a convection–diffusion problem with exponential layers.
  East-West J. Numer. Math. 9(2001), 59-76
  On a Shishkin-type mesh

$$\|u^I - u^N\|_{SD} = \mathcal{O}(\varepsilon^{1/2} N^{-1} \ln N + N^{-3/2})$$

Advantage: Stability in the stronger SD-norm

$$\|\cdot\|_{SD} := \left(\|\cdot\|_{1,\varepsilon}^2 + \sum_K \delta_K \|b \cdot \nabla \cdot\|_{0,K}^2 \right)^{1/2}$$

Question: Do you have to pay the price $\mathcal{O}(N^{-2} \ln^2 N) \mapsto \mathcal{O}(N^{-3/2})$ for getting the improved stability?

SDFEM on a Shishkin mesh

$$V^N := \{v \in C(\bar{\Omega}) \ : \ v|_\Gamma = 0 \text{ and } v|_K \in Q_1(K) \ \forall K \in \mathcal{T}_N\}.$$

Bilinear form

$$B_{SD}(w,v) = B_{GAL}(w,v) + B_{STAB}(w,v)$$

$$\begin{aligned} B_{GAL}(w,v) &:= \varepsilon(\nabla w, \nabla v) + (b \cdot \nabla w + cw, v) \\ B_{STAB}(w,v) &= \sum_{K \subset \Omega_{11}} \delta_K(-\varepsilon \Delta w + b \cdot \nabla w + cw, b \cdot \nabla v)_K \end{aligned}$$

Find $u^N \in V^N$ such that for all $v^N \in V^N$

$$B_{SD}(u^N, v^N) = (f, v^N) + \sum_{K \subset \Omega_{11}} \delta_K(f, b \cdot \nabla v^N)_K$$

SD-Norm

$$\|v\|_{SD} := \left(\varepsilon |v|_1^2 + \sum_{K \subset \Omega_{11}} \delta_K \|b \cdot \nabla v\|_{0,K}^2 + c_0 \|v\|_0^2 \right)^{1/2}$$

A-priori error estimate for $u^I - u^N$

Using coercivity for $0 \le \delta_K \le \dfrac{c_0}{\max\limits_K |c(x,y)|^2} \quad \forall K \subset \Omega_{11}$ and the choice $\delta_K \sim N^{-1}$

$$\begin{aligned}
\frac{1}{2}\|u^N - u^I\|_{SD}^2 &\le B_{SD}(u^N - u^I, u^N - u^I) = B_{SD}(u - u^I, u^N - u^I) \\
&= B_{GAL}(u - u^I, u^N - u^I) + B_{STAB}(u - u^I, u^N - u^I)
\end{aligned}$$

Zhang: Finite element superconvergence . . . Math. Comp. 72(2003), 1147-1177

$$\begin{aligned}
|B_{GAL}(u - u^I, v^N)| &\le C(\varepsilon N^{-3/2} + N^{-2}\ln^2 N)\, \|v^N\|_{1,\varepsilon} \\
&\le C(\varepsilon N^{-3/2} + N^{-2}\ln^2 N)\, \|v^N\|_{SD}
\end{aligned}$$

under the assumption $\|S\|_{3,\infty} \le C$. However a careful investigation shows that $\|S\|_3 \le C$ is sufficient.

Ingredients

Anisotropic interpolation error estimates

$$\|v - v^I\|_{0,K} \le C\left(h_{x,K}^2\|v_{xx}\|_{0,K} + h_{x,K}h_{y,K}\|v_{xy}\|_{0,K} + h_{y,K}^2\|v_{yy}\|_{0,K}\right),$$

$$\|(v - v^I)_x\|_{0,K} \le C\left(h_{x,K}\|v_{xx}\|_{0,K} + h_{y,K}\|v_{xy}\|_{0,K}\right),$$

$$\|(v - v^I)_y\|_{0,K} \le C\left(h_{x,K}\|v_{xy}\|_{0,K} + h_{y,K}\|v_{yy}\|_{0,K}\right).$$

Properties of the terms in the decomposition ($E = E_{21}$, E_{12}, E_{22})

$$\|S - S^I\|_{0,\Omega} \le C\,N^{-2},$$

$$\|E\|_{0,\Omega_{11}} \le C\,\varepsilon^{1/2}N^{-5/2},$$

$$\varepsilon\|\Delta E\|_{L^1(\Omega_{11})} + \|\nabla E\|_{L^1(\Omega_{11})} \le C\,N^{-5/2},$$

$$N^{-1}\|\nabla E^I\|_{0,\Omega_{11}} + \|E^I\|_{0,\Omega_{11}} \le C\left(\varepsilon^{1/2}N^{-5/2} + N^{-3}\right),$$

$$\|E - E^I\|_{0,\Omega} \le CN^{-2}\ln^2 N.$$

Lin's identities

Let the east, north, west and south edges of K denoted by $l_{i,K}$ for $i = 1, \ldots, 4$, and

$$G_K(x) = \frac{1}{2}\left[(x - x_K)^2 - \left(\frac{h_{x,K}}{2}\right)^2\right], \quad F_K(y) = \frac{1}{2}\left[(y - y_K)^2 - \left(\frac{h_{y,K}}{2}\right)^2\right].$$

$$\int_K (w - w^I)_x v_x^N \, dx\, dy = \int_K w_{xyy} \left(F_K v_x^N - \frac{1}{3}\left(F_K^2\right)' v_{xy}^N\right) dxdy$$

$$\int_K (w - w^I)_x v_y^N \, dx\, dy = \int_K \left(F_K w_{xyy}(v_y^N - G_K' v_{xy}^N) + G_K w_{xxy} v_x^N\right) dx\, dy$$

$$- \int_{l_{2,K}} G_K w_{xx} v_x^N \, dx + \int_{l_{4,K}} G_K w_{xx} v_x^N \, dx$$

$$\int_K (w - w^I)_y v_x^N \, dx\, dy = \int_K \left(G_K w_{xxy}(v_x^N - F_K' v_{xy}^N) + F_K w_{xyy} v_y^N\right) dx\, dy$$

$$- \int_{l_{1,K}} F_K w_{yy} v_y^N \, dy + \int_{l_{3,K}} F_K w_{yy} v_y^N \, dy$$

$$\int_K (w - w^I)_y v_y^N \, dxdy = \int_K w_{xxy} \left(G_K v_y^N - \frac{1}{3}\left(G_K^2\right)' v_{xy}^N\right) dx\, dy$$

Final result of estimating $u^N - u^I$

Under the assumption we have

$$\left| B_{STAB}(u-u^I, v^N) \right| \leq C\, N^{-2} \ln^{1/2} N \| v^N \|_{SD} \qquad \forall v^N \in V^N.$$

Zhang's estimate

$$\left| B_{GAL}(u-u^I, v^N) \right| \leq C\, \left(\varepsilon N^{-3/2} + N^{-2} \ln^2 N \right) \| v^N \|_{SD} \qquad \forall v^N \in V^N.$$

Theorem 1: The SDFEM solution on a Shishkin-type mesh satisfies

$$\| u^N - u^I \|_{SD} \leq C\, (\varepsilon N^{-3/2} + N^{-2} \ln^2 N).$$

Superconvergence of the derivatives in the streamline direction in Ω_{11}

$$\left(\sum_{K \subset \Omega_{11}} \| b \cdot \nabla (u^I - u^N) \|^2_{0,K} \right)^{1/2} \leq C N^{-3/2} \ln^2 N,$$

A-priori error estimate for $u - u^N$?

Triangle inequality

$$\begin{aligned} \|u - u^N\|_{SD} &\leq \|u - u^I\|_{SD} + \|u^I - u^N\|_{SD} \\ &\leq \|u - u^I\|_{SD} + C\,(\varepsilon N^{-3/2} + N^{-2}\ln^2 N) \end{aligned}$$

Unfortunately, $\varepsilon^{1/2}|u - u^I|_1 \leq C\, N^{-1}\ln N$ is optimal! Thus,

$$\|u - u^N\|_{SD} \leq C\, N^{-1}\ln N$$

Two way for getting higher order estimates in H^1-type norms

- use a discrete (weaker) norm satisfying $\|\cdot\|_{SD,d} \leq C\,\|\cdot\|_{SD}$ on V^N
- construct a local postprocessing operator P and meassure the error $u - Pu^N$

Bound in discrete SD-norm

Integrals of derivatives in $\|\cdot\|_{SD}$ are replaced by a barycentric one-point quadrature rule

$$\|v\|_{SD,d} = \Bigg[\sum_{K\in\mathcal{T}_N} \varepsilon\,(\mathrm{area}\,K)|\nabla v(x_K,y_K)|^2 + \sum_{K\subset\Omega_{11}} \delta_K\,(\mathrm{area}\,K)|(b\cdot\nabla v)(x_K,y_K)|^2 + \|v\|_0^2 \Bigg]^{1/2},$$

Then, we have

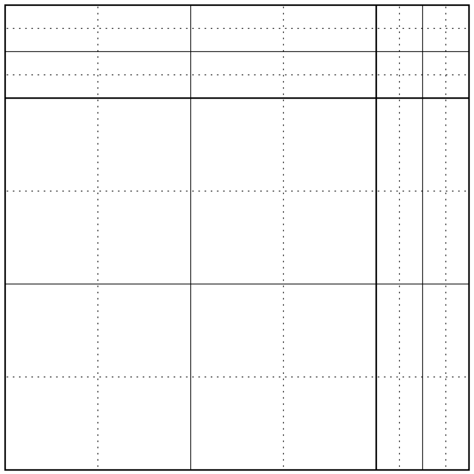
$$\|v^N\|_{SD,d} \le C\,\|v^N\|_{SD} \quad \text{for all } v^N \in V^N$$

$$\|u - u^I\|_{SD,d} \le C\,N^{-2}\ln^2 N$$

Theorem 2: Under additional regularity assumptions

$$\|u - u^N\|_{SD,d} \le C\,(\varepsilon N^{-3/2} + N^{-2}\ln^2 N)$$

Local postprocessing $u^N \mapsto Pu^N$



Construct the piecewise biquadratic interpolation P on each $4\times$ 4 macroelement M, then

$$P(v^I) = P(v) \qquad \forall v \in C(\bar{\Omega})$$

$$\|Pv^N\|_{1,\varepsilon} \leq C\|v^N\|_{1,\varepsilon} \qquad \forall v^N \in V^N$$

$$\|Pu - u\|_{1,\varepsilon} \leq C\left(\varepsilon N^{-3/2} + N^{-2}\ln^2 N\right)$$

Theorem 3: Under additional regularity assumptions

$$\|u - Pu^N\|_{1,\varepsilon} \leq C\,(\varepsilon N^{-3/2} + N^{-2}\ln^2 N)$$

Proof:

$$\begin{aligned}\|u - Pu^N\|_{1,\varepsilon} &\leq \|u - Pu\|_{1,\varepsilon} + \|P(u^I - u^N)\|_{1,\varepsilon}\\ &\leq C\left(\varepsilon N^{-3/2} + N^{-2}\ln^2 N + \|u^I - u^N\|_{1,\varepsilon}\right)\\ &\leq C\left(\varepsilon N^{-3/2} + N^{-2}\ln^2 N\right)\end{aligned}$$