Robust a-posteriori analysis of the (1D) advection-diffusion-reaction problems

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A-posteriori estimates (adaptivity) $\leftrightarrow$ multiscale

- adaptivity is effective for capturing localized small-scales
- Araya R. and Valentin F., "A Multiscale A Posteriori Error Estimate", to appear in CMAME.
- Sangalli G., "A robust a posteriori estimator for the Residual-free Bubbles method applied to advection-diffusion problems", Numer. Math., Vol. 89 (2), pp. 379-399, 2001.

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Outline:

- one-dimensional adv.-diff.-reac. differential problem
- residual-based a-posteriori estimators known in literature
- robustness (symmetric vs. non-symmetric problem)
- costruction of a new estimator (for a new norm of the error)
- numerical tests

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We consider the very simple 1D advection-diffusion-reaction problem:

$$\mathcal{L}u := -\varepsilon u'' + \beta u' + \rho u = f \text{ in } (0,1), \qquad u(0) = u(1) = 0.$$

Introducing the bilinear form $a(w,v) := {}_{H^{-1}}\langle \mathcal{L}w, v\rangle_{H_0^1}, \quad \forall w, v \in V,$
we have the variational formulation: find $u_h \in V$ such that:

$$a(u,v) = \langle f, v\rangle, \forall v \in V;$$

and the *standard* and *stabilized* FEM:

1) Standard FEM: Find $u_h \in V_h \in V$ (...piecewise polynomials) s.t.:

$$a(u_h, v_h) = \langle f, v_h\rangle, \forall v_h \in V_h;$$

2) SUPG: Find $u_h \in V_h \in V$ such that, given $\tau_T \approx h/|\beta|$:

$$a(u_h, v_h) + \sum_T \tau_T \int_T \mathcal{L}u_h \, \beta v_h' = \langle f, v_h\rangle + \sum_T \tau_T \int_T f \, \beta v_h', \qquad \forall v_h \in V_h,$$

1 Residual-based a posteriori estimators

An a posteriori estimator is an expression $\eta(f, u_h)$ which estimates the numerical error $\|u - u_h\|$. It is reliable and efficient when the

$$\text{e.i.} = \text{effectivity index} := \frac{\eta(f, u_h)}{\|u - u_h\|}$$

is close to 1. It is called *robust* (in this context) when e.i. $\simeq 1$ uniformly with respect to the operator coefficients ε, β and ρ.

estimator for $\mathcal{L}w = -w''$

I. Babuška and A. Miller, *A feedback finite element method with a posteriori error estimation. I. The finite element method and some basic properties of the a posteriori error estimator*, CMAME, 61 ('87)

$\downarrow$

robust estimator for $\mathcal{L}w = -\varepsilon w'' + \rho w$

R. Verfürth, *Robust a posteriori error estimators for a singularly perturbed reaction-diffusion equation*, Numer. Math., 78 ('98)

$\downarrow$

non-robust estimator for $\mathcal{L}w = -\varepsilon w'' + \beta w' + \rho w$

R. Verfürth, *A posteriori error estimators for convection-diffusion equations*, Numer. Math., 80 ('98), pp. 641–663

$\downarrow$

the present quasi-robust estimator for $\mathcal{L}w = -\varepsilon w'' + \beta w' + \rho w$ (1D).

The loss of robustness is confirmed in numerical tests:

S. Berrone, *Robustness in a posteriori error analysis for FEM flow models*, Numer. Math., 91 (2002), pp. 389–422.

V. John, *A numerical study of a posteriori error estimators for convection-diffusion equations*, Comput. Methods Appl. Mech. Engrg., 190 (2000), pp. 757–781.

A. Papastavrou and R. Verfürth, *A posteriori error estimators for stationary convection-diffusion problems: a computational comparison*, Comput. Methods Appl. Mech. Engrg., 189 (2000), pp. 449–462.

1.1 case $\mathcal{L}w = -w''$ (Babuska, Miller)

$$\|u - u_h\|_{H_0^1} \overset{\boxed{1}}{\simeq} \|f - \mathcal{L}u_h\|_{H^{-1}}$$

$$\overset{\boxed{2}}{\simeq} \left(\sum_T h_T^2 \|f - \mathcal{L}u_h\|_{L^2(T)}^2 \right)^{1/2} =: \eta$$

$\boxed{1}$ because $\mathcal{L}$ is an isometry: $\|w\|_{H_0^1} = \|\mathcal{L}w\|_{H^{-1}}$;

$\boxed{2}$ "$\leq C$" for the Galerkin ortogonality

"$\geq C$" because $f - \mathcal{L}u_h$ is "discrete", assuming f piece-wise polynomial.

1.2 case $\mathcal{L}w = -\varepsilon w'' + \rho w$ (Verfürth 1998)

1 is ok with the *energy norm* $\|w\|_E = \left(\varepsilon |w|^2_{H^1_0} + \rho \|w\|^2_{L^2}\right)^{1/2}$, since

$$\|u - u_h\|_E \overset{\boxed{1}}{=} \|f - \mathcal{L}u_h\|_{E^*} := \sup_{v \in H^1_0} \frac{\langle f - \mathcal{L}u_h, v\rangle}{\|v\|_E};$$

2 this step is similar to the previous case, leading to

$$\|f - \mathcal{L}u_h\|_{E^*} \overset{\boxed{2}}{\simeq} \left[\sum_T \min\{\rho^{-1}, \varepsilon^{-1} h_T^2\} \|f - \mathcal{L}u_h\|^2_{L^2(T)}\right]^{1/2} + \left[\sum_z \min\{\rho^{-1/2}\varepsilon^{-1/2}, \varepsilon^{-1} h_z\} [\varepsilon u_h']^2(z)\right]^{1/2}$$

1.3 case $\mathcal{L}w = -\varepsilon w'' + \beta w' + \rho w$ (Verfürth 1998)

Now we have:

$$\|u - u_h\|_E \overset{\boxed{1}}{\le} \|f - \mathcal{L}u_h\|_{E^*}$$

$$\overset{\boxed{2}}{\simeq} \left[\sum_T \min\{\rho^{-1}, \varepsilon^{-1} h_T^2\} \|f - \mathcal{L}u_h\|_{L^2(T)}^2\right]^{1/2}$$

$$+ \left[\sum_z \min\{\rho^{-1/2}\varepsilon^{-1/2}, \varepsilon^{-1} h_z\}[\varepsilon u_h']^2(z)\right]^{1/2} =: \hat{\eta}_{old}$$

$\boxed{1}$ this is no more an equivalence: indeed $\mathcal{L}$ is coercive w.r.t. the energy norm $\|\cdot\|_E$ ($\Rightarrow \|w\|_E \le \|\mathcal{L}w\|_{E^*}$) but is not continuous uniformly w.r.t. the coefficients ($\|\mathcal{L}w\|_{E^*} \not\le C\|w\|_E$);

$\boxed{2}$ still ok.

if we want an estimator from below, then we need a different expression; the situation is

$$\check{\eta}_{old} \leq \|u - u_h\|_E \leq \hat{\eta}_{old},$$

where $\hat{\eta}_{old}/\check{\eta}_{old}$ depends on the coefficients.

The weak step in the previous estimate is $\boxed{1}$ where we used the energy norm:

$$\|u - u_h\|_E \overset{\boxed{1}}{\leq} \|f - \mathcal{L}u_h\|_{E^*},$$

we have to use instead a *natural* norm $\|\cdot\|_V$

$$\|u - u_h\|_V \overset{\boxed{1}}{\simeq} \|f - \mathcal{L}u_h\|_{V^*}.$$

2 Obtaining $\|w\|_V \simeq \|\mathcal{L}w\|_{V^*}$

The construction of a possible $\|\cdot\|_V$ has been addressed in:

G. Sangalli, *A uniform analysis of non-symmetric and coercive linear operators*, Tech. Rep. 23-PV, I.M.A.T.I.-C.N.R., 2003.

Rougly speaking:

$$\mathcal{L}w = -\varepsilon w'' + \rho w \qquad \|w\|_E{}^2 = \varepsilon |w|^2_{H^1_0} + \rho \|w\|^2_{L^2}$$

$$\mathcal{L}w = -\varepsilon w'' + \beta w' + \rho w \qquad \|w\|_V{}^2 = \varepsilon |w|^2_{H^1_0} + |\beta| \cdot |w|^2_{1/2} + \rho \|w\|^2_{L^2}.$$

Then we can reason as before, deriving

$$\|u - u_h\|_V \overset{\boxed{1}}{\simeq} \|f - \mathcal{L}u_h\|_{V^*}$$

$$\overset{\boxed{2}}{\simeq} \text{ new a-posteriori estimator.}$$

$$
\begin{aligned}
C_1 \check{\eta}_{new} &:= C_1 \left(\sum_T \check{\eta}_T^2 + \sum_z \check{\eta}_z^2 \right)^{1/2} \\
&\leq \|f - \mathcal{L}u_h\|_{V^*} \\
&\leq C_2 \left(\sum_T \hat{\eta}_T^2 + \sum_z \hat{\eta}_z^2 \right)^{1/2} =: C_2 \hat{\eta}_{new}.
\end{aligned}
$$

$$
\check{\eta}_T^2 = \min\{\rho^{-1}, \beta^{-1} h_T, \varepsilon^{-1} h_T^2\} \|f - \mathcal{L}u_h\|_{L^2(T)}^2
$$

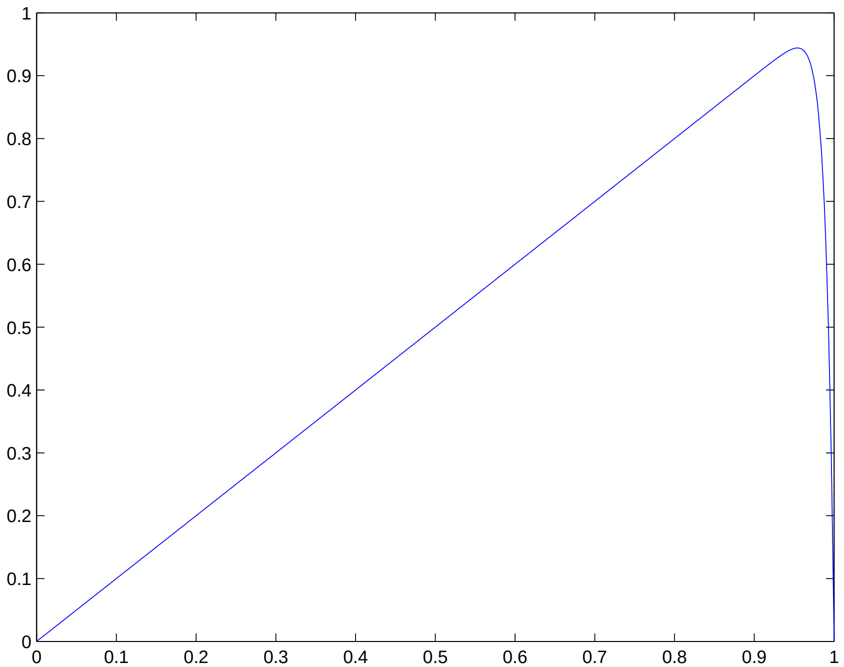
$$
\check{\eta}_z^2 = \min\{\rho^{-1/2}\varepsilon^{-1/2}, \beta^{-1}, \varepsilon^{-1} h_T\} \left(\varepsilon \left[u_h'\right](z)\right)^2
$$

$$
\hat{\eta}_T^2 = \min\{\rho^{-1}, \beta^{-1}(1 + \log(Pe)) h_T, \varepsilon^{-1} h_T^2\} \|f - \mathcal{L}u_h\|_{L^2(T)}^2
$$

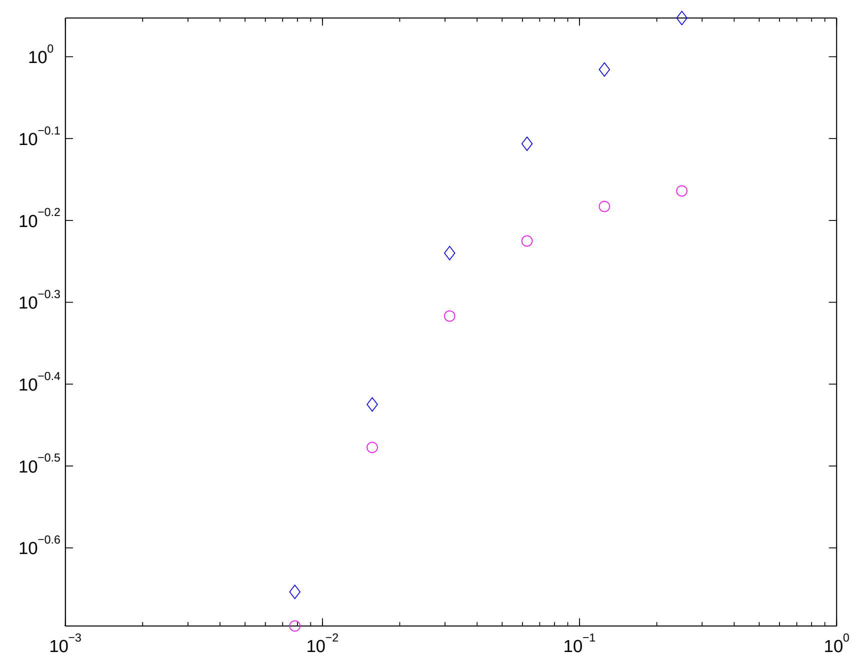
$$
\hat{\eta}_z^2 = \min\{\rho^{-1/2}\varepsilon^{-1/2}, \beta^{-1}(1 + \log(Pe)), \varepsilon^{-1} h_T\} \left(\varepsilon \left[u_h'\right](z)\right)^2
$$

$$
\mathrm{Pe} := \frac{\mathrm{diam}(\Omega)\|\beta\|}{\varepsilon}.
$$

test n.1: no reaction

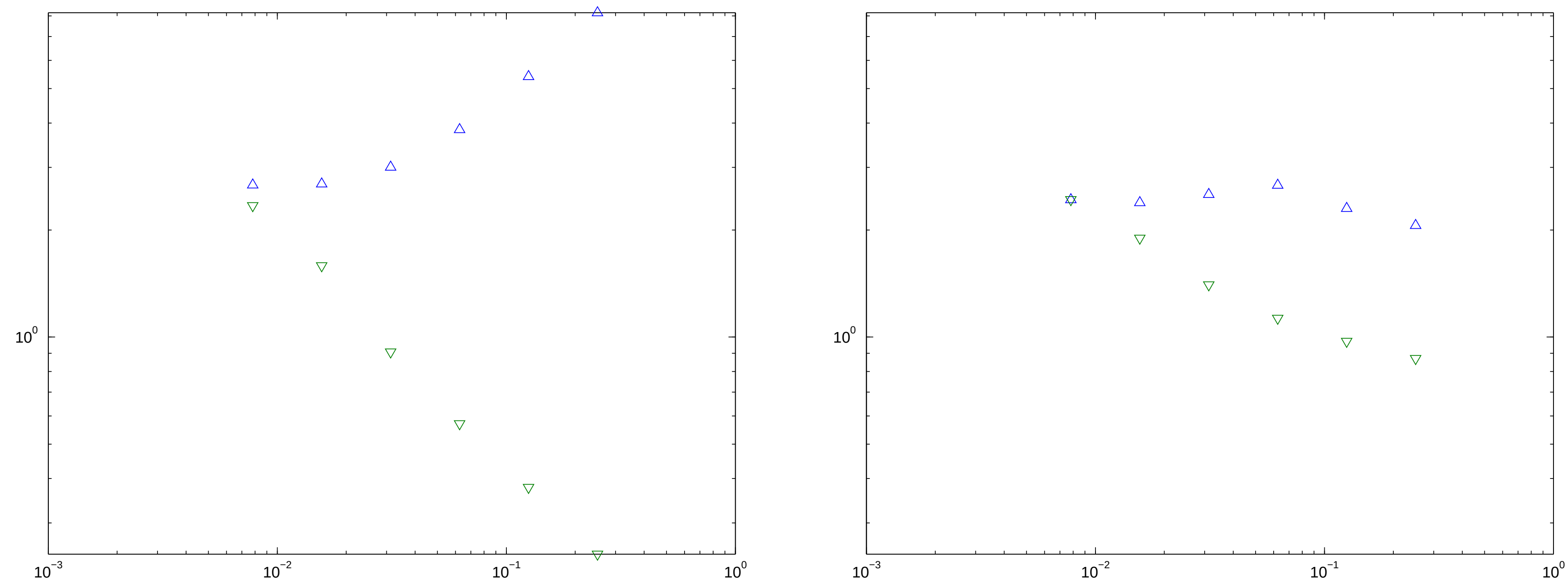


u for $\varepsilon = 10^{-2}$, $\beta = 1$ and $\rho = 0$.



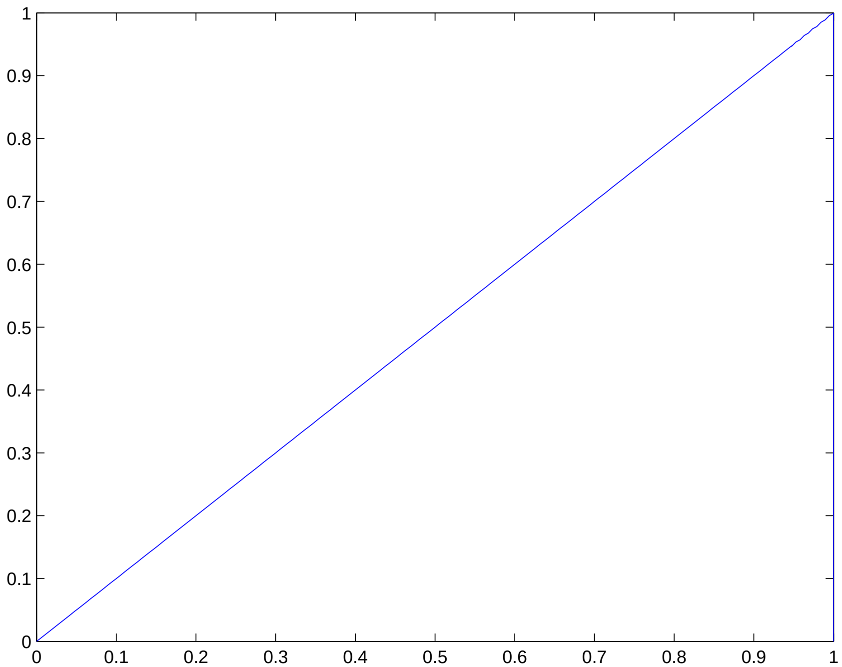
$\circ = \|u - u_h\|_E$ and $\diamond = \|u - u_h\|_V$ vs. h.

effectivity indexes for test n.1

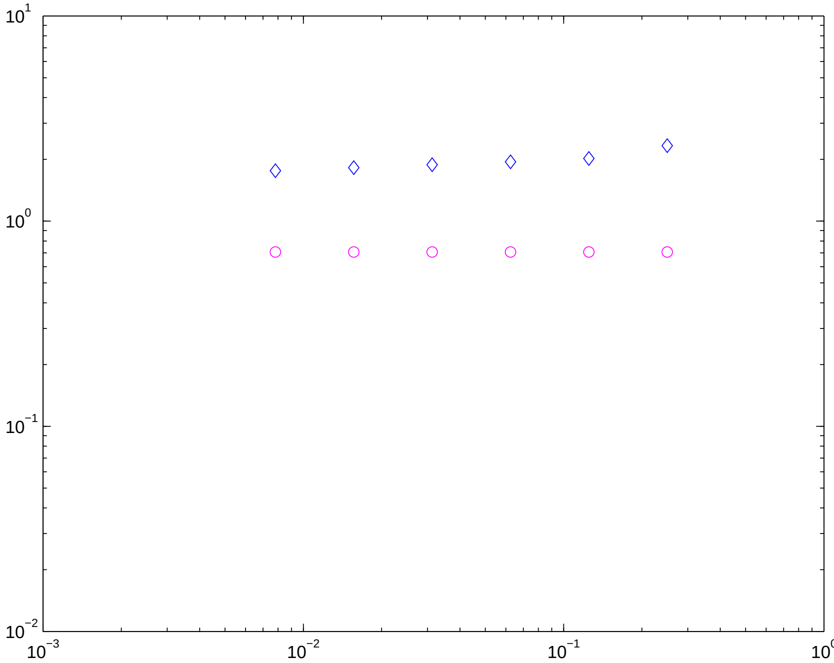


e.i.: ▽ for $\check{\eta}_{old}$ and △ for $\hat{\eta}_{old}$, vs. h; ▽ for $\check{\eta}_{new}$ and △ for $\hat{\eta}_{new}$, vs. h

test n.2: no reaction

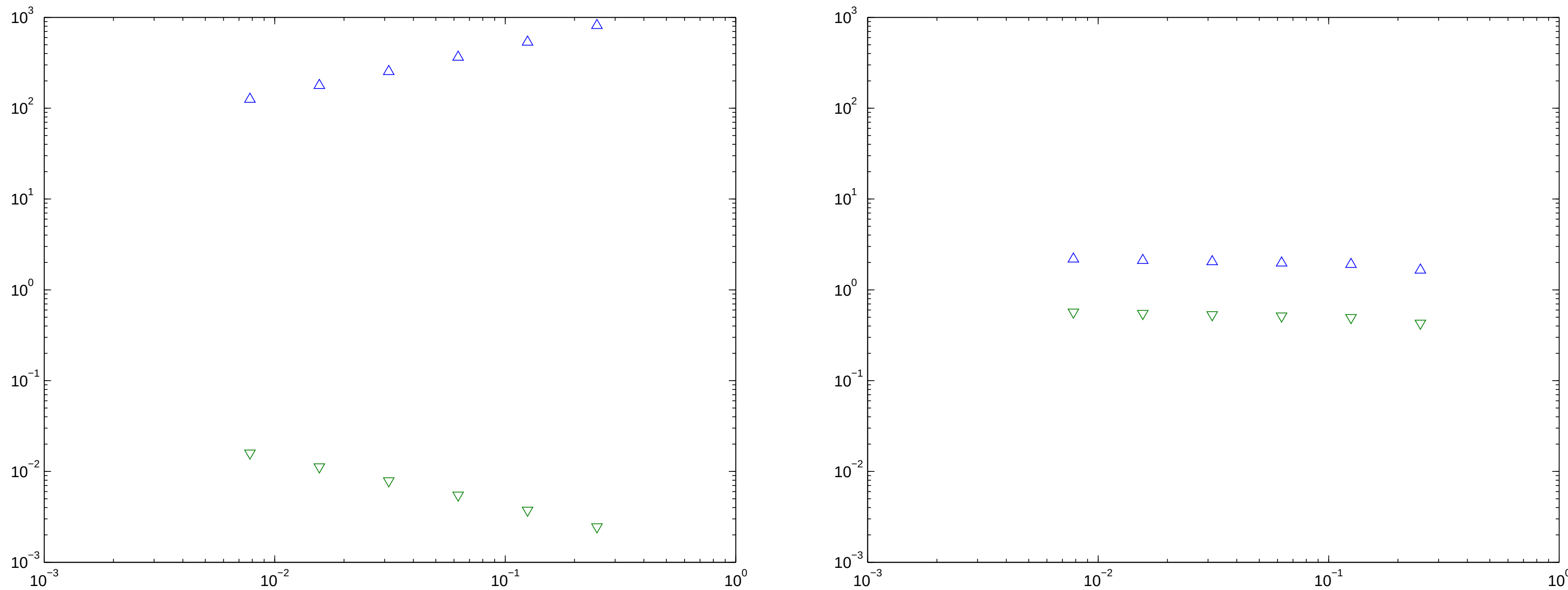


u for $\varepsilon = 10^{-6}$, $\beta = 1$ and $\rho = 0$.



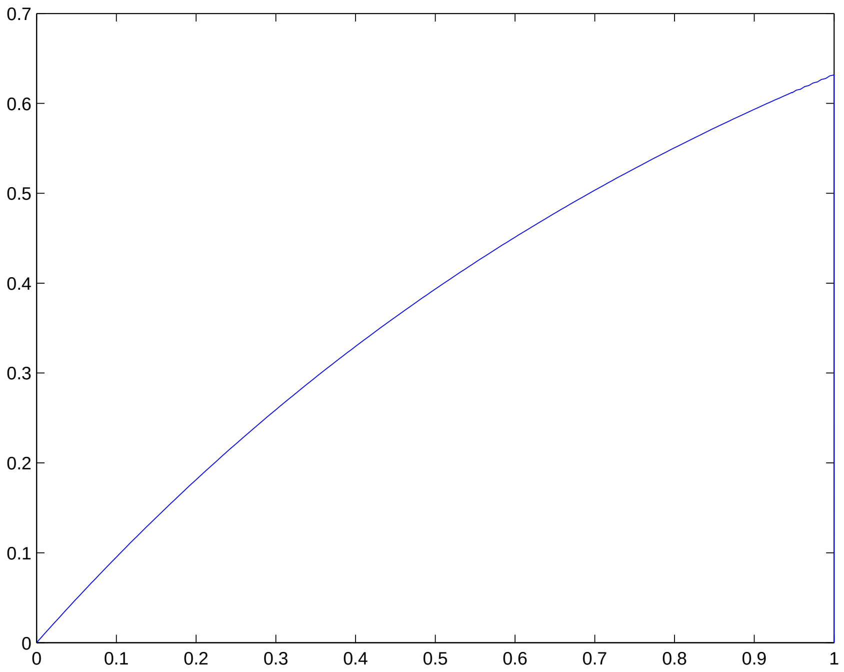
$\circ = \|u - u_h\|_E$, $\diamond = \|u - u_h\|_V$ vs. h.

effectivity indexes for test n.2

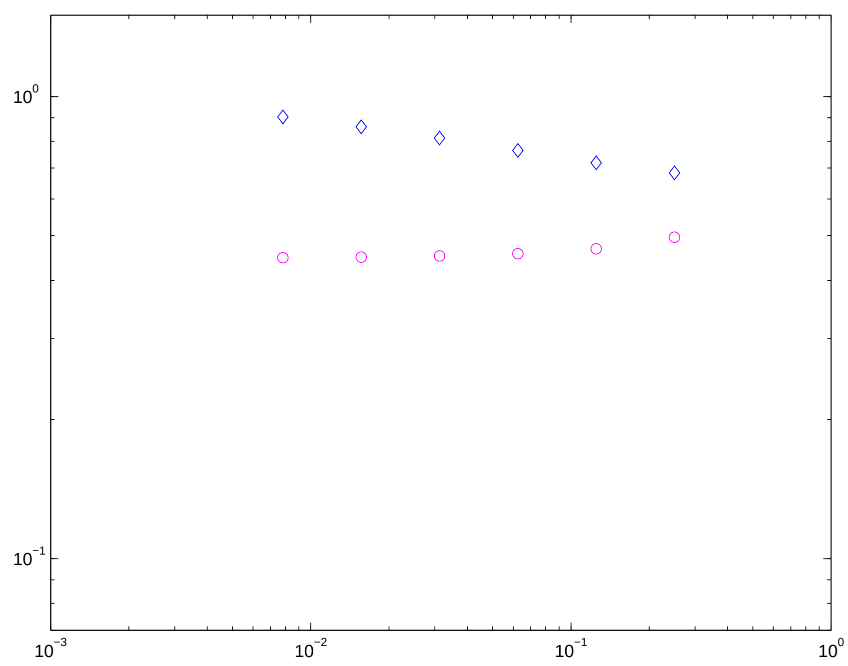


e.i.: $\bigtriangledown$ for $\check{\eta}_{old}$ and $\bigtriangleup$ for $\hat{\eta}_{old}$, vs. h; $\bigtriangledown$ for $\check{\eta}_{new}$ and $\bigtriangleup$ for $\hat{\eta}_{new}$, vs. h

test n.3: with reaction

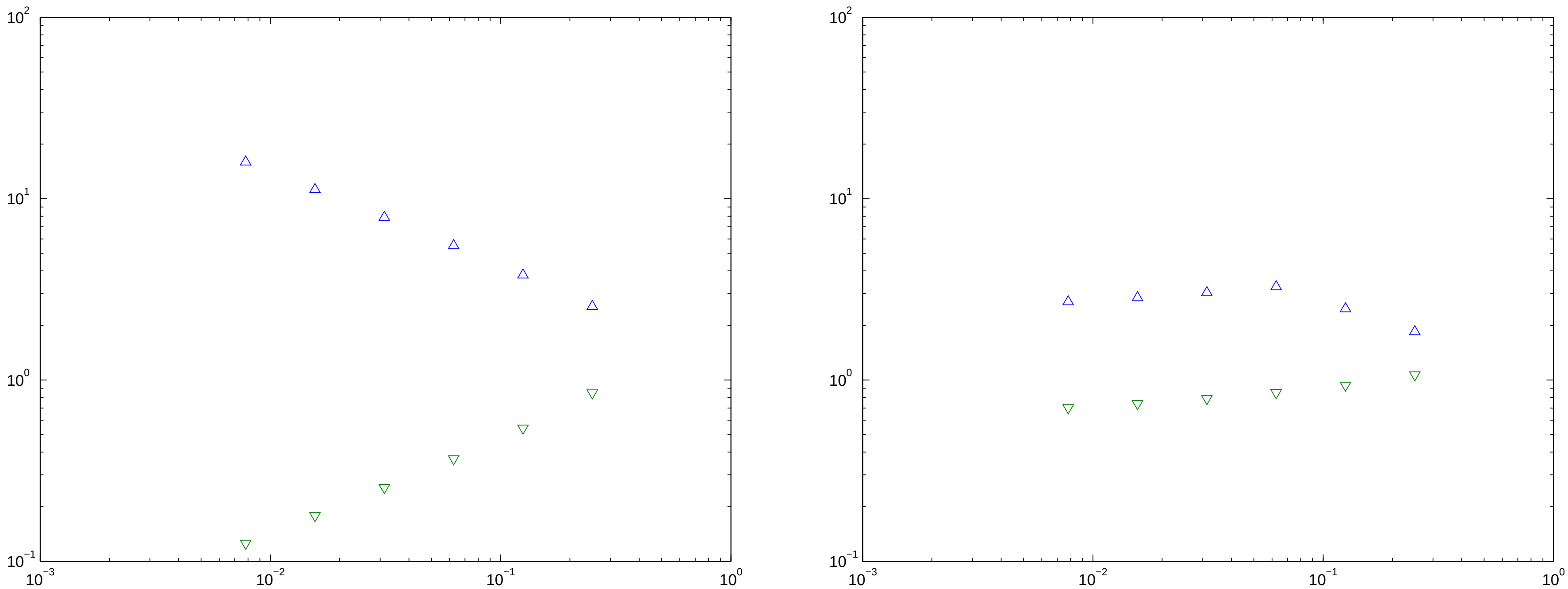


u for $\varepsilon = 10^{-6}$, $\beta = 1$ and $\rho = 1$.



$\circ = \|u - u_h\|_E$, $\diamond = \|u - u_h\|_V$, vs. h.

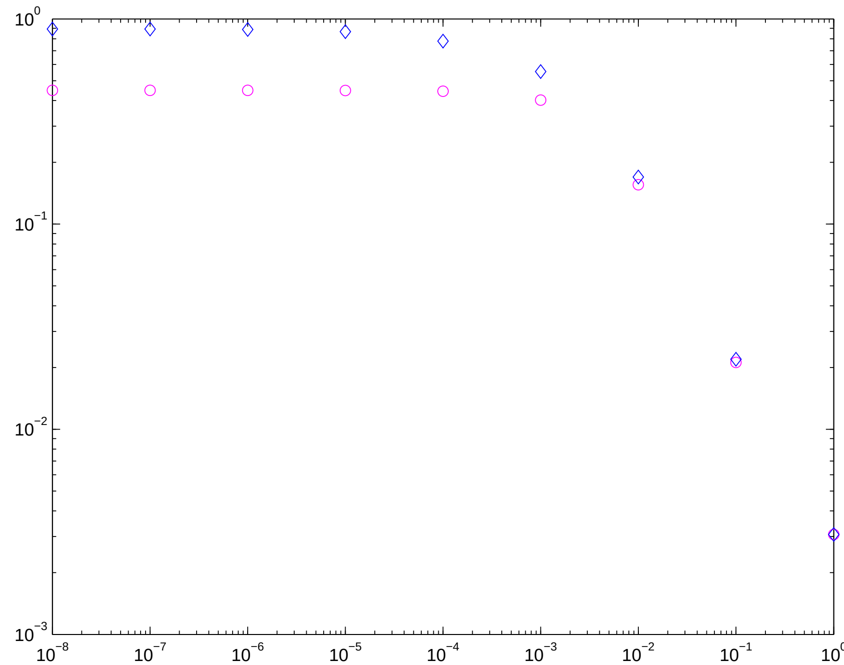
effectivity indexes for test n.3



e.i.: ▽ for $\check{\eta}_{old}$ and △ for $\hat{\eta}_{old}$, vs. h; ▽ for $\check{\eta}_{new}$ and △ for $\hat{\eta}_{new}$, vs. h

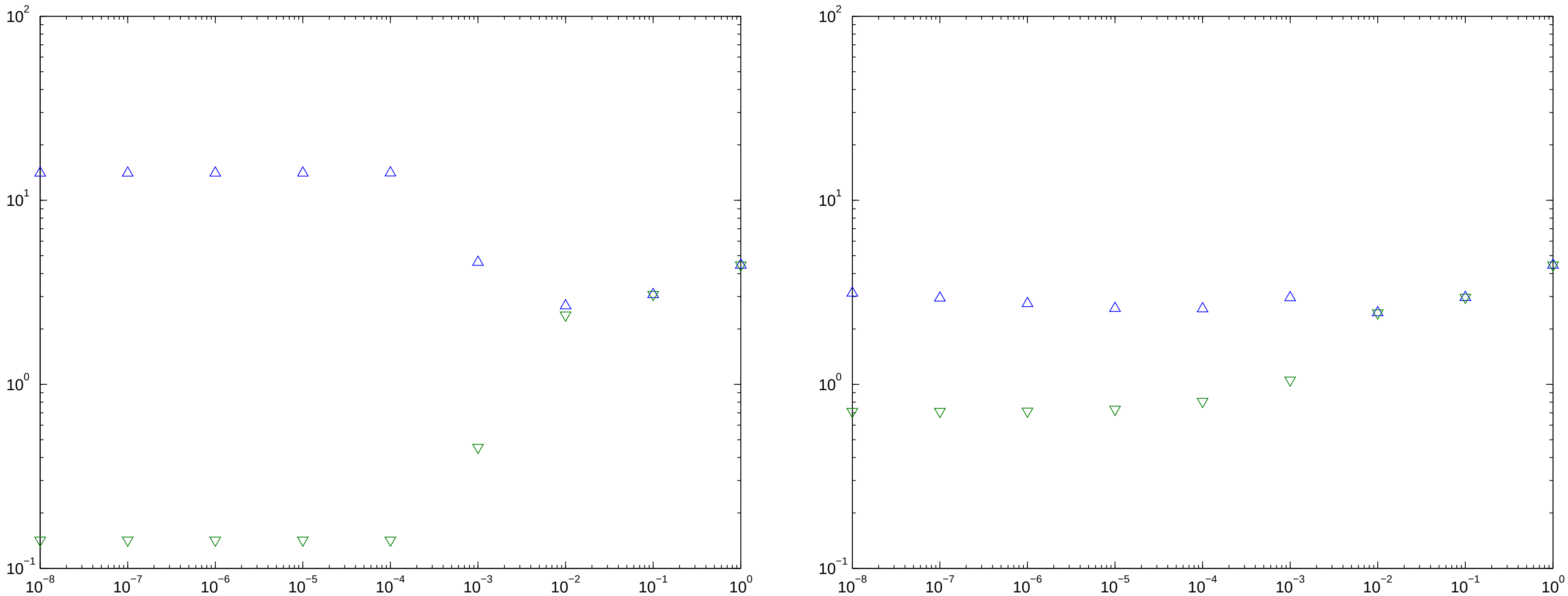
test n.4: varying ε

keep a fixed meshsize h
and let $\varepsilon \to 0$



$\circ = \|u - u_h\|_E$, $\diamond = \|u - u_h\|_V$, vs. ε.

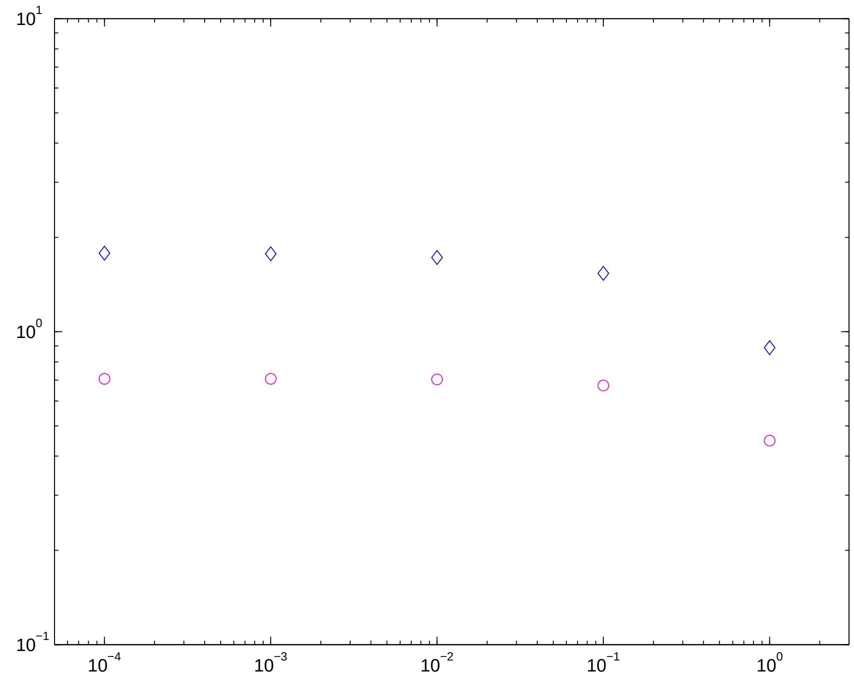
effectivity indexes for test n.4



e.i.: $\bigtriangledown$ for $\check{\eta}_{old}$ and $\bigtriangleup$ for $\hat{\eta}_{old}$, ε; $\bigtriangledown$ for $\check{\eta}_{new}$ and $\bigtriangleup$ for $\hat{\eta}_{new}$, vs. ε.

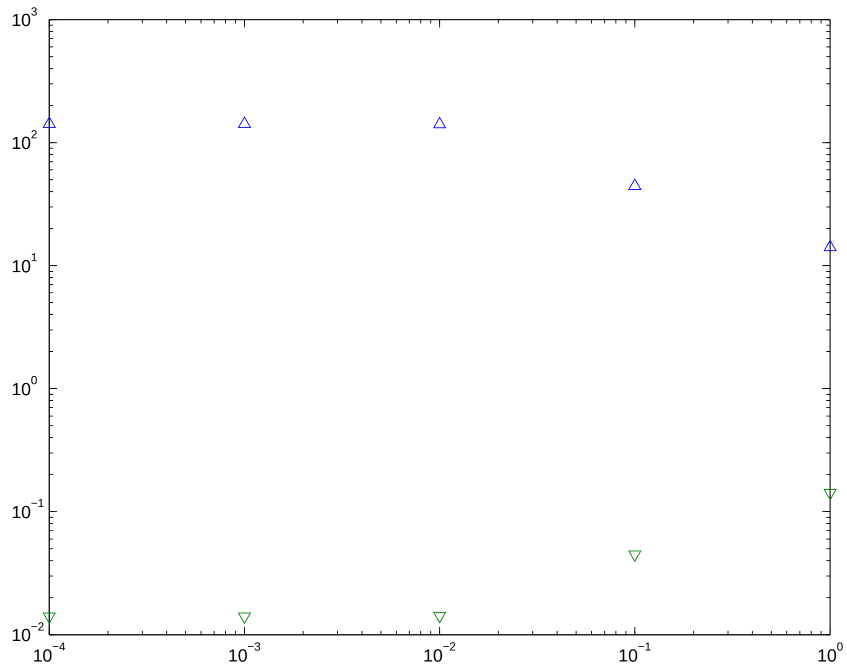
test n.5: varying ρ

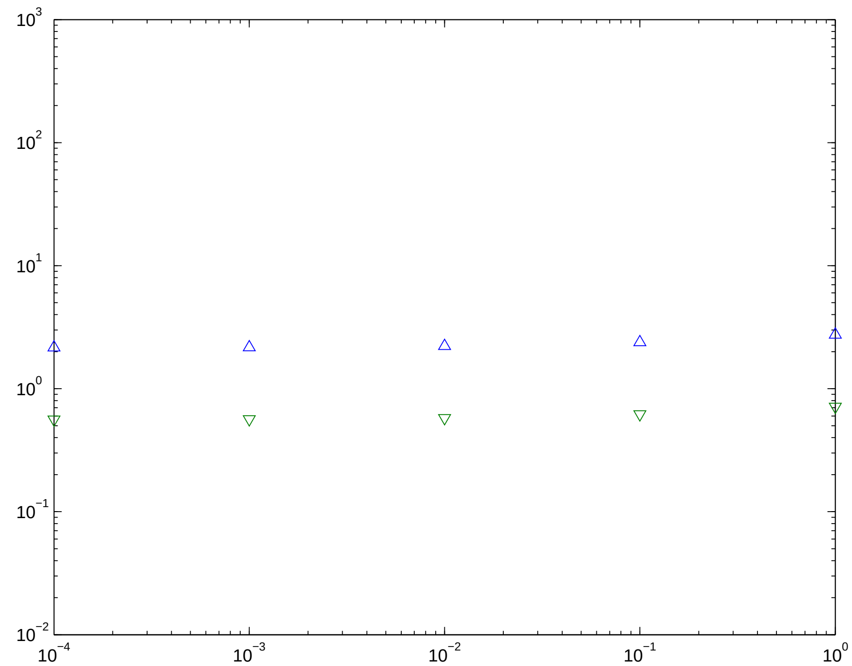
keep a fixed meshsize h
and let $\rho \to 0$



$\circ = \| u - u_h \|_E$, $\diamond = \| u - u_h \|_V$, vs. ρ.

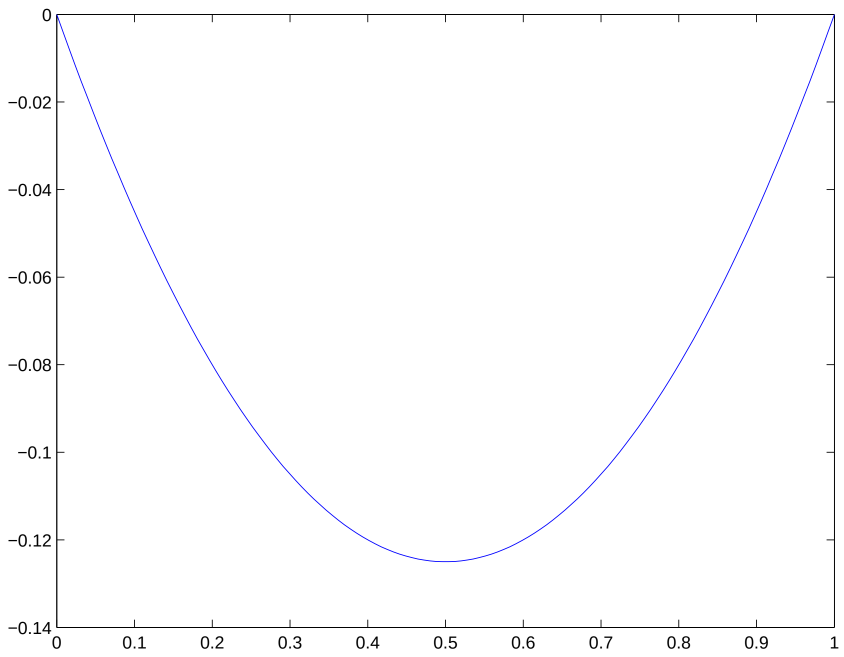
effectivity indexes for test n.5: varying ρ

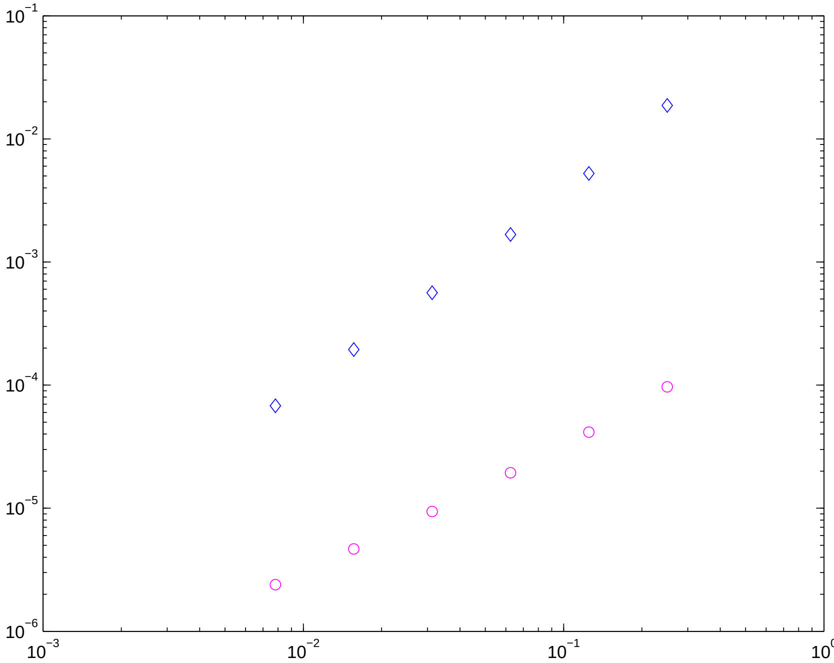




e.i.: ▽ for $\check{\eta}_{old}$ and △ for $\hat{\eta}_{old}$, ρ; ▽ for $\check{\eta}_{new}$ and △ for $\hat{\eta}_{new}$, vs. ρ. .

test n.6: smooth u

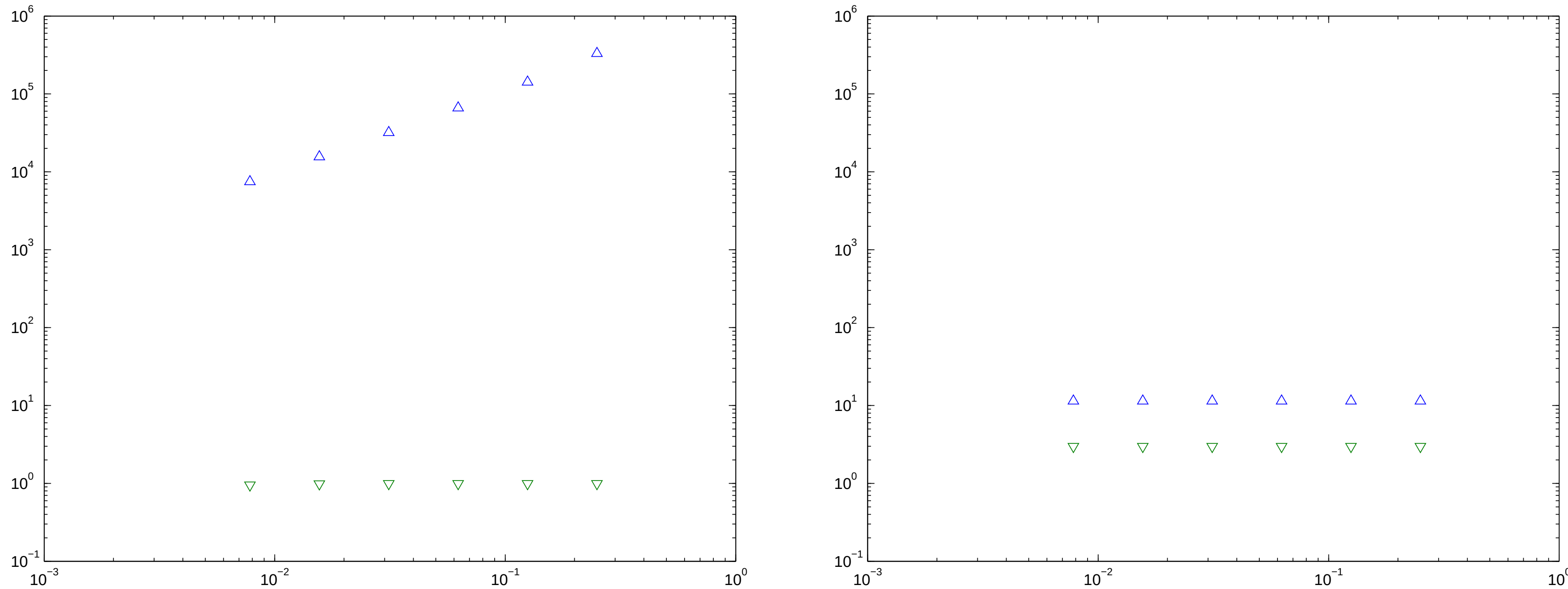




u for $\varepsilon = 10^{-6}$, $\beta = 1$ and $\rho = 0$.

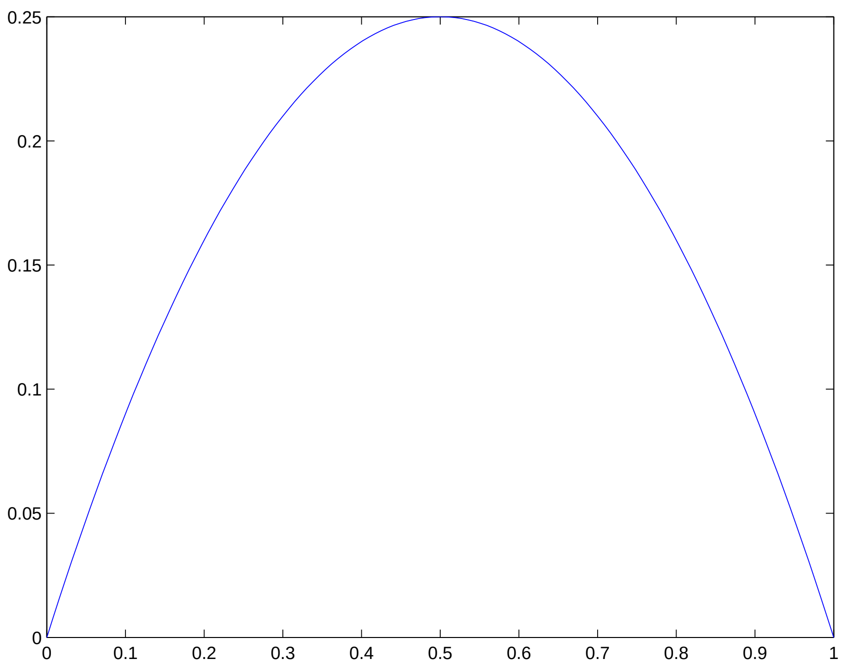
$\circ = \|u - u_h\|_E$, $\diamond = \|u - u_h\|_V$ vs. h.

effectivity indexes for test n.6

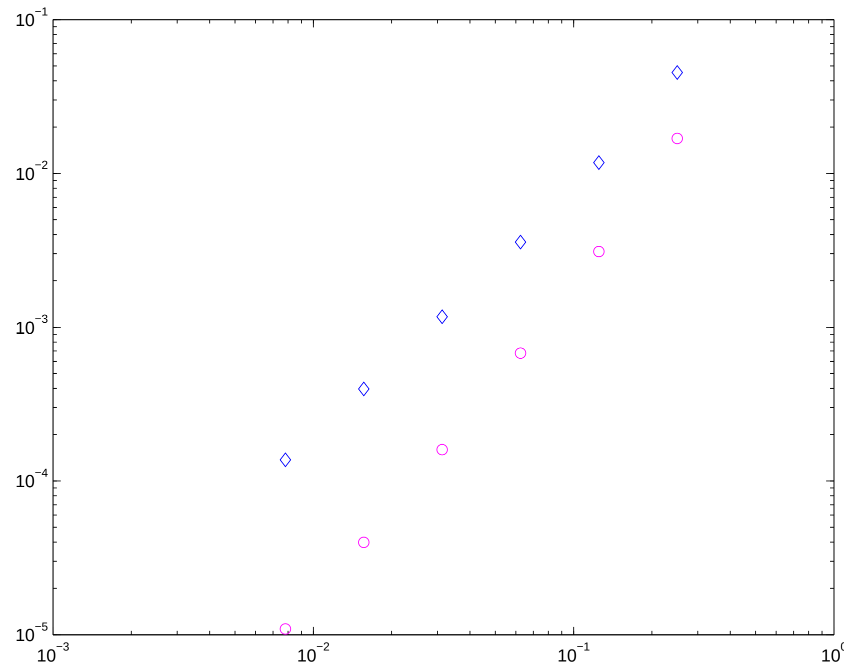


e.i.: $\bigtriangledown$ for $\check{\eta}_{old}$ and $\bigtriangleup$ for $\hat{\eta}_{old}$, vs. h; $\bigtriangledown$ for $\check{\eta}_{new}$ and $\bigtriangleup$ for $\hat{\eta}_{new}$, vs. h

test n.7: another smooth u,

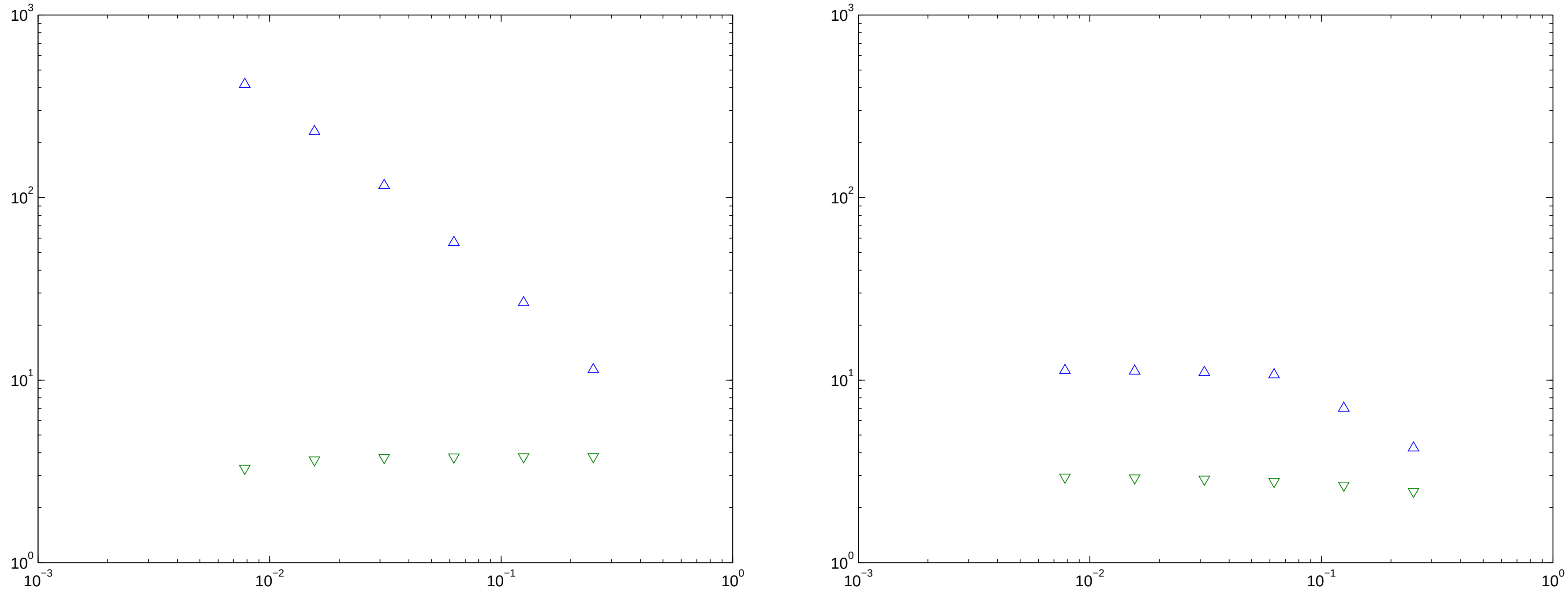


u for $\varepsilon = 10^{-6}$, $\beta = 1$ and $\rho = 1$.



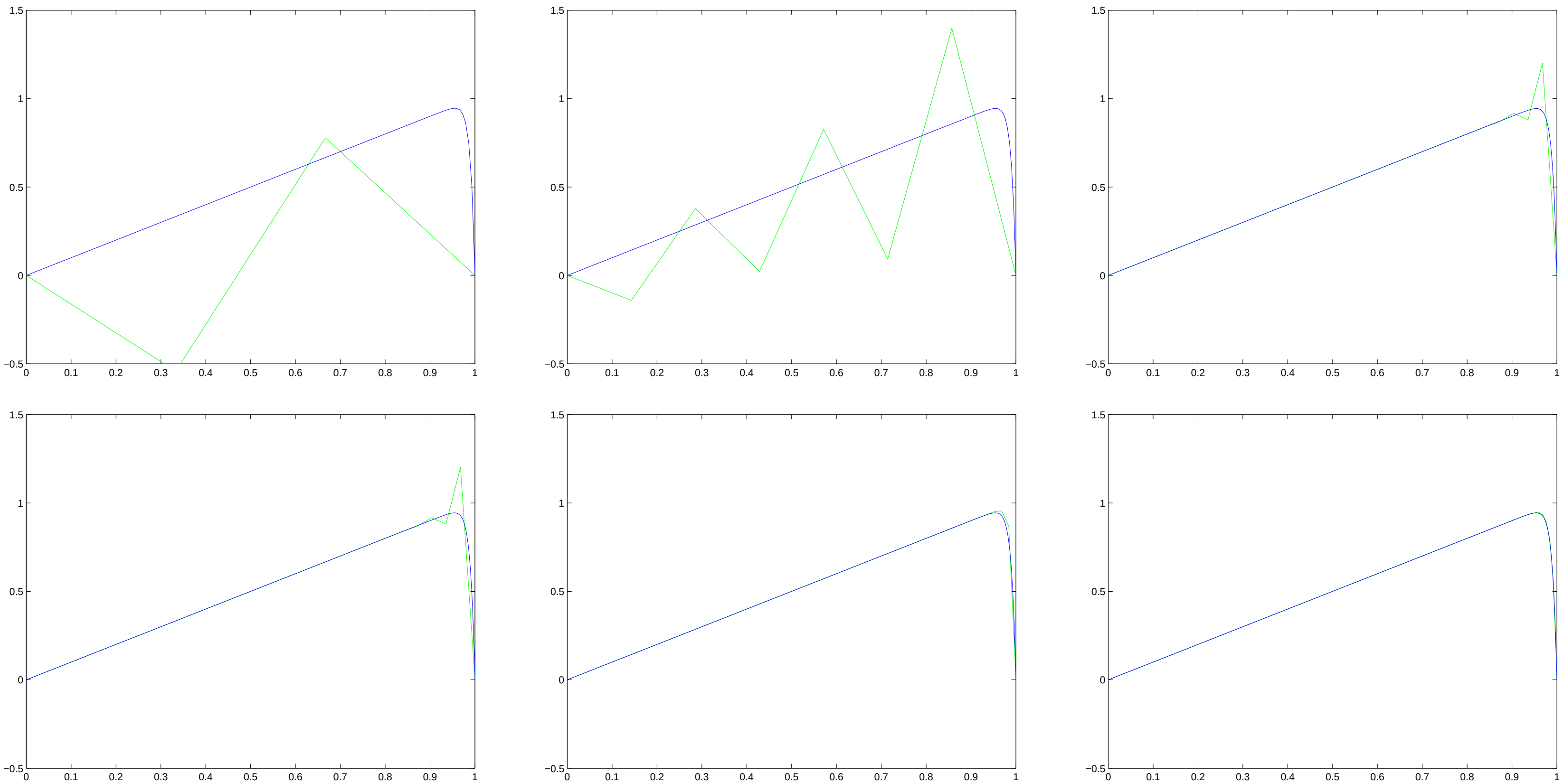
$\circ = \|u - u_h\|_E$, $\diamond = \|u - u_h\|_V$ vs. h.

effectivity indexes for test n.7



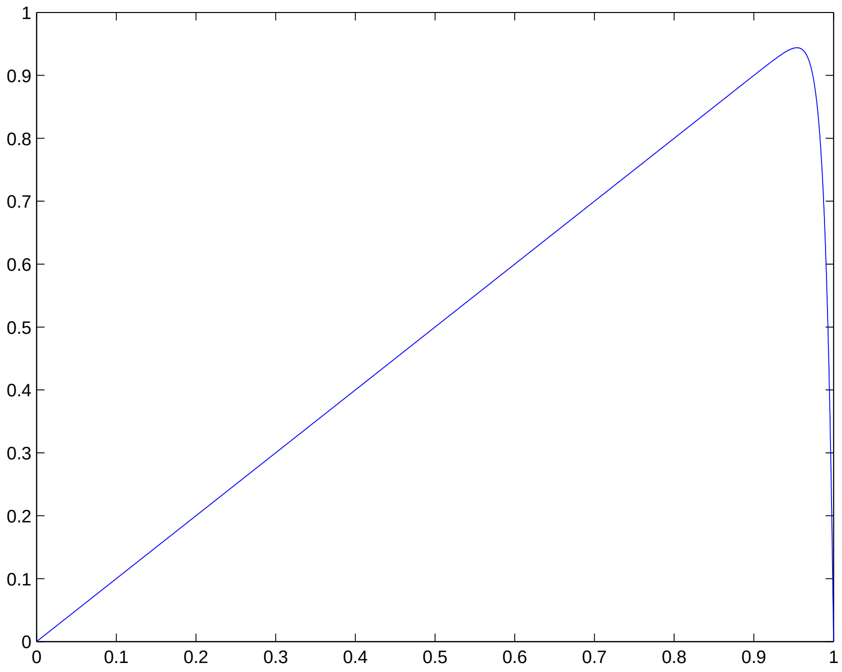
e.i.: ▽ for $\check{\eta}_{old}$ and △ for $\hat{\eta}_{old}$, vs. h; ▽ for $\check{\eta}_{new}$ and △ for $\hat{\eta}_{new}$, vs. h

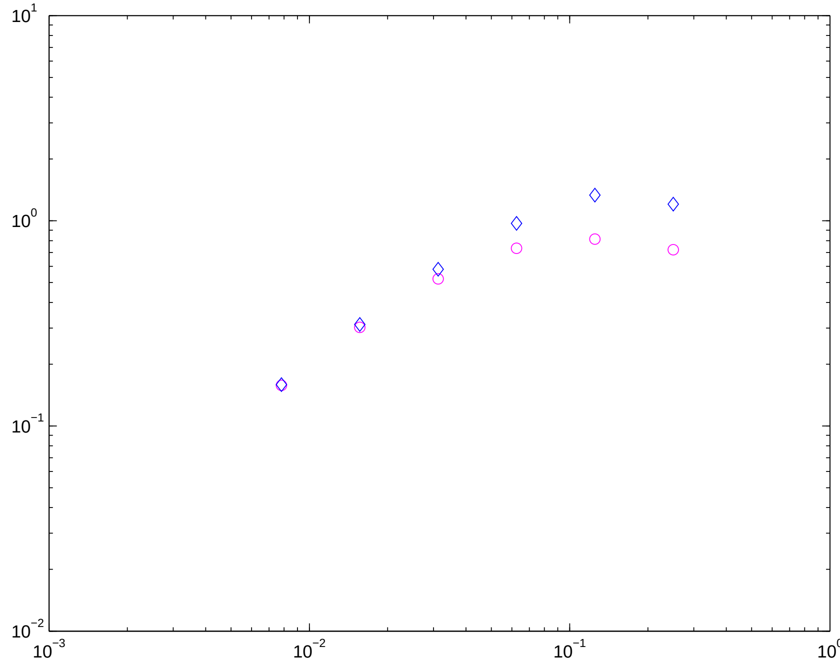
test n.8: Standard Galerkin (unstable when $Pe_T := \beta h_T/\varepsilon > 1$)



u and a sequence of discrete solutions u_h, for $\varepsilon = 10^{-2}$, $\beta = 1$, $\rho = 0$;

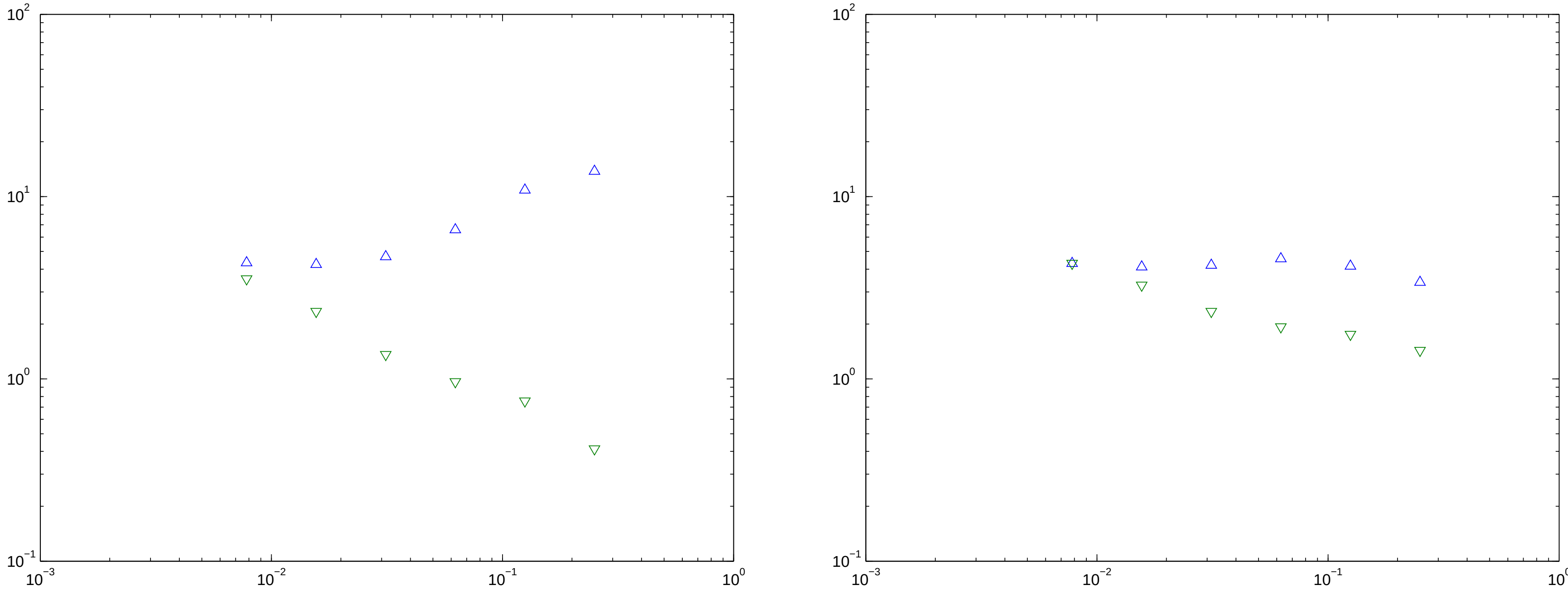
test n.8: Standard Galerkin (unstable when $Pe_T := \beta h_T/\varepsilon > 1$)





u for $\varepsilon = 10^{-2}$, $\beta = 1$, $\rho = 0$; $\circ = \|u - u_h\|_E$, $\diamond = \|u - u_h\|_V$, vs. ρ.

effectivity indexes for test n.8

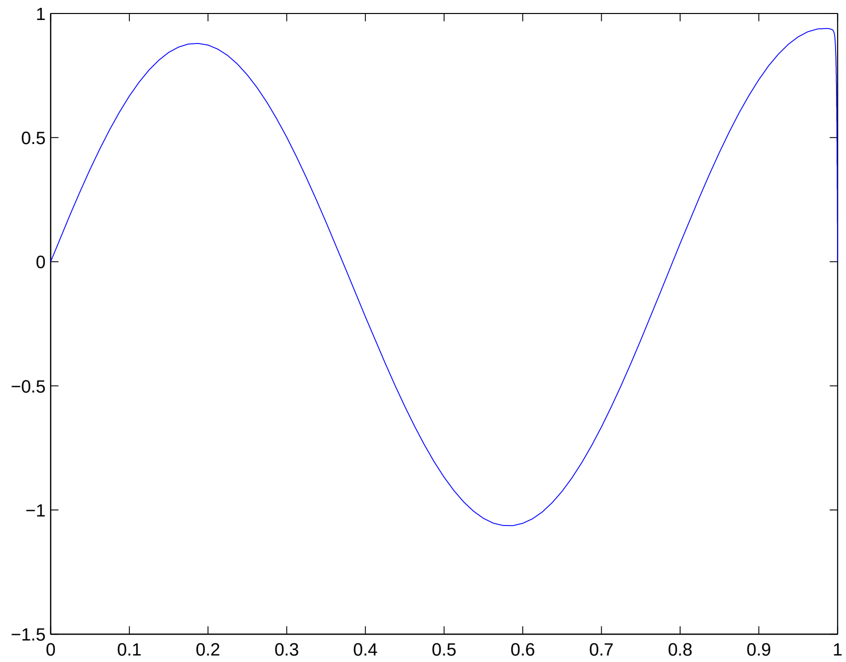


e.i.: ▽ for $\check{\eta}_{old}$ and △ for $\hat{\eta}_{old}$, vs. h; ▽ for $\check{\eta}_{new}$ and △ for $\hat{\eta}_{new}$, vs. h

test n.9: first example of adaptivity

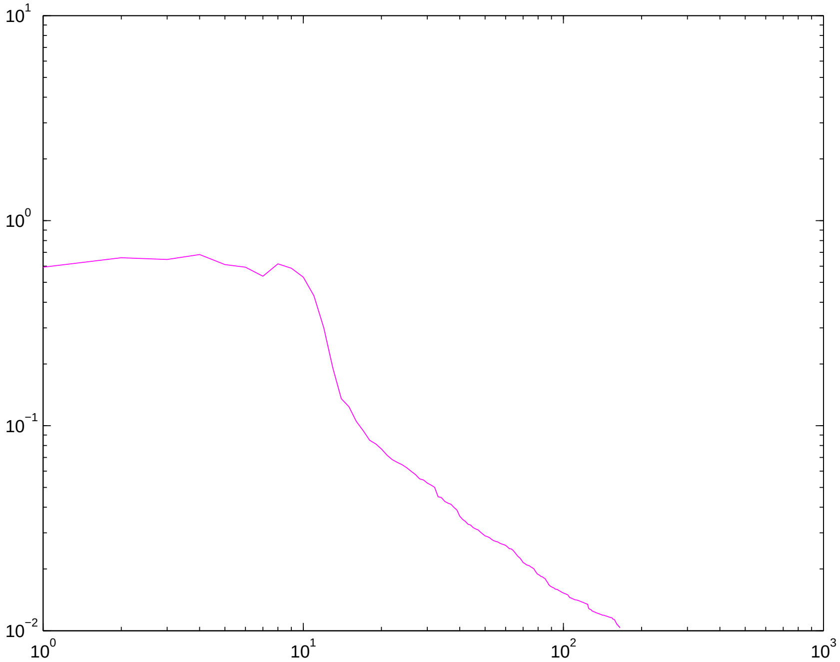
$\varepsilon = 10^{-3}$, $\beta = 1$, $\rho = 1$ and $f(x) = \frac{5}{2}\pi \cos\left(\frac{5}{2}\pi x\right)$

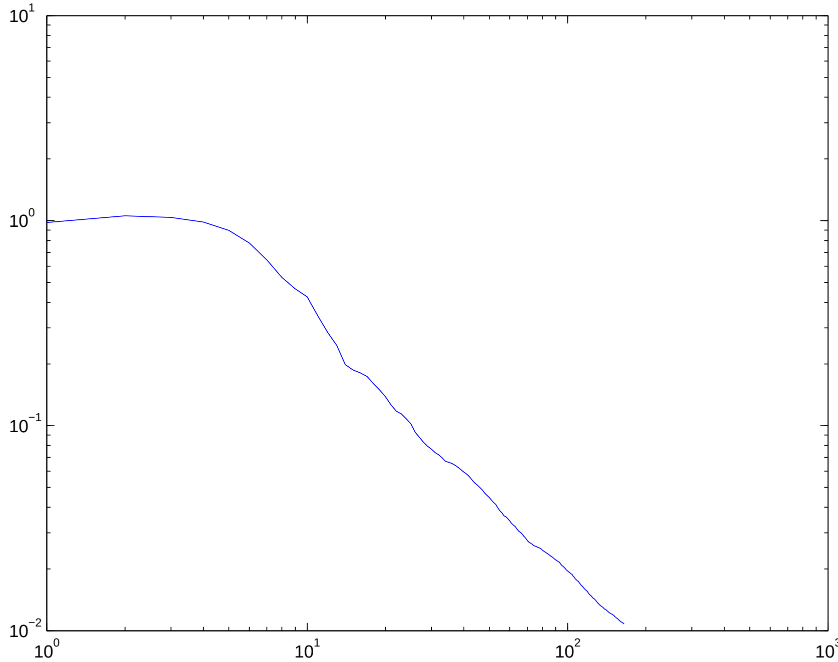
the adaptivity is driven by the (local) estimator from below.



the exact solution u.

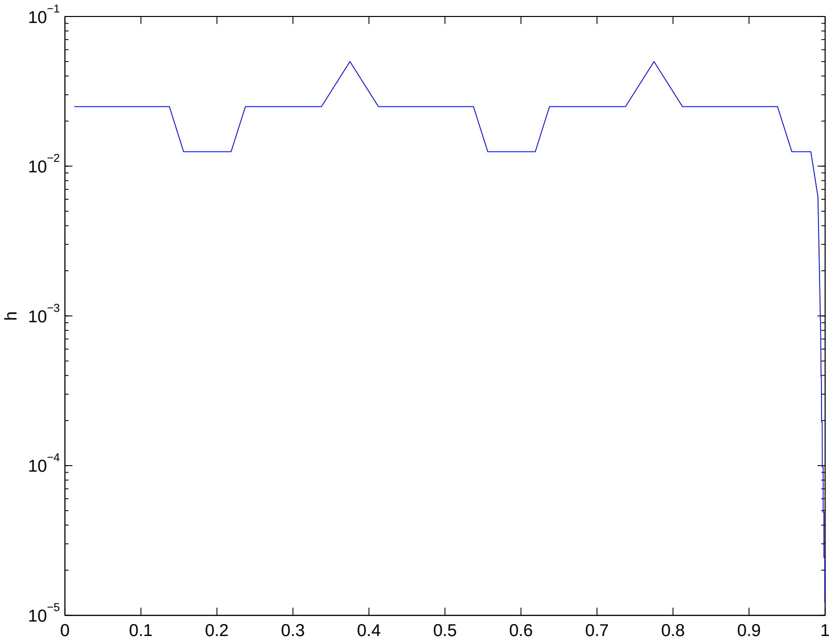
test n.9: error plots



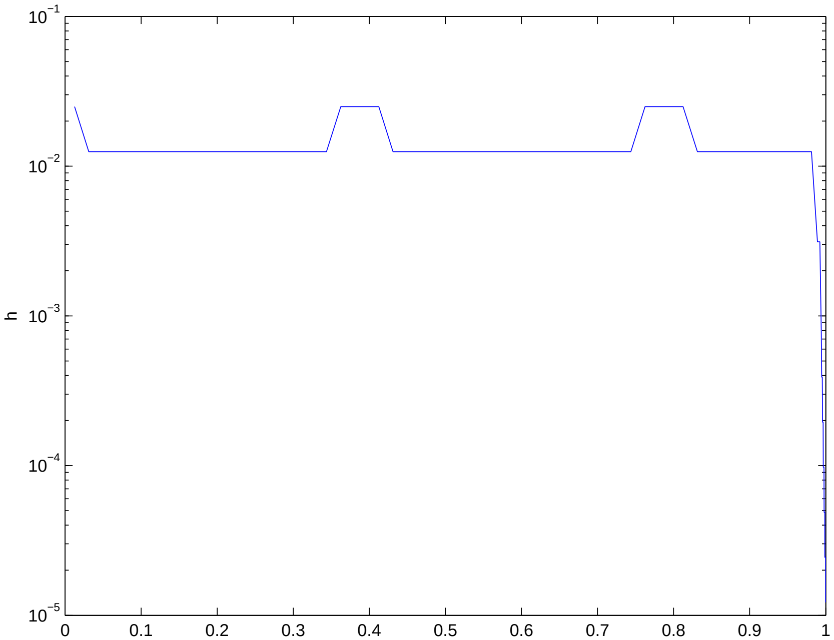


$\|u - u_h\|_E$ vs. the number of d.o.f.; $\|u - u_h\|_V$, vs. the number of d.o.f..

test n.9: final meshes

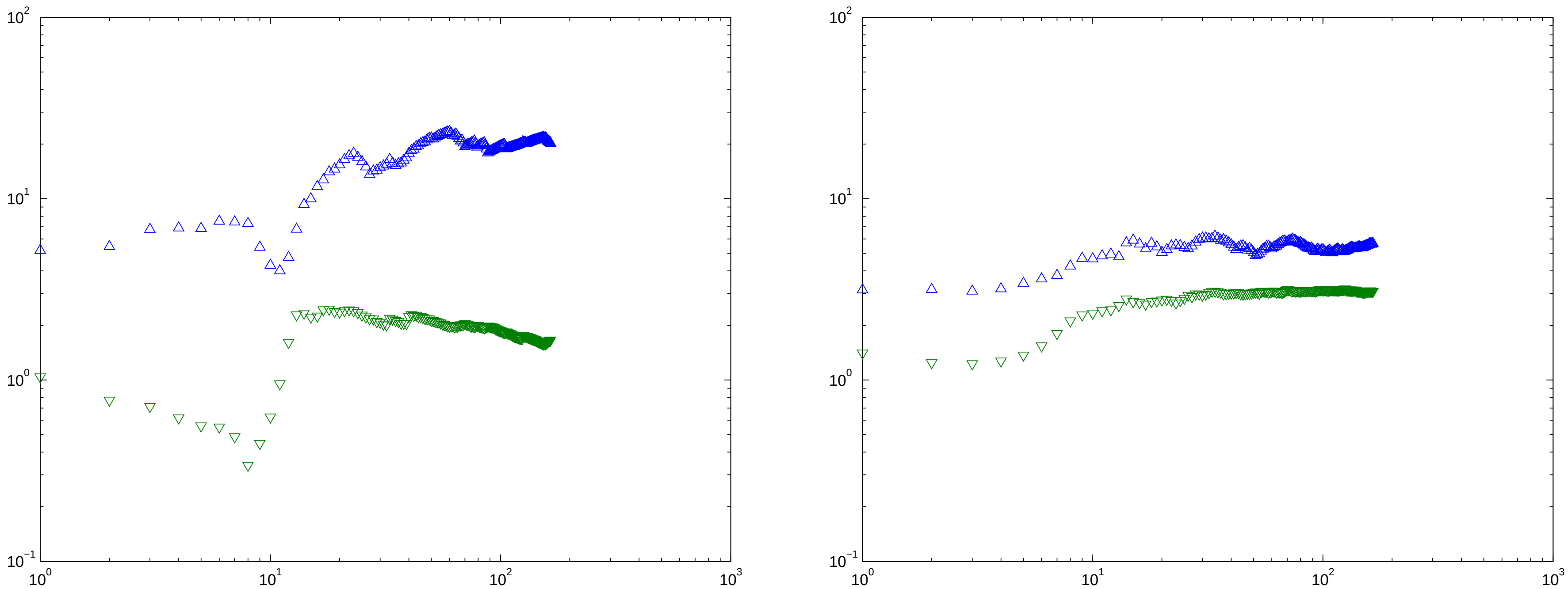


final mesh-sizes using $\check{\eta}_{old}$;



final mesh-sizes using $\check{\eta}_{new}$.

test n.9: effectivity indexes

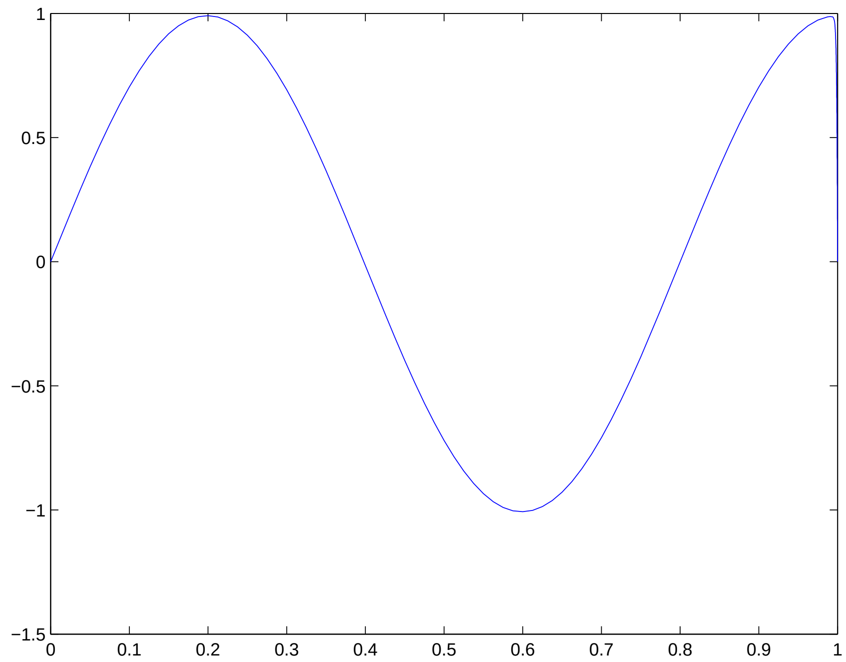


e.i.: $\bigtriangledown$, $\check{\eta}_{old}$ and $\bigtriangleup$, $\hat{\eta}_{old}$, vs. # d.o.f. $\bigtriangledown$, $\check{\eta}_{new}$ and $\bigtriangleup$, $\hat{\eta}_{new}$, vs.# d.o.f.

test n.10: second example of adaptivity

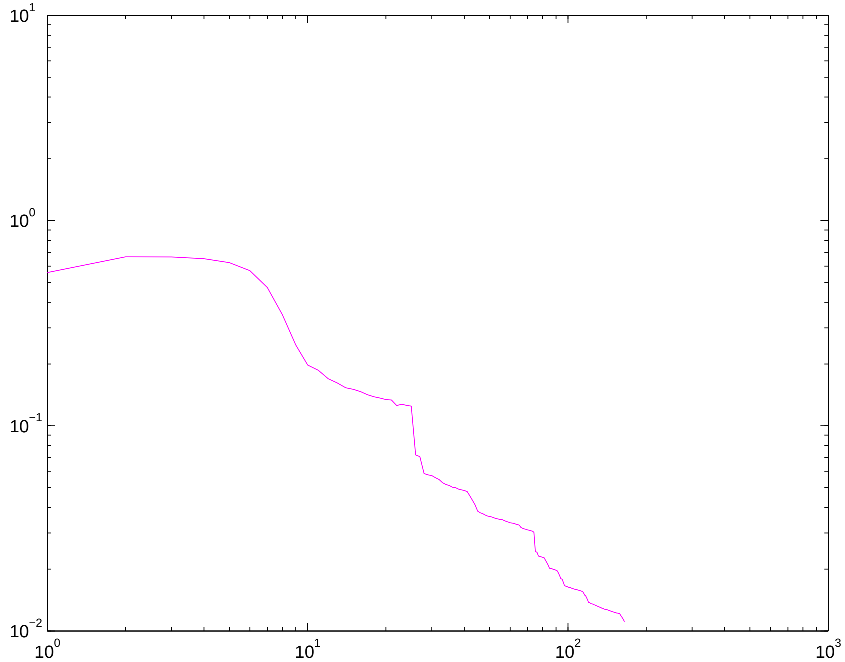
$\varepsilon = 10^{-3}$, $\beta = 1$, $\rho = 0$ and $f(x) = \frac{5}{2}\pi \cos\left(\frac{5}{2}\pi x\right)$

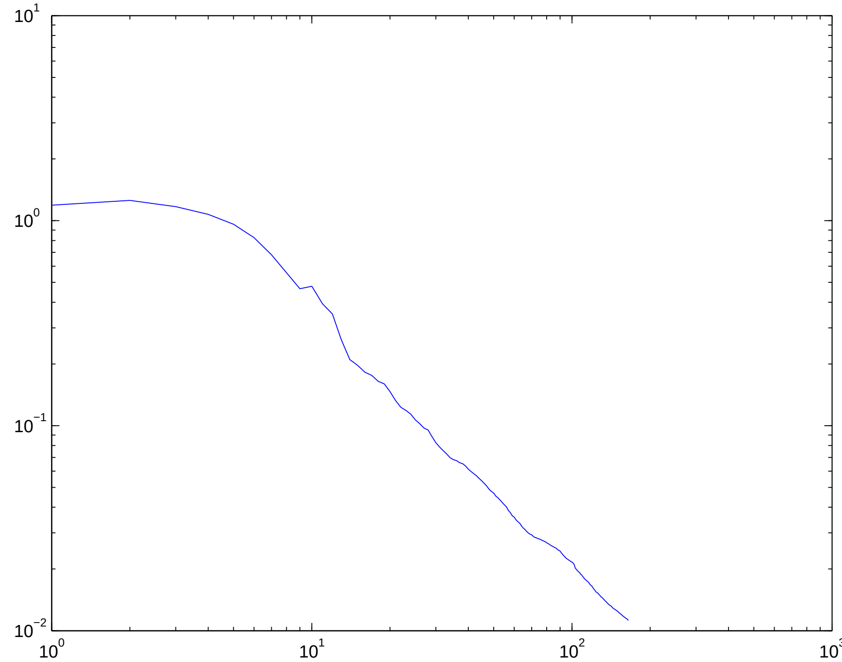
the adaptivity is driven by the (local) estimator from below.



the exact solution u.

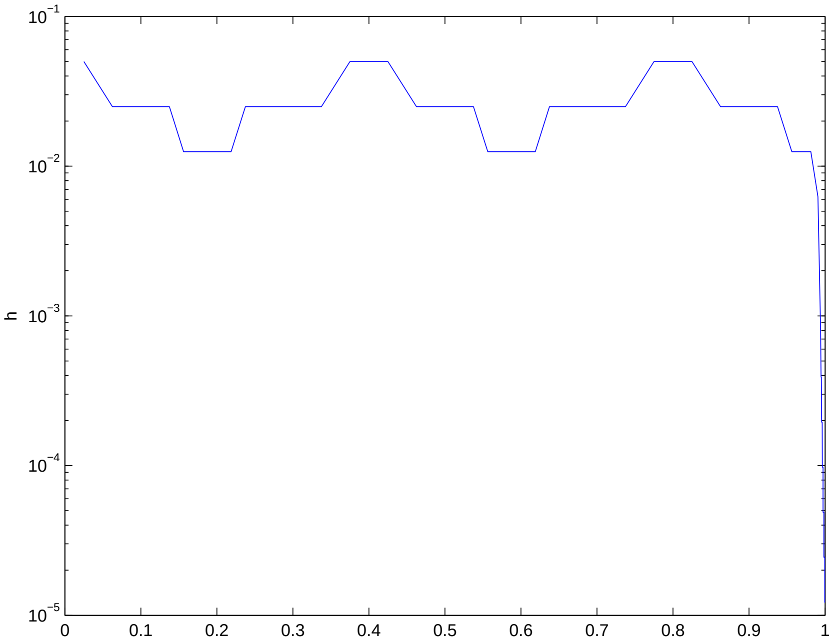
test n.10: error plots



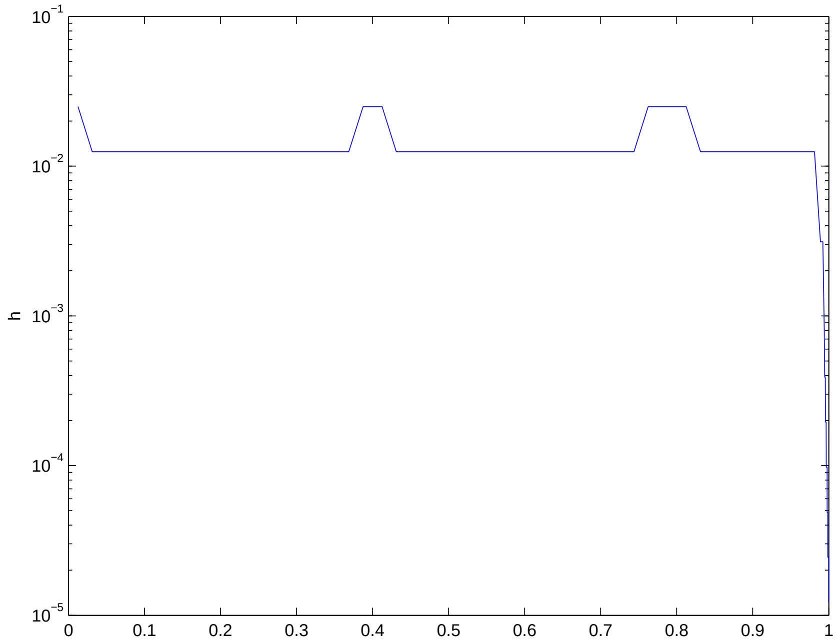


$\|u - u_h\|_E$ vs. the number of d.o.f.; $\|u - u_h\|_V$, vs. the number of d.o.f..

test n.10: final meshes

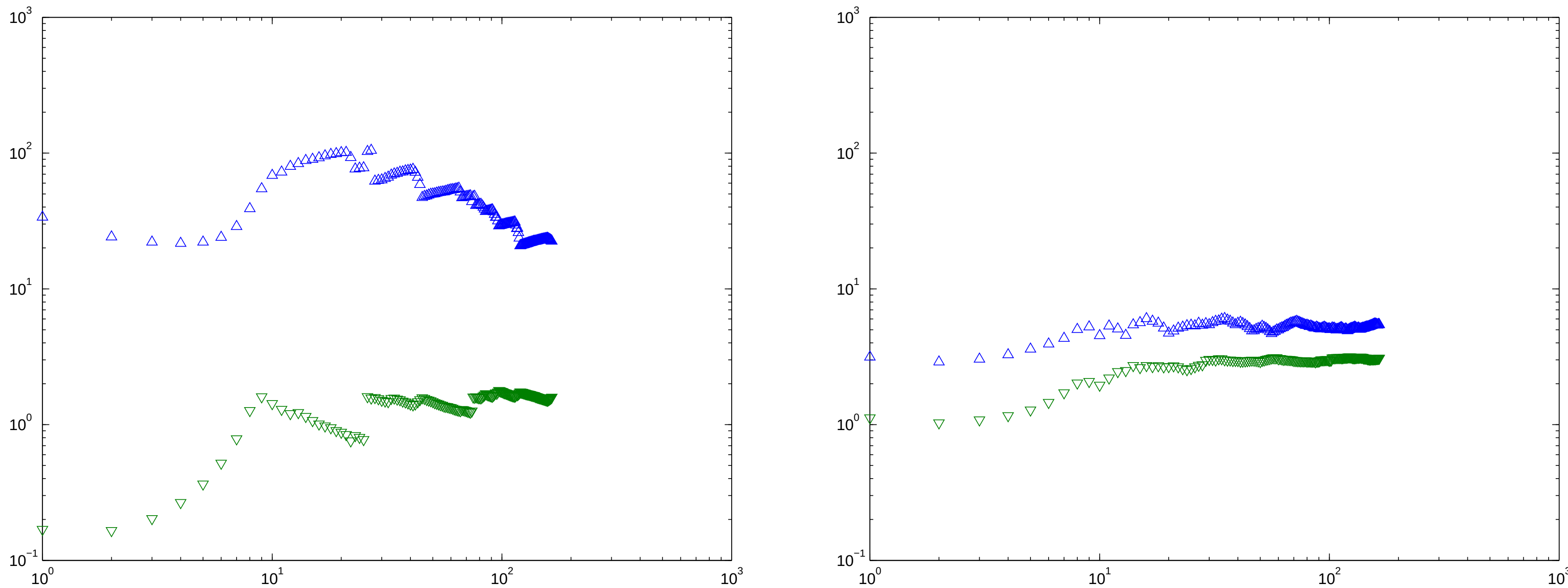


final mesh-sizes using $\check{\eta}_{old}$;



final mesh-sizes using $\check{\eta}_{new}$.

test n.10: effectivity indexes



e.i.: ▽, $\check{\eta}_{old}$ and △, $\hat{\eta}_{old}$, vs. # d.o.f.; ▽, $\check{\eta}_{new}$ and △, $\hat{\eta}_{new}$, vs. # d.o.f.